PARALLEL MECHANISMS WITH VARIABLE COMPLIANCE

By

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This dissertation is dedicated to my wife, Eyun Jung Lee and son, Sung Jae.

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Compliant mechanisms can be considered as planar/spatial springs having multiple degrees of freedom rather than one freedom as line springs have. The compliance of the mechanism can be well described by the stiffness matrix of the mechanism which relates a small twist applied to the mechanism to the corresponding wrench exerted on the mechanism.

A derivative of the spring wrench connecting two moving rigid bodies is derived. By using the derivative of the spring wrench, the stiffness matrices of compliant mechanisms which consist of rigid bodies connected to each other by line springs are obtained. It is shown that the resultant compliance of two compliant parallel mechanisms that are serially arranged is not the summation of the compliances of the constituent mechanisms unless the external wrench applied to the mechanism is zero.

A derivative of the stiffness matrix of planar compliant mechanisms with respect to the twists of the constituent rigid bodies and the spring parameters such as the stiffness coefficient and free length is obtained. It is shown that the compliance and the resultant

wrench of a compliant mechanism may be controlled at the same time by using adjustable line springs.

CHAPTER 1
INTRODUCTION

1.1 Motivation

Robots have been employed successfully in applications that do not require interaction between the robot and the environment but require only position control schemes. For instance, arc welding and painting belong to this category of application. There are many other operations involving contact of the robot and its environment. A small amount of positional error of the robot system, which is almost inevitable, may cause serious damage to the robot or the object with which it is in contact. Compliant mechanisms, which may be inserted between the end effecter and the last link of the robotic manipulator, can be a solution to this problem.

Compliant mechanisms can be considered as spatial springs having multiple degrees of freedom rather than one freedom as line springs have. A small force/torque applied to the compliant mechanism generates a small displacement of the compliant mechanism. This relation is well described by the compliance matrix of the mechanism. RCC (Remote Center of Compliance) devices, developed by Whitney (1982), are one of the most successful compliant mechanisms. They have a unique compliant property at a specific operation point and are mainly used to compensate positional errors during tasks such as inserting a peg into a chamfered hole. Compliant mechanisms can also be employed for force control applications by using the theory of Kinestatic Control which was proposed by Griffis (1991). Kinestatic Control varies the position of the last link of the manipulator to control the position and contact force of the distal end of the robotic manipulator at the same time with the compliance of the mechanism in mind.

Mechanisms with variable compliance, which is the topic of this dissertation, are believed to have several advantages over mechanisms having fixed compliance. Since RCC devices typically have a specific operation point, if the length of the peg to be inserted is changed, a different RCC device should be employed to do insertion tasks unless the RCC device has variable compliance. As for force control tasks, each task may have an optimal compliance. With variable compliant mechanisms, several different tasks involving different force ranges can be accomplished without having to physically change the compliant mechanism. Variable compliant mechanisms also can improve the performance of humanoid robot parts such as ankles and wrists, and animals are believed to have physically variable leg compliance and utilize it when running and hopping (see Hurst et al. 2004).

Many compliant mechanisms including RCC devices have been designed typically based on parallel kinematic mechanisms. Parallel kinematic mechanisms contain positive features compared to serial mechanisms such as higher stiffness, compactness, and smaller positional errors at the cost of a smaller workspace and increased complexity of analysis. In this dissertation mechanisms having two compliant parallel mechanisms in a serial arrangement as well as compliant parallel mechanisms are investigated. These mechanisms may have a trade-off of characteristics relative to traditional parallel and serial mechanisms.

1.2 Literature Review

The concepts of twists and wrenches were introduced by Ball (1900) in his groundbreaking work *A Treatise on the Theory of Screws*. These concepts are employed throughout this dissertation to describe a small (or instantaneous) displacement of a rigid body and a force/torque applied to a body (Crane et al. 2006).

The compliance of a mechanism can be well described by the stiffness matrix which is a 6×6 matrix for a spatial mechanism and a 3×3 matrix for a planar mechanism. Using screw theory, Dimentberg (1965) studied properties of an elastically suspended body. Loncaric (1985) used Lie groups rather than screw theory to study symmetric spatial stiffness matrices of compliant mechanisms assuming that the springs are in an equilibrium position and derived a constraint that makes the number of independent elements of symmetric 6×6 stiffness matrices 20 rather than 21. Loncaric (1987) also defined a normal form of the stiffness matrix in which rotational and translational parts of the stiffness matrix are maximally decoupled.

Griffis (1991) presented a global stiffness model for compliant parallel mechanisms where he used the term global to state that the springs are not restricted to an unloaded equilibrium position. Griffis (1991) also showed that the stiffness matrix is not symmetric when the springs are deflected from the equilibrium positions due to an external wrench. Ciblak and Lipkin (1994) decomposed a stiffness matrix into a symmetric and a skew symmetric part and showed the skew symmetric part is negative one-half the externally applied load expressed as a spatial cross product operator.

Compliant parallel mechanisms have been investigated by a number of researchers to realize desired compliances because of its high stiffness, compactness, and small positional errors. Huang and Schimmels (1998) obtained the bounds of the stiffness matrix of compliant parallel mechanisms which consist of simple elastic devices and proposed an algorithm for synthesizing a realizable stiffness matrix with at most seven simple elastic devices. Roberts (1999) and Ciblak and Lipkin (1999) independently developed algorithms for implementing a realizable stiffness matrix with *r* number of springs where *r* is the rank of the stiffness matrix.

As for serial robot manipulators, Salisbury (1980) derived the stiffness mapping between the joint space and the Cartesian space. Chen and Kao (2000) showed that the formulation of Salisbury (1980) is only valid in the unloaded equilibrium pose and derived the conservative congruence transformation for stiffness mapping accounting for the effect of an external force.

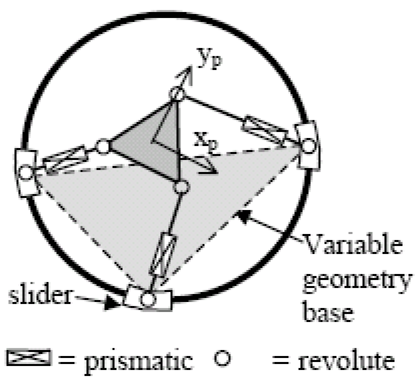

Figure 1-1. Planar robot with variable geometry base platform (from Simaan and Shoham 2002).

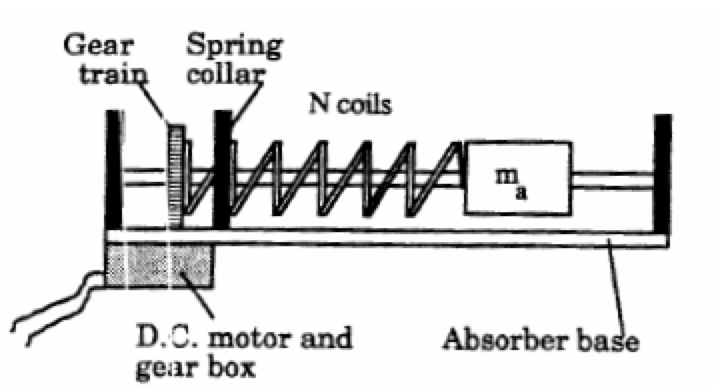

Figure 1-2. Adaptive vibration absorber (from Ryan et al. 1994).

Planar/spatial compliant mechanisms are in general constructed with rigid bodies which are connected to each other by simple springs. The stiffness matrix of the mechanism depends on the geometry of the mechanism and the properties of the constituent springs such as stiffness coefficient and free length. To realize variable compliant mechanisms, variable geometry or adjustable springs have been investigated. Simaan and Shoham (2002) studied the stiffness synthesis problem using a variable geometry planar mechanism. They changed the geometry of the base using sliding joints on the circular base (see Figure 1-1). Ryan et al. (1994) designed a variable spring by changing the effective number of coils of the spring for adaptive-passive vibration control (see Figure 1-2).

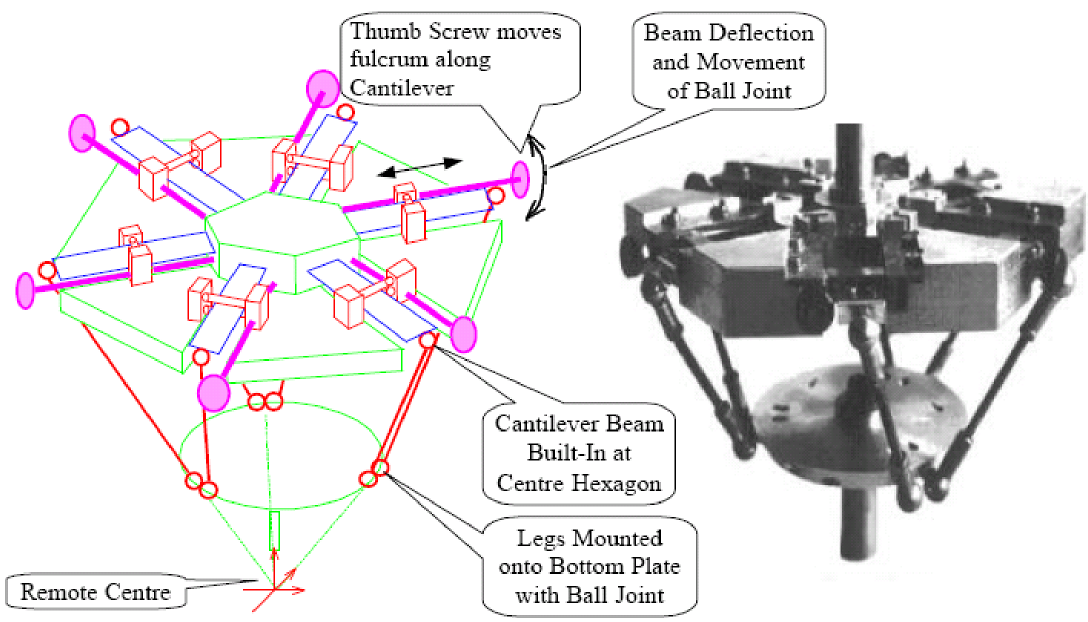

Figure 1-3. Parallel topology 6DOF with adjustable compliance (from McLachlan and Hall 1999).

Cantilever beam-based variable compliant devices have been studied by a few researchers. Under an external force, a cantilever beam deflects and its deflection depends on the length of the beam and the Young's modulus of the material. Henrie (1997) investigated a cantilever beam which is filled with magneto-rheological material and changed the Young's modulus by changing the magnetic field. McLachlan and Hall (1999) devised a programmable passive device by changing the length of the cantilever beam as shown in Figure 1-3. Hurst et al. (2004) presented an actuator with physically variable stiffness by using two motors and analyzed it for application to legged locomotion.

1.3 Problem Statement

Planar/spatial compliant mechanisms consisting of rigid bodies which are connected to each other by adjustable compliant couplings are investigated. For spatial mechanisms, each adjustable compliant coupling is assumed to have a spherical joint at each end and a prismatic joint with an adjustable line spring in the middle. For planar cases, spherical joints are replaced with revolute joints. Mechanisms having two compliant parallel mechanisms that are serially arranged are mainly investigated. The compliant mechanisms are not restricted to be in unloaded equilibrium configuration and this makes the analysis of the mechanism more complicated.

Firstly a stiffness mapping of a line spring connecting two moving bodies is derived for planar and spatial cases. The line spring is assumed to have a fixed stiffness coefficient and free length at this stage. This stiffness mapping leads to the derivation of the stiffness matrix of compliant mechanisms consisting of rigid bodies connected to each other by line springs.

A derivative of the stiffness matrix of a compliant mechanism with respect to the twists of the constituent rigid bodies and the spring properties such as spring constant and free length is obtained. Since the compliant mechanism is assumed initially in static equilibrium under an external wrench, changing the spring constants and the free lengths of the constituent springs may result in the change of the resultant wrench and it may change the position of the compliant mechanism. Stiffness modulation methods, which utilize adjustable line springs and vary the position of the robot where the compliant mechanism is attached, are investigated to realize a desired compliance and to regulate the position of the compliant mechanism.

CHAPTER 2
STIFFNESS MAPPING OF PLANAR COMPLIANT MECHANISMS

When a rigid body supported by a compliant coupling moves, the deflection and/or the directional change of the coupling may lead to a change of the force. In this chapter, a planar stiffness mapping model which maps a small twist of the body into the corresponding wrench variation is studied. To describe a small (or instantaneous) displacement of a rigid body and a force/torque applied to a body, the concepts of *twist* and *wrench* from screw theory are used throughout this dissertation (see Ball 1900 and Crane et al. 2006). Further, the notations of Kane and Levinson are also employed (see Kane and Levinson 1985) to describe spatial motions of rigid bodies.

Specifically, as part of the notation, the position of a point P embedded in body B measured with respect to a reference system embedded in body A will be written as ${}^{A}\underline{\mathbf{r}}_{P}^{B}$. The derivative of the displacement of this point P (embedded in body B in terms of a reference coordinate system embedded in body A) is denoted as ${}^{A}\delta\underline{\mathbf{r}}_{P}^{B}$. The derivative of an angle of body B with respect to a body A is denoted by ${}^{A}\delta\underline{\boldsymbol{\theta}}^{B}$ and its magnitude is denoted by ${}_{A}\delta\theta_{B}$. The twist of a body B with respect to a body A will be denoted by ${}^{A}\delta\underline{\mathbf{D}}^{B}$.

2.1 Spring in a Line Space

The analysis of rigid bodies which are constrained to move in a line space and connected to each other by line springs is presented because it is simple and intuitive and a similar approach can be applied for planar and spatial compliant mechanisms. Figure

2-1 illustrates a line spring connecting body A to ground. Body A is allowed to move only on a line along the axis of the spring. The spring has a spring constant k and a free length x_o. The position of body A can be expressed by a scalar x and the force from the spring by a scalar f.

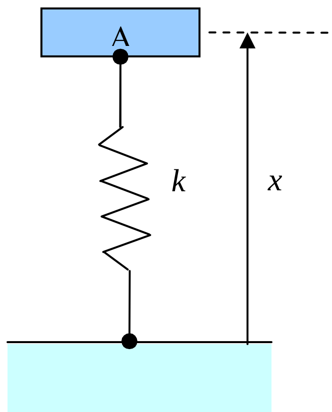

Figure 2-1. Spring in a line space.

The spring force can be written as

$$f = k(x - x_o) \ . \tag{2.1}$$

The relation between a small change of the position of body A and the corresponding small force variation can be obtained by taking a derivative of Eq. (2.1) as

$$\delta f = k\delta x . \tag{2.2}$$

When springs are arranged in parallel as shown in Figure 2-2 (a), the resultant spring constant k_R may be derived as Eq. (2.3).

$$\delta f = k_R \delta x = k_1 \delta x + k_2 \delta x$$

$$k_R = k_1 + k_2 \ . \tag{2.3}$$

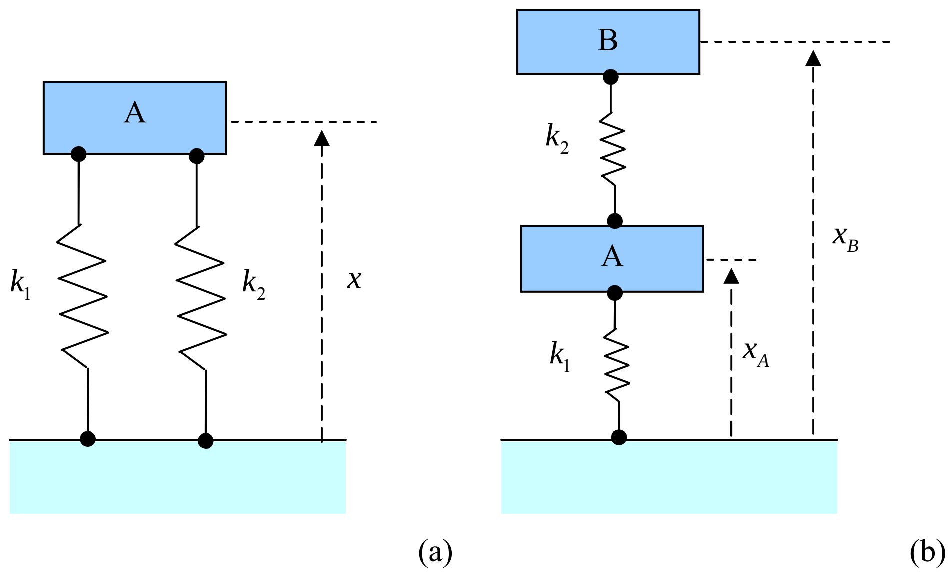

Figure 2-2. Spring arrangements in a line space. (a) parallel and (b) series.

For a serial arrangement as shown in Figure 2-2 (b), the resultant spring constant k_R which maps a small change of position of body B into a small force variation upon body B may be written as Eq. (2.4).

$$\delta f = k_R \delta x_B = k_1 \delta x_A = k_2 \left(\delta x_B - \delta x_A \right)$$

$$\delta x_A = \frac{k_2}{k_1 + k_2} \delta x_B$$

$$k_R = \frac{k_1 k_2}{k_1 + k_2} \quad or \quad k_R^{-1} = k_1^{-1} + k_2^{-1} \ . \tag{2.4}$$

It is obvious from Eqs. (2.3) and (2.4) that the resultant spring constant of springs in parallel is the summation of each spring constant and that the resultant compliance of springs in series is the summation of each spring compliance. This statement is valid for springs in a line space.

2.2 A Derivative of Planar Spring Wrench Joining a Moving Body and Ground

In this section, a derivative of the planar spring wrench joining a moving body and ground, which was presented by Pigoski (1993) and led to the stiffness mapping of a planar parallel mechanism, is restated. Figure 2-3 illustrates a rigid body connected to ground by a compliant coupling. The compliant coupling has a revolute joint at each end and a prismatic joint with a spring in the middle part. Body A can translate and rotate in a planar space.

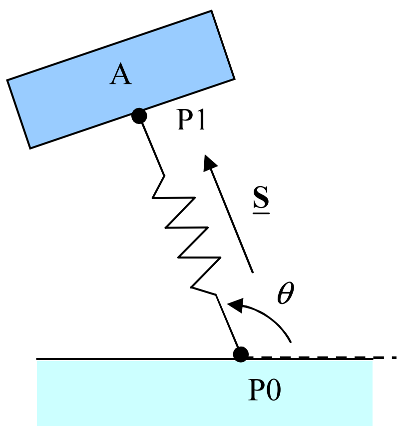

Figure 2-3. Planar compliant coupling connecting body A and the ground.

The force which the spring exerts on body A can be written as

$$\underline{\mathbf{f}} = k(l - l_o)\underline{\$} \tag{2.5}$$

where k, l, and l_o are the spring constant, current spring length, and spring free length of the compliant coupling, respectively. Also $\underline{\$}$ represents the unitized Plücker coordinates of the line along the compliant coupling which may be written as

$$\underline{\$} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P0}^{E} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} \end{bmatrix} \tag{2.6}$$

where $\underline{\mathbf{S}}$ is the unit vector along the compliant coupling and ${}^{E}\underline{\mathbf{r}}_{P0}^{E}$ and ${}^{E}\underline{\mathbf{r}}_{P1}^{A}$ are the position of the pivot point PO in the ground body and that of P1 in body A, respectively, measured with respect to a reference coordinate system attached to ground. To obtain the stiffness mapping, a small twist ${}^{E}\delta\underline{\mathbf{D}}^{A}$ is applied to body A and the corresponding change of the spring force will be obtained. The twist ${}^{E}\delta\underline{\mathbf{D}}^{A}$ may be written in axis coordinates as

$$ {}^{E}\delta\underline{\mathbf{D}}^{A} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{r}}_{0}^{A} \\ {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \end{bmatrix} \tag{2.7} $$

where ${}^{E}\delta\underline{\mathbf{r}}_{o}{}^{A}$ is the differential of the position of point O in body A which is coincident with the origin of the inertial frame E measured with respect to the inertial frame. In addition ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}$ is the differential of the angle of body A with respect to the inertial frame. Taking a derivative of Eq. (2.5) with the consideration that $\underline{\mathbf{\$}}$ is a function of θ in planar cases yields

$$ \begin{aligned} \delta\underline{\mathbf{f}} &= k\delta l\underline{\mathbf{\$}} + k(l - l_o)\delta\underline{\mathbf{\$}} \\ &= k\delta l\underline{\mathbf{\$}} + k(1 - \frac{l_o}{l})\frac{\partial\underline{\mathbf{\$}}}{\partial\theta} l\,\delta\theta \end{aligned} \tag{2.8} $$

where

$$ \frac{\partial\underline{\mathbf{\$}}}{\partial\theta} = \begin{bmatrix} \dfrac{\partial\underline{\mathbf{S}}}{\partial\theta} \\ {}^{E}\underline{\mathbf{r}}_{P0}^{E} \times \dfrac{\partial\underline{\mathbf{S}}}{\partial\theta} \end{bmatrix} \tag{2.9} $$

and where $\dfrac{\partial\underline{\mathbf{S}}}{\partial\theta}$ is a unit vector perpendicular to $\underline{\mathbf{S}}$.

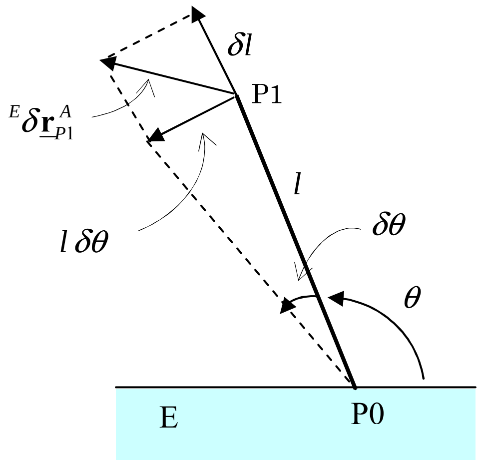

Figure 2-4. Small change of position of P1 due to a small twist of body A.

Using screw theory, the variation of position P1 can be written as

$$ {}^E\delta\underline{\mathbf{r}}^A_{P1} = {}^E\delta\underline{\mathbf{r}}^A_o + {}^E\delta\underline{\boldsymbol{\varphi}}^A \times {}^E\underline{\mathbf{r}}^A_{P1} \ . \tag{2.10}$$

It may be decomposed into two perpendicular vectors, one along $\underline{\mathbf{S}}$ and one along $\frac{\partial \underline{\mathbf{S}}}{\partial \theta}$.

These vectors correspond to the change of the spring length δl and the change of the direction of the spring $l\delta\theta$ as shown in Figure 2-4. The change of the position of point P1 may thus also be written as

$$\begin{aligned} {}^E\delta\underline{\mathbf{r}}^A_{P1} &= \left({}^E\delta\underline{\mathbf{r}}^A_{P1}\cdot\underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^E\delta\underline{\mathbf{r}}^A_{P1}\cdot\frac{\partial \underline{\mathbf{S}}}{\partial \theta}\right)\frac{\partial \underline{\mathbf{S}}}{\partial \theta} \\ &= \delta l\ \underline{\mathbf{S}} + l\delta\theta\frac{\partial \underline{\mathbf{S}}}{\partial \theta} \end{aligned} \ . \tag{2.11}$$

From Eqs. (2.10), (2.11), (2.6), and (2.7) expressions for δl and $l\delta\theta$ may be obtained as

$$\begin{aligned} \delta l &= {}^E\delta\underline{\mathbf{r}}^A_{P1}\cdot\underline{\mathbf{S}} = {}^E\delta\underline{\mathbf{r}}^A_o\cdot\underline{\mathbf{S}} + {}^E\delta\underline{\boldsymbol{\varphi}}^A\times{}^E\underline{\mathbf{r}}^A_{P1}\cdot\underline{\mathbf{S}} \\ &= {}^E\delta\underline{\mathbf{r}}^A_o\cdot\underline{\mathbf{S}} + {}^E\delta\underline{\boldsymbol{\varphi}}^A\cdot{}^E\underline{\mathbf{r}}^A_{P1}\times\underline{\mathbf{S}} \\ &= \underline{\$}^T\,{}^E\delta\underline{\mathbf{D}}^A \end{aligned} \tag{2.12}$$

$$l\delta\theta = {}^E\delta\underline{\mathbf{r}}_{P1}^A \cdot \frac{\partial \underline{\mathbf{S}}}{\partial\theta} = {}^E\delta\underline{\mathbf{r}}_o^A \cdot \frac{\partial \underline{\mathbf{S}}}{\partial\theta} + {}^E\delta\underline{\boldsymbol{\varphi}}^A \times {}^E\underline{\mathbf{r}}_{P1}^A \cdot \frac{\partial \underline{\mathbf{S}}}{\partial\theta}$$
$$= {}^E\delta\underline{\mathbf{r}}_o^A \cdot \frac{\partial \underline{\mathbf{S}}}{\partial\theta} + {}^E\delta\underline{\boldsymbol{\varphi}} \cdot {}^E\underline{\mathbf{r}}_{P1}^A \times \frac{\partial \underline{\mathbf{S}}}{\partial\theta} \tag{2.13}$$
$$= \frac{\partial \underline{\mathbf{\$}}'^T}{\partial\theta}\, {}^E\delta\underline{\mathbf{D}}^A$$

and where

$$\frac{\partial \underline{\mathbf{\$}}'}{\partial\theta} = \begin{bmatrix} \dfrac{\partial \underline{\mathbf{S}}}{\partial\theta} \\ {}^E\underline{\mathbf{r}}_{P1}^A \times \dfrac{\partial \underline{\mathbf{S}}}{\partial\theta} \end{bmatrix}. \tag{2.14}$$

All terms of Eq. (2.14) are known.

From Eqs. (2.8), (2.12), and (2.13), a derivative of the spring force may be written as

$$\delta\,\underline{\mathbf{f}} = k\delta l \underline{\mathbf{\$}} + k(1-\frac{l_o}{l})\frac{\partial \underline{\mathbf{\$}}}{\partial\theta}\, l\,\delta\theta$$
$$= k\underline{\mathbf{\$}}\underline{\mathbf{\$}}^T\, {}^E\delta\underline{\mathbf{D}}^A + k(1-\frac{l_o}{l})\frac{\partial \underline{\mathbf{\$}}}{\partial\theta}\frac{\partial \underline{\mathbf{\$}}'^T}{\partial\theta}\, {}^E\delta\underline{\mathbf{D}}^A \tag{2.15}$$
$$= \left[K_F\right]{}^E\delta\underline{\mathbf{D}}^A$$

where

$$\left[K_F\right] = k\underline{\mathbf{\$}}\underline{\mathbf{\$}}^T + k(1-\frac{l_o}{l})\frac{\partial \underline{\mathbf{\$}}}{\partial\theta}\frac{\partial \underline{\mathbf{\$}}'^T}{\partial\theta}. \tag{2.16}$$

$\left[K_F\right]$ is the stiffness matrix of a planar compliant coupling and maps a small twist of body A into the corresponding variation of the wrench. The first term of Eq. (2.16) is always symmetric and the second term is not. When the spring deviates from its equilibrium position due to an external wrench, the second term of Eq. (2.16) doesn’t vanish and it makes the stiffness matrix asymmetric.

2.3 A Derivative of Spring Wrench Joining Two Moving Bodies

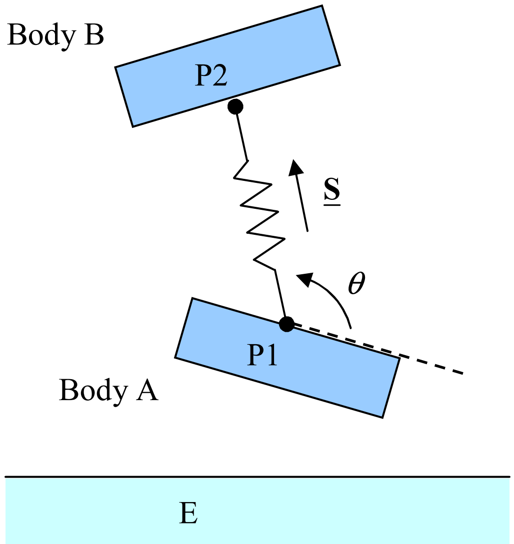

Figure 2-5. Planar compliant coupling joining two moving bodies.

In this section a derivative of the spring wrench joining two moving bodies is derived, which supersedes the result of the previous section and is essential to obtain a stiffness mapping of springs in complicated arrangement.

Figure 2-5 illustrates two rigid bodies connected to each other by a compliant coupling with a spring constant k, a free length l_o, and a current length l. Body A can move in a planar space and the compliant coupling exerts a force $\underline{\mathbf{f}}$ to body B which is in equilibrium. The spring force may be written by

$$\underline{\mathbf{f}} = k(l - l_o)\underline{\mathbf{\$}} \tag{2.17}$$

where

$$\underline{\mathbf{\$}} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P2}^{B} \times \underline{\mathbf{S}} \end{bmatrix} \tag{2.18}$$

and where $\underline{\mathbf{S}}$ is a unit vector along the compliant coupling and ${}^{E}\underline{\mathbf{r}}_{P1}^{A}$ and ${}^{E}\underline{\mathbf{r}}_{P2}^{B}$ are the position vector of the point P1 in body A and that of point P2 in body B, respectively, measured with respect to the reference system embedded in ground (body E).

A small twist of body B with respect to an inertial frame E ${}^{E}\delta\underline{\mathbf{D}}^{B}$ is applied and it is desired to find the corresponding change of the spring force. The twist ${}^{E}\delta\underline{\mathbf{D}}^{B}$ may be written as

$$ {}^{E}\delta\underline{\mathbf{D}}^{B} = {}^{E}\delta\underline{\mathbf{D}}^{A} + {}^{A}\delta\underline{\mathbf{D}}^{B} \tag{2.19} $$

where

$$ {}^{E}\delta\underline{\mathbf{D}}^{B} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{r}}_{o}^{B} \\ {}^{E}\delta\underline{\boldsymbol{\varphi}}^{B} \end{bmatrix} \tag{2.20} $$

$$ {}^{E}\delta\underline{\mathbf{D}}^{A} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \\ {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \end{bmatrix} \tag{2.21} $$

$$ {}^{A}\delta\underline{\mathbf{D}}^{B} = \begin{bmatrix} {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \\ {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \end{bmatrix} \tag{2.22} $$

and where the notation from Kane and Levinson (1985) is employed as stated in the beginning of this chapter. For example, ${}^{E}\delta\underline{\mathbf{r}}_{o}{}^{B}$ is the differential of the coordinates of point O, which is in body B and coincident with the origin of the inertial frame, measured with respect to the inertial frame and ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}$ is the differential of angle of body A with respect to the inertial frame.

The derivative of the spring force, Eq. (2.17), can be written as

$$ {}^{E}\delta\underline{\mathbf{f}} = k\,\delta l\,\underline{\mathbf{\$}} + k\,(l - l_{o})\,{}^{E}\delta\underline{\mathbf{\$}}\,. \tag{2.23} $$

From the twist equation, the variation of the position of point P2 in body B with respect to body A can be expressed as

$$ {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B} = {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \times {}^{A}\underline{\mathbf{r}}_{P2}^{B} \tag{2.24} $$

where ${}^{A}\underline{\mathbf{r}}_{P2}^{B}$ is the position of P2, which is embedded in body B, measured with respect to a coordinate system embedded in body A which at this instant is coincident and aligned with the reference system attached to ground. It can also be decomposed into two perpendicular vectors along $\underline{\mathbf{S}}$ and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$ which is a known unit vector perpendicular to $\underline{\mathbf{S}}$. These two vectors correspond to the change of the spring length δl and the directional change of the spring $l\delta\theta$ in terms of body A in a way that is analogous to that shown in Figure 2-4. Thus the variation of position of point P2 in body B in terms of body A can be written as

$$ \begin{aligned} {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B} &= \left({}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot\underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\ &= \delta l\,\underline{\mathbf{S}} + l\delta\theta\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \end{aligned} \tag{2.25} $$

where

$$ \frac{{}^{A}\partial\underline{\$}}{\partial\theta} = \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\ {}^{A}\underline{\mathbf{r}}_{P1}^{A}\times\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \end{bmatrix}. \tag{2.26} $$

From Eqs. (2.24) and (2.25), δl and $l\delta\theta$ can be obtained as

$$ \begin{aligned} \delta l &= {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot\underline{\mathbf{S}} = {}^{A}\delta\underline{\mathbf{r}}_{o}^{B}\cdot\underline{\mathbf{S}} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\times{}^{A}\underline{\mathbf{r}}_{P2}^{B}\cdot\underline{\mathbf{S}} \\ &= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B}\cdot\underline{\mathbf{S}} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\cdot{}^{A}\underline{\mathbf{r}}_{P2}^{B}\times\underline{\mathbf{S}} \\ &= \underline{\$}^{T}\,{}^{A}\delta\underline{\mathbf{D}}^{B} \end{aligned} \tag{2.27} $$

$$l\delta\theta = {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} = {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \times {}^{A}\underline{\mathbf{r}}_{P2}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$$
$$= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \cdot {}^{A}\underline{\mathbf{r}}_{P2}^{B} \times \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \tag{2.28}$$
$$= \frac{{}^{A}\partial\underline{\mathbf{\$}}'^{T}}{\partial\theta}\, {}^{A}\delta\underline{\mathbf{D}}^{B}$$

where

$$\frac{{}^{A}\partial\underline{\mathbf{\$}}'}{\partial\theta} = \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\ {}^{A}\underline{\mathbf{r}}_{P2}^{B} \times \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \end{bmatrix}. \tag{2.29}$$

It is important to note that screw $\frac{{}^{A}\partial\underline{\mathbf{\$}}'}{\partial\theta}$ has the same direction as $\frac{{}^{A}\partial\underline{\mathbf{\$}}}{\partial\theta}$ but has a different moment term.

Only ${}^{E}\delta\underline{\mathbf{\$}}$ is unknown in Eq. (2.23). It is a derivative of the unit screw along the spring in terms of the inertial frame and may be written as

$${}^{E}\delta\underline{\mathbf{\$}} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} + {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times {}^{E}\delta\underline{\mathbf{S}} \end{bmatrix}. \tag{2.30}$$

Using an intermediate frame attached to body A, a derivative of the direction cosine vector may be written as

$${}^{E}\delta\underline{\mathbf{S}} = {}^{A}\delta\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}}. \tag{2.31}$$

Then, ${}^{E}\delta\underline{\mathbf{\$}}$ may be decomposed into three screws as

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{\$}} &= \begin{bmatrix} {}^{E}\delta\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}}+{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{E}\delta\underline{\mathbf{S}} \end{bmatrix} \\
&= \begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}}+{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left({}^{A}\delta\underline{\mathbf{S}}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right) \end{bmatrix} \\
&= \begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{A}\delta\underline{\mathbf{S}} \end{bmatrix} + \begin{bmatrix} {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right) \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}} \end{bmatrix}
\end{aligned}
\quad . \tag{2.32}
$$

Since $\underline{\mathbf{S}}$ is a function of θ alone from the vantage of body A and $l\delta\theta$ is already described in Eq. (2.28), the first screw in Eq. (2.32) can be written as

$$
\begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{A}\delta\underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\delta\theta \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\delta\theta \end{bmatrix} = \frac{{}^{A}\partial\underline{\mathbf{\$}}}{\partial\theta}\frac{1}{l}l\delta\theta = \frac{1}{l}\frac{{}^{A}\partial\underline{\mathbf{\$}}}{\partial\theta}\frac{{}^{A}\partial\underline{\mathbf{\$}}'^{T}}{\partial\theta}\,{}^{A}\delta\underline{\mathbf{D}}^{B}\,. \tag{2.33}
$$

As for the second screw in Eq. (2.32), ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}$ has the same direction with $\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$ and a magnitude of ${}_{E}\delta\phi_{A}$ and thus may be written as

$$
{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} = {}_{E}\delta\phi_{A}\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\,. \tag{2.34}
$$

Then the second screw in Eq. (2.32) can be expressed as

$$
\begin{aligned}
\begin{bmatrix} {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right) \end{bmatrix} &= \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\,{}_{E}\delta\phi_{A} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\,{}_{E}\delta\phi_{A} \end{bmatrix} \\
&= \frac{{}^{A}\partial\underline{\mathbf{\$}}}{\partial\theta}\,{}_{E}\delta\phi_{A} = \frac{{}^{A}\partial\underline{\mathbf{\$}}}{\partial\theta}\begin{bmatrix}0 & 0 & 1\end{bmatrix}{}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned}
\quad . \tag{2.35}
$$

As to the third screw in Eq. (2.32), ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$ can be decomposed into two perpendicular vectors along $\underline{\mathbf{S}}$ and $\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$ respectively and may be written as

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times {}^{E}\underline{\mathbf{r}}_{P1}^{A} \\
&= \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}
\end{aligned} \tag{2.36}
$$

where

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \underline{\mathbf{S}} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{r}}_{P1}^{A} \cdot \underline{\mathbf{S}} \\
&= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \cdot \underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} \\
&= \underline{\mathbf{\$}}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned} \tag{2.37}
$$

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\
&= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \cdot \underline{\mathbf{r}}_{P1}^{A} \times \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}. \\
&= \frac{{}^{A}\partial\underline{\mathbf{\$}}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned} \tag{2.38}
$$

By combining Eqs. (2.36), (2.37), and (2.38) ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$ can be written as

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} &= \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\
&= \left(\underline{\mathbf{\$}}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\underline{\mathbf{S}} + \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}
\end{aligned}. \tag{2.39}
$$

The third screw in Eq. (2.32) can now be written as

$$
\begin{aligned}
\begin{bmatrix} \underline{\mathbf{0}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} \end{bmatrix} &= \begin{bmatrix} \underline{\mathbf{0}} \\ \left\{\left(\underline{\mathbf{\$}}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\underline{\mathbf{S}} + \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\right\} \times \underline{\mathbf{S}} \end{bmatrix} \\
&= \begin{bmatrix} \underline{\mathbf{0}} \\ \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{0}} \\ -\frac{{}^{A}\partial\underline{\mathbf{\$}}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A} \end{bmatrix} \\
&= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{{}^{A}\partial\underline{\mathbf{\$}}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A} = -\left(\frac{{}^{A}\partial\underline{\mathbf{\$}}}{\partial\theta}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right)^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned} \tag{2.40}
$$

since $\frac{{}^{A}\partial \underline{\mathbf{S}}}{\partial \theta} \times \underline{\mathbf{S}} = -1(\underline{\mathbf{k}})$.

Among all unknowns in Eq. (2.23), δl was obtained in Eq. (2.27) and all the terms of ${}^{E}\delta\underline{\$}$ were obtained through Eqs. (2.33), (2.35), and (2.40). Hence the derivative of the spring force can be rewritten as

$$\begin{aligned} {}^{E}\delta\underline{\mathbf{f}} &= k\delta l\underline{\$} + k(l-l_o)\,{}^{E}\delta\underline{\$} \\ &= k\underline{\$}\underline{\$}^{T}\,{}^{A}\delta\underline{\mathbf{D}}^{B} + k(l-l_o)\left(\frac{1}{l}\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\frac{{}^{A}\partial\underline{\$}'^{T}}{\partial\theta}\,{}^{A}\delta\underline{\mathbf{D}}^{B} + \frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix}0 & 0 & 1\end{bmatrix}{}^{E}\delta\underline{\mathbf{D}}^{A} - \begin{bmatrix}0\\0\\1\end{bmatrix}\frac{{}^{A}\partial\underline{\$}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right) \\ &= \left(k\underline{\$}\underline{\$}^{T} + k(1-\frac{l_o}{l})\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\frac{{}^{A}\partial\underline{\$}'^{T}}{\partial\theta}\right){}^{A}\delta\underline{\mathbf{D}}^{B} + k(l-l_o)\left(\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix}0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0\\0\\1\end{bmatrix}\frac{{}^{A}\partial\underline{\$}}{\partial\theta}^{T}\right){}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= \left[K_F\right]{}^{A}\delta\underline{\mathbf{D}}^{B} + \left[K_M\right]{}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{2.41}$$

where

$$\left[K_F\right] = k\underline{\$}\underline{\$}^{T} + k(1-\frac{l_o}{l})\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\frac{{}^{A}\partial\underline{\$}'^{T}}{\partial\theta} \tag{2.42}$$

$$\left[K_M\right] = k(l-l_o)\left(\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix}0 & 0 & 1\end{bmatrix} - \left(\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix}0 & 0 & 1\end{bmatrix}\right)^{T}\right). \tag{2.43}$$

It is important to note that $\left[K_M\right]$ is a function of the external wrench. To prove it, Eq. (2.18) is explicitly expressed in a planar coordinate system and ${}^{E}\underline{\mathbf{r}}_{P1}^{A} = \begin{bmatrix}p_x & p_y\end{bmatrix}^{T}$ to yield

$$\underline{\$} = \begin{bmatrix} c_\theta \\ s_\theta \\ c_\theta\, p_x - s_\theta\, p_y \end{bmatrix} \tag{2.44}$$

$$\frac{{}^{A}\partial \underline{\underline{\$}}}{\partial \theta} = \begin{bmatrix} -s_\theta \\ c_\theta \\ -s_\theta\, p_x - c_\theta\, p_y \end{bmatrix} \tag{2.45}$$

where $c_\theta = \cos(\theta)$ and $s_\theta = \sin(\theta)$.

By substituting Eq. (2.45) for $\frac{{}^{A}\partial \underline{\underline{\$}}}{\partial \theta}$ in Eq. (2.43), $[K_M]$ can be expressed as

$$[K_M] = k(l - l_o)\begin{bmatrix} 0 & 0 & -s_\theta \\ 0 & 0 & c_\theta \\ s_\theta & -c_\theta & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -f_y \\ 0 & 0 & f_x \\ f_y & -f_x & 0 \end{bmatrix} \tag{2.46}$$

where $\underline{\mathbf{f}} = \begin{bmatrix} f_x & f_y & m_z \end{bmatrix}^T$ is the initial spring wrench.

As shown in Eq. (2.41), the derivative of the spring wrench joining two rigid bodies depends not only on a relative twist between two bodies but also on the twist of the intermediate body, in this case body A, in terms of the inertial frame. $[K_F]$ which maps a small twist of body B in terms of body A into the corresponding change of wrench upon body B is identical to the stiffness matrix of the spring assuming the body A is stationary. $[K_M]$ is newly introduced from this research and results from the motion of the base frame, in this case body A, and is a function of the initial external wrench.

2.4 Stiffness Mapping of Planar Compliant Parallel Mechanisms in Series

The derivative of the spring wrench derived in the previous section is applied to obtain the stiffness mapping of compliant parallel mechanisms in series as shown in Figure 2-6.[1] Body A is connected to ground by three compliant couplings and body B is connected to body A in the same way. Each compliant coupling has a revolute joint at

[1] Figure 2-6 shows a coordinate system attached to each of three bodies for illustration purposes. In this analysis, the three coordinate systems are assumed to be coincident and aligned at each instant.

each end and a prismatic joint with a spring in the middle part. It is assumed that an external wrench $\underline{\mathbf{w}}_{ext}$ is applied to body B and that both body B and body A are in static equilibrium. The positions and orientations of bodies A and B and the spring constants and free lengths of all constituent springs are given. The stiffness matrix $[K]$ which maps a small twist of body B with respect to the ground ${}^{E}\delta\underline{\mathbf{D}}^{B}$ into a small wrench variation $\delta\underline{\mathbf{w}}_{ext}$ is desired to obtain.

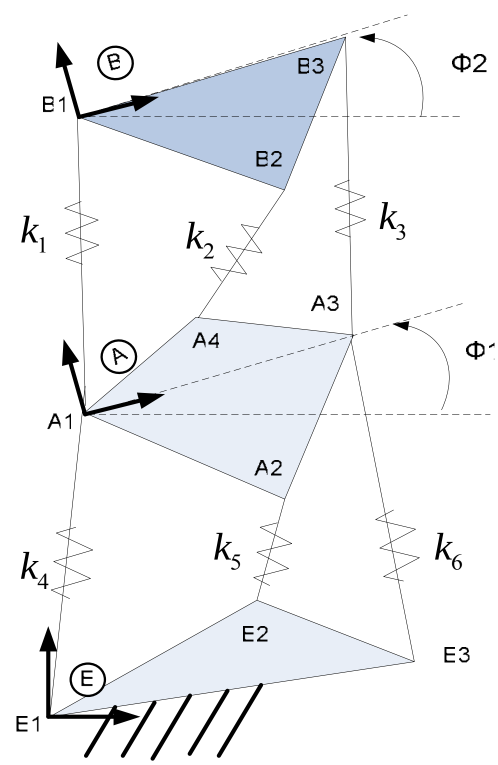

Figure 2-6. Mechanism having two compliant mechanisms in series.

The static equilibrium equation of bodies B and A can be written by

$$\begin{aligned}\underline{\mathbf{w}}_{ext} &= \underline{\mathbf{f}}_1 + \underline{\mathbf{f}}_2 + \underline{\mathbf{f}}_3 \\ &= \underline{\mathbf{f}}_4 + \underline{\mathbf{f}}_5 + \underline{\mathbf{f}}_6\end{aligned} \tag{2.47}$$

where $\underline{\mathbf{f}}_i$ are the forces from the compliant couplings.

The stiffness matrix is derived by taking a derivative of the static equilibrium equation, Eq. (2.47), to yield

$$\begin{aligned} \delta \underline{\mathbf{w}}_{ext} &= [K]\, {}^{E}\delta\underline{\mathbf{D}}^{B} \\ &= \delta \underline{\mathbf{f}}_1 + \delta \underline{\mathbf{f}}_2 + \delta \underline{\mathbf{f}}_3 \\ &= \delta \underline{\mathbf{f}}_4 + \delta \underline{\mathbf{f}}_5 + \delta \underline{\mathbf{f}}_6 \end{aligned} \quad . \tag{2.48}$$

The derivatives of spring forces can be written by Eqs. (2.49) and (2.50) since springs 4, 5, and 6 connect body A and ground and springs 1, 2, and 3 join two moving bodies.

$$\begin{aligned} \delta\underline{\mathbf{f}}_4 + \delta\underline{\mathbf{f}}_5 + \delta\underline{\mathbf{f}}_6 &= [K_F]_4\, {}^{E}\delta\underline{\mathbf{D}}^{A} + [K_F]_5\, {}^{E}\delta\underline{\mathbf{D}}^{A} + [K_F]_6\, {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= [K_F]_{R,L}\, {}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{2.49}$$

$$\begin{aligned} \delta\underline{\mathbf{f}}_1 + \delta\underline{\mathbf{f}}_2 + \delta\underline{\mathbf{f}}_3 &= [K_F]_1\, {}^{A}\delta\underline{\mathbf{D}}^{B} + [K_F]_2\, {}^{A}\delta\underline{\mathbf{D}}^{B} + [K_F]_3\, {}^{A}\delta\underline{\mathbf{D}}^{B} \\ &\quad + [K_M]_1\, {}^{E}\delta\underline{\mathbf{D}}^{A} + [K_M]_2\, {}^{E}\delta\underline{\mathbf{D}}^{A} + [K_M]_3\, {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= [K_F]_{R,U}\, {}^{A}\delta\underline{\mathbf{D}}^{B} + [K_M]_{R,U}\, {}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{2.50}$$

where

$$[K_F]_{R,L} = \sum_{i=4}^{6} [K_F]_i$$

$$[K_F]_{R,U} = \sum_{i=1}^{3} [K_F]_i$$

$$[K_M]_{R,U} = \sum_{i=1}^{3} [K_M]_i \ .$$

From Eqs. (2.49), (2.50), and (2.19) twist ${}^{E}\delta\underline{\mathbf{D}}^{A}$ can be written as

$$\begin{aligned} [K_F]_{R,L}\, {}^{E}\delta\underline{\mathbf{D}}^{A} &= [K_F]_{R,U}\, {}^{A}\delta\underline{\mathbf{D}}^{B} + [K_M]_{R,U}\, {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= [K_F]_{R,U} \left({}^{E}\delta\underline{\mathbf{D}}^{B} - {}^{E}\delta\underline{\mathbf{D}}^{A} \right) + [K_M]_{R,U}\, {}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{2.51}$$

$${}^{E}\delta\underline{\mathbf{D}}^{A} = \left([K_F]_{R,L} + [K_F]_{R,U} - [K_M]_{R,U} \right)^{-1} [K_F]_{R,U}\, {}^{E}\delta\underline{\mathbf{D}}^{B} \ . \tag{2.52}$$

Substituting Eq. (2.52) for ${}^{E}\delta\underline{\mathbf{D}}^{A}$ in Eq. (2.49) and comparing it with Eq. (2.48) yields the stiffness matrix as

$$\begin{aligned}[K]\,{}^{E}\delta\underline{\mathbf{D}}^{B} &= [K_F]_{R,L}\;{}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= [K_F]_{R,L}\left([K_F]_{R,L}+[K_F]_{R,U}-[K_M]_{R,U}\right)^{-1}[K_F]_{R,U}\;{}^{E}\delta\underline{\mathbf{D}}^{B}\end{aligned} \tag{2.53}$$

$$[K]=[K_F]_{R,L}\left([K_F]_{R,L}+[K_F]_{R,U}-[K_M]_{R,U}\right)^{-1}[K_F]_{R,U}\,. \tag{2.54}$$

It was generally accepted that the resultant compliance, which is the inverse of the stiffness, of serially connected mechanisms is the summation of the compliances of all constituent mechanisms (see Griffis 1991). However, the stiffness matrix derived from this research shows a different result. Taking an inverse of the stiffness matrix Eq. (2.54) yields

$$[K]^{-1}=[K_F]_{R,L}^{\;\;-1}+[K_F]_{R,U}^{\;\;-1}-[K_F]_{R,U}^{\;\;-1}[K_M]_{R,U}[K_F]_{R,L}^{\;\;-1} \tag{2.55}$$

The third term in Eq. (2.55) is newly introduced in this research and it does not vanish unless the external wrench is zero.

A numerical example is presented to support the derived stiffness mapping model. The geometry information, spring properties of the mechanism shown in Figure 2-6, and the external wrench $\underline{\mathbf{w}}_{ext}$ are given in Tables 2-1 and 2-2.

$$\underline{\mathbf{w}}_{ext}=\left[0.01\,N \quad -0.02\,N \quad 0.03\,Ncm\right]^{T}$$

Table 2-1. Spring properties of the compliant couplings in Figure 2-6.

Spring No.	1	2	3	4	5	6
Stiffness constant k	0.2	0.3	0.4	0.5	0.6	0.7
Free length l_o	5.0040	2.2860	4.9458	5.5145	3.1573	5.2568

(Unit: N/cm for k, cm for l_o)

Table 2-2. Positions of pivot points in terms of the inertial frame in Figure 2-6.

Pivot points	E1	E2	E3	B1	B2	B3
X	0.0000	1.5000	3.0000	0.0903	1.7063	1.9185
Y	0.0000	1.2000	0.5000	9.8612	8.6833	10.6721

(Unit: cm)

Table 2-2. Continued.

A1	A2	A3	A4
0.9036	2.5318	2.7236	1.6063
4.5962	3.4347	5.4255	5.4659

Two stiffness matrices are obtained. $[K_1]$ is from Eq. (2.54) and $[K_2]$ from the same equation ignoring $[K_M]_{R,U}$.

$$[K_1]=\begin{bmatrix} 0.0108 \quad N/cm & -0.0172 \quad N/cm & -0.0797 \quad N \\ -0.0172 \quad N/cm & 0.3447 \quad N/cm & 0.8351 \quad N \\ -0.0997 \quad N & 0.8251 \quad N & 2.6567 \quad Ncm \end{bmatrix}$$

$$[K_2]=\begin{bmatrix} 0.0111 \quad N/cm & -0.0157 \quad N/cm & -0.0874 \quad N \\ -0.0162 \quad N/cm & 0.3462 \quad N/cm & 0.8124 \quad N \\ -0.0969 \quad N & 0.8150 \quad N & 2.6129 \quad Ncm \end{bmatrix}$$

The result is evaluated in the following way:

1. A small wrench $\delta \underline{\mathbf{w}}_T$ is applied in addition to $\underline{\mathbf{w}}_{ext}$ to body B and twists ${}^E\delta\underline{\mathbf{D}}_1^B$ and ${}^E\delta\underline{\mathbf{D}}_2^B$ are obtained by multiplying the inverse matrices of the stiffness matrices, $[K]_1$ and $[K]_2$, respectively, by $\delta \underline{\mathbf{w}}_T$ as of Eq. (2.48). Corresponding positions for body B are then determined, based on the calculated twists ${}^E\delta\underline{\mathbf{D}}_1^B$ and ${}^E\delta\underline{\mathbf{D}}_2^B$.

2. ${}^E\delta\underline{\mathbf{D}}^A$ is calculated by multiplying the inverse matrix of $[K_F]_{R,L}$ by $\delta \underline{\mathbf{w}}_T$ as of Eq. (2.49). The position of body A is then determined from this twist.

3. The wrench between body B and body A is calculated for the two cases based on knowledge of the positions of bodies A and B and the spring parameters. The change in wrench for the two cases is determined as the difference between the new equilibrium wrench and the original. The changes in the wrenches are named $\delta \underline{\mathbf{w}}_{AB,1}$ and $\delta \underline{\mathbf{w}}_{AB,2}$ which correspond to the matrices $[K]_1$ and $[K]_2$.

4. The given change in wrench $\delta \underline{\mathbf{w}}_T$ is compared to $\delta \underline{\mathbf{w}}_{AB,1}$ and $\delta \underline{\mathbf{w}}_{AB,2}$.

The given wrench $\delta \underline{\mathbf{w}}_T$ and the numerical results are presented as below.

$$\delta \underline{\mathbf{w}}_T = 10^{-5} \times [0.5 \quad 0.2 \quad 0.4]$$

$${}^E\delta \underline{\mathbf{D}}_1^B = 10^{-3} \times [0.7674 \quad -0.1186 \quad 0.0672]^T$$

$${}^E\delta \underline{\mathbf{D}}_2^B = 10^{-3} \times [0.8208 \quad -0.1152 \quad 0.0679]^T$$

$${}^E\delta \underline{\mathbf{D}}^A = 10^{-3} \times [0.2350 \quad -0.0898 \quad 0.0330]^T$$

$$\delta \underline{\mathbf{w}}_{EA} = 10^{-5} \times [0.5000 \quad 0.1998 \quad 0.3996]^T$$

$$\delta \underline{\mathbf{w}}_{AB,1} = 10^{-5} \times [0.5004 \quad 0.1969 \quad 0.3919]^T$$

$$\delta \underline{\mathbf{w}}_{AB,2} = 10^{-5} \times [0.5666 \quad 0.2300 \quad 0.0117]^T$$

where $\delta \underline{\mathbf{w}}_{EA}$ is the wrench between body A and ground. The unit for the wrenches is $[N \quad N \quad Ncm]^T$ and that of the twists is $[cm \quad cm \quad rad]^T$. The difference between $\delta \underline{\mathbf{w}}_{EA}$ and $\delta \underline{\mathbf{w}}_T$ is small and is due to the fact that the twist was not infinitesimal. The difference between $\delta \underline{\mathbf{w}}_{AB,1}$ and $\delta \underline{\mathbf{w}}_T$ is also small and is most likely attributed to the same fact. However, the difference between $\delta \underline{\mathbf{w}}_{AB,2}$ and $\delta \underline{\mathbf{w}}_T$ is not negligible. This indicates that the stiffness matrix formula derived in this research produces the proper result and that the term $[K_M]_{R,U}$ cannot be neglected in Eq. (2.54).

2.5 Stiffness Mapping of Planar Compliant Parallel Mechanisms in a Hybrid Arrangement

Figure 2-7 depicts a compliant mechanism having compliant couplings in a serial/parallel arrangement. Each compliant coupling has a revolute joint at each end and a prismatic joint with a spring in the middle part. An external wrench $\underline{\mathbf{w}}_{ext}$ is applied to body T and body T is separately connected to body B, body C, and body D by three compliant couplings. Body B, body C, and body D are connected to ground by two compliant couplings. It is assumed that all bodies are in static equilibrium. It is desired to find the stiffness matrix which maps a small twist of body T in terms of ground ${}^{E}\delta\underline{\mathbf{D}}^{T}$ to the corresponding wrench variation $\delta\underline{\mathbf{w}}_{ext}$. The stiffness constants and free lengths of all constituent springs and the geometry of the mechanism are assumed to be known.

The stiffness matrix of the mechanism can be derived by taking a derivative of the static equilibrium equations. The static equilibrium equations may be written as

$$\underline{\mathbf{w}}_{ext} = \underline{\mathbf{f}}_7 + \underline{\mathbf{f}}_8 + \underline{\mathbf{f}}_9 \tag{2.56}$$

$$\underline{\mathbf{f}}_7 = \underline{\mathbf{f}}_1 + \underline{\mathbf{f}}_2 \tag{2.57}$$

$$\underline{\mathbf{f}}_8 = \underline{\mathbf{f}}_3 + \underline{\mathbf{f}}_4 \tag{2.58}$$

$$\underline{\mathbf{f}}_9 = \underline{\mathbf{f}}_5 + \underline{\mathbf{f}}_6 \tag{2.59}$$

where $\underline{\mathbf{w}}_{ext}$ is the external wrench and $\underline{\mathbf{f}}_i$ is the force of the i-th spring.

Derivatives of Eqs. (2.56)-(2.59) can be written as

$$\begin{aligned}\delta\underline{\mathbf{w}}_{ext} &= \delta\underline{\mathbf{f}}_7 + \delta\underline{\mathbf{f}}_8 + \delta\underline{\mathbf{f}}_9 \\ &= \left[K\right]_R {}^{E}\delta\underline{\mathbf{D}}^{T}\end{aligned} \tag{2.60}$$

$$\delta\underline{\mathbf{f}}_7 = \delta\underline{\mathbf{f}}_1 + \delta\underline{\mathbf{f}}_2 \tag{2.61}$$

$$\delta\underline{\mathbf{f}}_8 = \delta\underline{\mathbf{f}}_3 + \delta\underline{\mathbf{f}}_4 \tag{2.62}$$

$$\delta\underline{\mathbf{f}}_9 = \delta\underline{\mathbf{f}}_5 + \delta\underline{\mathbf{f}}_6 \tag{2.63}$$

where $\left[K\right]_R$ is the stiffness matrix and ${}^{E}\delta\underline{\mathbf{D}}^{T}$ is a small twist of body T in terms of the inertial frame attached to the ground.

Using Eqs. (2.15) and (2.41), Eq. (2.61) can be rewritten as

$$\begin{aligned}\delta \underline{\mathbf{f}}_7 &= [K_F]_7 \, {}^B\delta\underline{\mathbf{D}}^T + [K_M]_7 \, {}^E\delta\underline{\mathbf{D}}^B \\ &= [K_F]_1 \, {}^E\delta\underline{\mathbf{D}}^B + [K_F]_2 \, {}^E\delta\underline{\mathbf{D}}^B = \left([K_F]_1 + [K_F]_2\right) {}^E\delta\underline{\mathbf{D}}^B \end{aligned} \tag{2.64}$$

where ${}^B\delta\underline{\mathbf{D}}^T$ is a small twist of body T in terms of body B and ${}^E\delta\underline{\mathbf{D}}^B$ is that of body B in terms of the inertial frame. $[K_F]_i$ and $[K_M]_i$ are the matrices for i-th spring defined by Eqs. (2.42) and (2.43) respectively.

The twist of body T can be decomposed as

$${}^E\delta\underline{\mathbf{D}}^T = {}^E\delta\underline{\mathbf{D}}^B + {}^B\delta\underline{\mathbf{D}}^T . \tag{2.65}$$

From Eqs. (2.64) and (2.65), ${}^E\delta\underline{\mathbf{D}}^B$ can be expressed in terms of ${}^E\delta\underline{\mathbf{D}}^T$ as Eq. (2.66).

$$[K_F]_7 \left({}^E\delta\underline{\mathbf{D}}^T - {}^E\delta\underline{\mathbf{D}}^B\right) + [K_M]_7 \, {}^E\delta\underline{\mathbf{D}}^B = \left([K_F]_1 + [K_F]_2\right) {}^E\delta\underline{\mathbf{D}}^B$$

$${}^E\delta\underline{\mathbf{D}}^B = \left([K_F]_1 + [K_F]_2 + [K_F]_7 - [K_M]_7\right)^{-1} [K_F]_7 \, {}^E\delta\underline{\mathbf{D}}^T \tag{2.66}$$

By substituting Eq. (2.66) for ${}^E\delta\underline{\mathbf{D}}^B$ in Eq. (2.64), $\delta \underline{\mathbf{f}}_7$ can be expressed in terms of ${}^E\delta\underline{\mathbf{D}}^T$ as

$$\delta \underline{\mathbf{f}}_7 = \left([K_F]_1 + [K_F]_2\right)\left([K_F]_1 + [K_F]_2 + [K_F]_7 - [K_M]_7\right)^{-1} [K_F]_7 \, {}^E\delta\underline{\mathbf{D}}^T . \tag{2.67}$$

Analogously, $\delta \underline{\mathbf{f}}_8$ and $\delta \underline{\mathbf{f}}_9$ can be written respectively as

$$\delta \underline{\mathbf{f}}_8 = \left([K_F]_3 + [K_F]_4\right)\left([K_F]_3 + [K_F]_4 + [K_F]_8 - [K_M]_8\right)^{-1} [K_F]_8 \, {}^E\delta\underline{\mathbf{D}}^T \tag{2.68}$$

$$\delta \underline{\mathbf{f}}_9 = \left([K_F]_5 + [K_F]_6\right)\left([K_F]_5 + [K_F]_6 + [K_F]_9 - [K_M]_9\right)^{-1} [K_F]_9 \, {}^E\delta\underline{\mathbf{D}}^T . \tag{2.69}$$

Finally from Eq. (2.60) and Eqs. (2.67)-(2.69), the stiffness matrix can be written as

$$\begin{aligned}[K]_R &= ([K_F]_1 + [K_F]_2)([K_F]_1 + [K_F]_2 + [K_F]_7 - [K_M]_7)^{-1}[K_F]_7 \\ &+ ([K_F]_3 + [K_F]_4)([K_F]_3 + [K_F]_4 + [K_F]_8 - [K_M]_8)^{-1}[K_F]_8 . \\ &+ ([K_F]_5 + [K_F]_6)([K_F]_5 + [K_F]_6 + [K_F]_9 - [K_M]_9)^{-1}[K_F]_9\end{aligned} \quad (2.70)$$

A numerical example of the compliant mechanism depicted in Figure 2-7 is presented. The four bodies are identical equilateral triangles whose edge length is 2 cm. Four coordinate systems, B, C, D, and T are attached to body B, C, D, and T, respectively and their positions of origin and orientations in terms of the inertial frame are given in Table 2-5. The spring properties and the positions of the fixed pivot points are given in Table 2-3 and Table 2-4, respectively. The external wrench is given as

$$\underline{\mathbf{w}}_{ext} = \begin{bmatrix} 0.1 \quad N \\ 0.1 \quad N \\ 0.2 \quad Ncm \end{bmatrix}.$$

Table 2-3. Spring properties of the compliant couplings in Figure 2-7.

Spring No.	1	2	3	4	5	6	7	8	9
Stiffness constant k	0.40	0.43	0.49	0.52	0.58	0.61	0.46	0.55	0.64
Free length l_o	2.2547	2.4014	1.5910	1.8450	1.7077	2.2695	2.3924	2.2200	1.8711

(Unit: N/cm for k and cm for l_o)

Table 2-4. Positions of the fixed pivot points of the compliant couplings in Figure 2-7.

	A1	A2	A3	A4	A5	A6
X	1.6700	4.4600	13.3449	14.6731	8.2300	4.9400
Y	4.4333	1.3964	3.2500	6.8400	14.1400	13.4943

(Unit: cm)

Table 2-5. Positions and orientations of the coordinates systems in Figure 2-7.

	Bo	Co	Do	To
X	4.0746	12.2367	7.2479	8.3174
Y	5.1447	4.4972	12.7430	6.9958
Φ	-0.8112	1.2283	3.8876	0.5818

(Unit: cm for x, y and radians for Φ)

Two stiffness matrices are obtained. $[K_1]$ is from Eq. (2.70) and $[K_2]$ from the same equation ignoring all $\left[K_M\right]$'s which are newly introduced in this research.

$$[K_1] = \begin{bmatrix} 0.2501 \quad N/cm & 0.0216 \quad N/cm & -1.7651 \quad N \\ 0.0216 \quad N/cm & 0.2910 \quad N/cm & 2.6661 \quad N \\ -1.6651 \quad N & 2.5661 \quad N & 38.5180 \quad Ncm \end{bmatrix}$$

$$[K_2] = \begin{bmatrix} 0.2463 \quad N/cm & 0.0172 \quad N/cm & -1.7844 \quad N \\ 0.0315 \quad N/cm & 0.2888 \quad N/cm & 2.5749 \quad N \\ -1.6139 \quad N & 2.5730 \quad N & 38.2221 \quad Ncm \end{bmatrix}$$

To evaluate the result, a small wrench $\delta \underline{\mathbf{w}}$ is applied to body T and the static equilibrium pose of the mechanism is obtained by a numerically iterative method. From the equilibrium pose of the mechanism, the twist of body T with respect to ground ${}^E\delta\underline{\mathbf{D}}^T$ is obtained as

$$\delta \underline{\mathbf{w}} = 10^{-4} \times \begin{bmatrix} 0.5 \quad N \\ 0.2 \quad N \\ 0.3 \quad Ncm \end{bmatrix}$$

$${}^E\delta\underline{\mathbf{D}}^T = \begin{bmatrix} 0.0050 \quad cm \\ -0.0058 \quad cm \\ 0.0006 \quad rad \end{bmatrix}.$$

Then the twist ${}^E\delta\underline{\mathbf{D}}^T$ is multiplied by both of the stiffness matrices to see if they result in the given small wrench $\delta \underline{\mathbf{w}}$.

$$\delta \underline{\mathbf{w}}_1 = [K_1]\,{}^E\delta\underline{\mathbf{D}}^T = 10^{-4} \times \begin{bmatrix} 0.4997 \quad N \\ 0.2000 \quad N \\ 0.3020 \quad Ncm \end{bmatrix}$$

$$\delta\underline{\mathbf{w}}_2 = [K_2]\,{}^E\delta\underline{\mathbf{D}}^T = 10^{-4} \times \begin{bmatrix} 0.4502 & N \\ 0.2726 & N \\ 0.6622 & Ncm \end{bmatrix}$$

The numerical example indicates that $[K_1]$ produces the given wrench $\delta\underline{\mathbf{w}}$ with high accuracy and that $[K_2]$ involves significant errors.

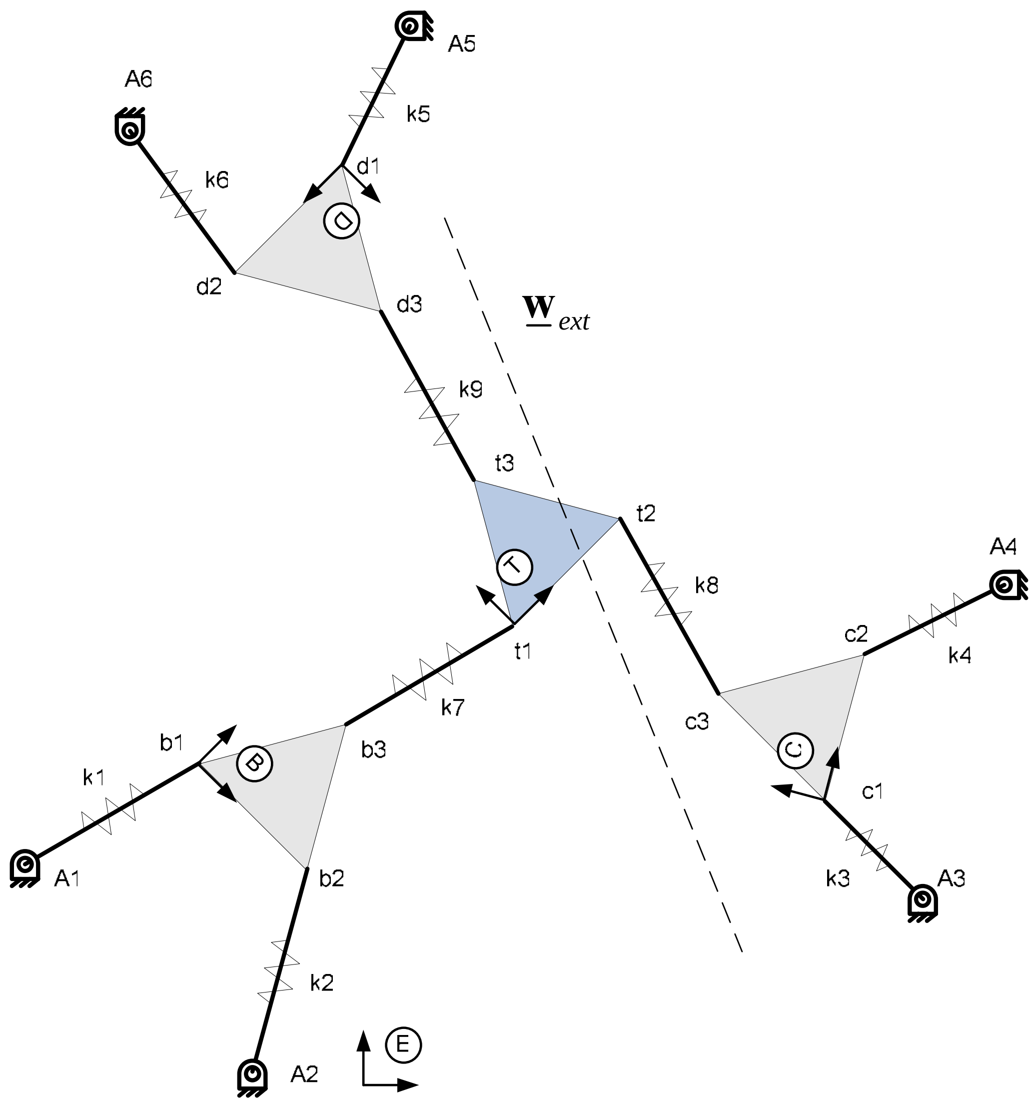

Figure 2-7. Mechanism consisting of four rigid bodies connected to each other by compliant couplings in a hybrid arrangement.

CHAPTER 3
STIFFNESS MAPPING OF SPATIAL COMPLIANT MECHANISMS

Taking a similar approach adopted for planar compliant mechanisms, a stiffness mapping of spatial compliant mechanisms is presented.

3.1 A Derivative of Spatial Spring Wrench Joining a Moving Body and Ground

Figure 3-1 depicts a rigid body and a compliant coupling connecting the body and the ground. The compliant coupling has a spherical joint at each end and a prismatic joint with a spring in the middle. Body A can translate and rotate in a spatial space. The wrench which the spring exerts on body A can be written as

$$\underline{\mathbf{w}} = k(l - l_o)\underline{\mathbf{\$}} \tag{3.1}$$

where k, l, and l_o are respectively the spring constant, current spring length, and spring free length of the compliant coupling. Further, $\underline{\mathbf{\$}}$ represents the unitized Plücker coordinates of the line along the compliant coupling which may be written by

$$\underline{\mathbf{\$}} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P0}^{E} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} \end{bmatrix} \tag{3.2}$$

where $\underline{\mathbf{S}}$ is the unit vector along the compliant coupling and ${}^{E}\underline{\mathbf{r}}_{P0}^{E}$ and ${}^{E}\underline{\mathbf{r}}_{P1}^{A}$ are the position of the pivot point PO in the ground body and that of P1 in body A, respectively, measured with respect to a reference coordinate system attached to ground.

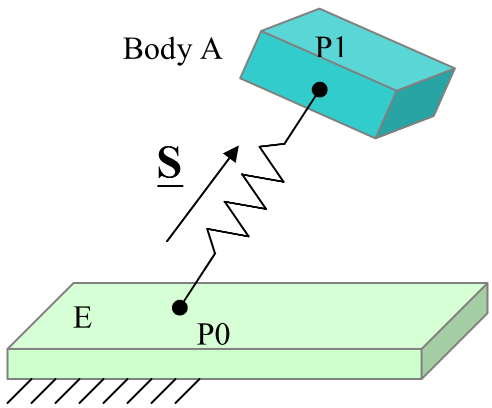

Figure 3-1. Spatial compliant coupling joining body A and the ground.

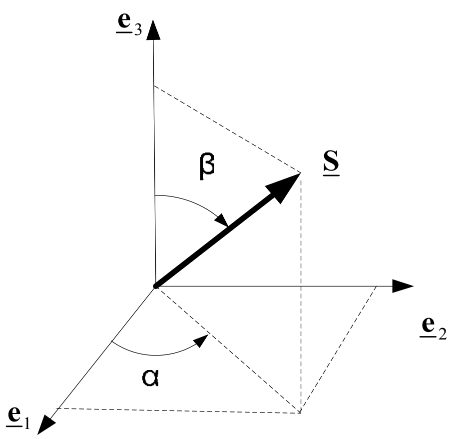

Figure 3-2. Unit vector expressed in a polar coordinates system.

A polar coordinates system can be used to express the unit vector $\underline{\mathbf{S}}$ (see Figure 3-2) as

$$\underline{\mathbf{S}} = \begin{bmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{bmatrix}. \tag{3.3}$$

It is obvious from Eqs. (3.2) and (3.3) that $\underline{\$}$ is a function of α and β since ${}^{E}\underline{\mathbf{r}}_{P0}^{E}$ is fixed on ground. Hence a derivative of the spring wrench can be written as

$$\begin{aligned} \delta\underline{\mathbf{w}} &= k\delta l\underline{\$} + k(l-l_o)\delta\underline{\$} \\ &= k\delta l\underline{\$} + k(1-\frac{l_o}{l})\left(\frac{\partial\underline{\$}}{\partial\alpha}\,l\,\delta\alpha + \frac{\partial\underline{\$}}{\partial\beta}\,l\,\delta\beta\right) \end{aligned} \tag{3.4}$$

where

$$\frac{\partial \underline{\$}}{\partial \alpha} = \begin{bmatrix} \dfrac{\partial \underline{\mathbf{S}}}{\partial \alpha} \\ {}^{E}\underline{\mathbf{r}}^{E}_{P0} \times \dfrac{\partial \underline{\mathbf{S}}}{\partial \alpha} \end{bmatrix} \tag{3.5}$$

$$\frac{\partial \underline{\$}}{\partial \beta} = \begin{bmatrix} \dfrac{\partial \underline{\mathbf{S}}}{\partial \beta} \\ {}^{E}\underline{\mathbf{r}}^{E}_{P0} \times \dfrac{\partial \underline{\mathbf{S}}}{\partial \beta} \end{bmatrix}. \tag{3.6}$$

By taking a derivative of Eq. (3.3), $\frac{\partial \underline{\mathbf{S}}}{\partial \alpha}$ and $\frac{\partial \underline{\mathbf{S}}}{\partial \beta}$ can be explicitly written by

$$\frac{\partial \underline{\mathbf{S}}}{\partial \alpha} = \begin{bmatrix} -\sin\beta\sin\alpha \\ \sin\beta\cos\alpha \\ 0 \end{bmatrix} \tag{3.7}$$

$$\frac{\partial \underline{\mathbf{S}}}{\partial \beta} = \begin{bmatrix} \cos\beta\cos\alpha \\ \cos\beta\sin\alpha \\ -\sin\beta \end{bmatrix}. \tag{3.8}$$

Since $\frac{\partial \underline{\mathbf{S}}}{\partial \alpha}$ is not a unit vector, a unit vector $\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha}$ is introduced as

$$\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix} \tag{3.9}$$

$$\frac{\partial \underline{\mathbf{S}}}{\partial \alpha} = \sin\beta \frac{\partial \underline{\mathbf{S}}'}{\partial \alpha}. \tag{3.10}$$

Hence Eq. (2.8) can be rewritten as

$$\delta \underline{\mathbf{w}} = k\delta l \underline{\$} + k(1 - \frac{l_o}{l})\left(\frac{\partial \underline{\$}'}{\partial \alpha} l \sin\beta\, \delta\alpha + \frac{\partial \underline{\$}}{\partial \beta} l\, \delta\beta \right) \tag{3.11}$$

where

$$\frac{\partial \underline{\$}'}{\partial \alpha} = \begin{bmatrix} \dfrac{\partial \underline{\mathbf{S}}'}{\partial \alpha} \\ {}^{E}\underline{\mathbf{r}}_{P0}^{E} \times \dfrac{\partial \underline{\mathbf{S}}'}{\partial \alpha} \end{bmatrix}. \tag{3.12}$$

It is important to note that $\frac{\partial \underline{\$}'}{\partial \alpha}$ and $\frac{\partial \underline{\$}}{\partial \beta}$ are the unitized Plücker coordinates of the lines perpendicular to $\underline{\mathbf{S}}$ and go through the pivot point P0.

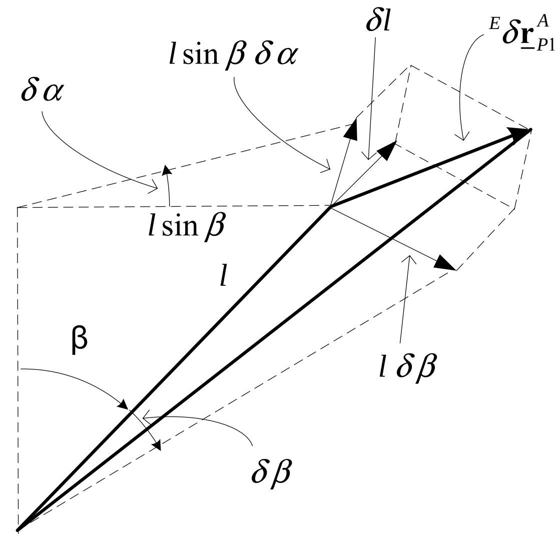

Figure 3-3. Small change of position of P1 due to a small twist of body A.

In Eq. (3.11) δl, $l \sin\beta\,\delta\alpha$, and $l\,\delta\beta$ can be considered as the change of the spring length and the changes of the direction of the spring (see Figure 3-3). These values correspond to the projections of the variation of position P1, ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$, onto the orthonormal vectors $\underline{\mathbf{S}}$, $\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha}$, and $\frac{\partial \underline{\mathbf{S}}}{\partial \beta}$, respectively. Thus ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$ can be rewritten as

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} &= \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\ \underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\ \frac{\partial\underline{\mathbf{S}}'}{\partial\alpha}\right)\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\ \frac{\partial\underline{\mathbf{S}}}{\partial\beta}\right)\frac{\partial\underline{\mathbf{S}}}{\partial\beta} \\
&= \delta l\ \underline{\mathbf{S}} + l\sin\beta\,\delta\alpha\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + l\,\delta\beta\frac{\partial\underline{\mathbf{S}}}{\partial\beta}
\end{aligned}
\quad . \tag{3.13}
$$

From the twist equation, the variation of position P1 can be written as

$$
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} = {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} + {}^{E}\delta\,\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A} \tag{3.14}
$$

where ${}^{E}\delta\underline{\mathbf{r}}_{0}^{A}$ is the differential of the position of point O in body A which is coincident with the origin of the inertial frame E measured with respect to the inertial frame. ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}$ is the differential of the angle of body A with respect to the inertial frame.

From Eqs. (2.11) and (2.10), δl, $l\sin\beta\,\delta\alpha$, and $l\,\delta\beta$ can be expressed as

$$
\begin{aligned}
\delta l &= {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}} = {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}} \\
&= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}} \\
&= \underline{\$}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned}
\tag{3.15}
$$

$$
\begin{aligned}
l\sin\beta\,\delta\alpha &= {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} = {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \\
&= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{E}\delta\underline{\boldsymbol{\varphi}}\cdot{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \\
&= \frac{\partial\underline{\$}''^{T}}{\partial\alpha}\,{}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned}
\tag{3.16}
$$

$$
\begin{aligned}
l\delta\beta &= {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} = {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} \\
&= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}\cdot{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{\partial\underline{\mathbf{S}}}{\partial\beta} \\
&= \frac{\partial\underline{\$}''^{T}}{\partial\beta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned}
\tag{3.17}
$$

where

$$ {}^{E}\delta\underline{\mathbf{D}}^{A} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{r}}_{0}^{A} \\ {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \end{bmatrix} \tag{3.18} $$

$$ \frac{\partial\underline{\$}''}{\partial\alpha} = \begin{bmatrix} \dfrac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\dfrac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix} \tag{3.19} $$

$$ \frac{\partial\underline{\$}''}{\partial\beta} = \begin{bmatrix} \dfrac{\partial\underline{\mathbf{S}}}{\partial\beta} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\dfrac{\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}. \tag{3.20} $$

It is important to note that $\frac{\partial\underline{\$}''}{\partial\alpha}$ and $\frac{\partial\underline{\$}''}{\partial\beta}$ are the unitized Plücker coordinates of lines perpendicular to $\underline{\mathbf{S}}$ which pass through the pivot point P1 in body A and ${}^{E}\delta\underline{\mathbf{D}}^{A}$ is a small twist of body A with respect to ground. Substituting Eqs. (2.12), (3.16), and (2.13) for δl, $l\sin\beta\,\delta\alpha$, and $l\,\delta\beta$ in Eq. (3.11) yields

$$ \begin{aligned} \delta\underline{\mathbf{w}} &= k\delta l\underline{\$} + k(1-\frac{l_o}{l})\left(\frac{\partial\underline{\$}'}{\partial\alpha}\,l\sin\beta\,\delta\alpha + \frac{\partial\underline{\$}}{\partial\beta}\,l\,\delta\beta\right) \\ &= k\underline{\$}\underline{\$}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A} + k(1-\frac{l_o}{l})\left(\frac{\partial\underline{\$}'}{\partial\alpha}\frac{\partial\underline{\$}''^{T}}{\partial\alpha} + \frac{\partial\underline{\$}}{\partial\beta}\frac{\partial\underline{\$}''^{T}}{\partial\beta}\right){}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= \left[K_F\right]{}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{3.21} $$

where

$$ \left[K_F\right] = k\underline{\$}\underline{\$}^{T} + k(1-\frac{l_o}{l})\left(\frac{\partial\underline{\$}'}{\partial\alpha}\frac{\partial\underline{\$}''^{T}}{\partial\alpha} + \frac{\partial\underline{\$}}{\partial\beta}\frac{\partial\underline{\$}''^{T}}{\partial\beta}\right). \tag{3.22} $$

$\left[K_F\right]$ is the stiffness matrix of a spatial compliant coupling and maps a small twist of body A into the corresponding variation of the wrench. The first term of Eq. (2.16) is always symmetric and the second term is not. When the spring deviates from its equilibrium position due to an external wrench, the second term of Eq. (2.16) doesn't vanish and it makes the stiffness matrix asymmetric. This result agrees with the works of Griffis (1991).

3.2 A Derivative of Spring Wrench Joining Two Moving Bodies

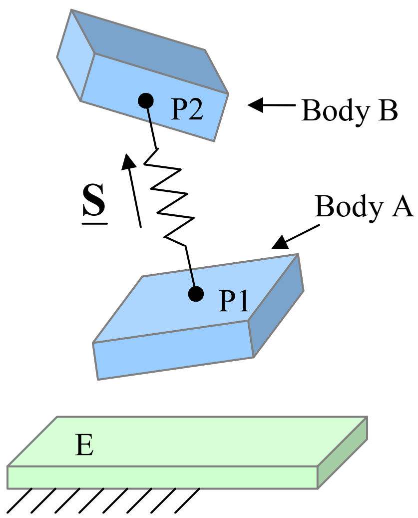

Figure 3-4. Spatial compliant coupling joining two moving bodies

Figure 3-4 illustrates two rigid bodies connected to each other by a compliant coupling with a spring constant k, a free length l_o, and a current length l. Both of body A and body B can move in a spatial space and the compliant coupling exerts a wrench $\underline{\mathbf{w}}$ to body B which is in equilibrium. The spring wrench may be written as

$$\underline{\mathbf{w}} = k(l - l_o)\underline{\$} \tag{3.23}$$

where

$$\underline{\$} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}^{A}_{P1} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}^{B}_{P2} \times \underline{\mathbf{S}} \end{bmatrix} \tag{3.24}$$

and where $\underline{\mathbf{S}}$ is a unit vector along the compliant coupling and ${}^{E}\underline{\mathbf{r}}_{P1}^{A}$ and ${}^{E}\underline{\mathbf{r}}_{P2}^{B}$ are the position vectors of the point P1 in body A and that of point P2 in body B, respectively, measured with respect to the reference system embedded in ground (body E). It is desired to express a derivative of the spring wrench in terms of the twist of body B ${}^{E}\delta\underline{\mathbf{D}}^{B}$ and that of body A ${}^{E}\delta\underline{\mathbf{D}}^{A}$. The twist ${}^{E}\delta\underline{\mathbf{D}}^{B}$ may be expressed as

$$ {}^{E}\delta\underline{\mathbf{D}}^{B} = {}^{E}\delta\underline{\mathbf{D}}^{A} + {}^{A}\delta\underline{\mathbf{D}}^{B} \tag{3.25} $$

where

$$ {}^{E}\delta\underline{\mathbf{D}}^{B} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{r}}_{o}^{B} \\ {}^{E}\delta\underline{\boldsymbol{\varphi}}^{B} \end{bmatrix} \tag{3.26} $$

$$ {}^{E}\delta\underline{\mathbf{D}}^{A} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \\ {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \end{bmatrix} \tag{3.27} $$

$$ {}^{A}\delta\underline{\mathbf{D}}^{B} = \begin{bmatrix} {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \\ {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \end{bmatrix} \tag{3.28} $$

and where ${}^{E}\delta\underline{\mathbf{r}}_{o}{}^{B}$ is the differential of point O, which is in body B and coincident with the origin of the inertial frame, measured with respect to the inertial frame and ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{B}$ is the differential of angle of body B with respect to the inertial frame. ${}^{E}\delta\underline{\mathbf{r}}_{o}{}^{A}$, ${}^{A}\delta\underline{\mathbf{r}}_{o}{}^{B}$, ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}$, and ${}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}$ are defined in the same way.

The derivative of the spring wrench in Eq. (2.17) can be written as

$$ {}^{E}\delta\underline{\mathbf{w}} = k\,\delta l\,\underline{\$} + k\,(l - l_{o})\,{}^{E}\delta\underline{\$} \tag{3.29} $$

and it is required to express δl and ${}^{E}\delta\underline{\$}$ in Eq. (2.23) in terms of the twists of the bodies. From the twist equation, the variation of position of point P2 in body B with respect to body A can be expressed as

$$ {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B} = {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \times {}^{A}\underline{\mathbf{r}}_{P2}^{B} \tag{3.30} $$

where ${}^{A}\underline{\mathbf{r}}_{P2}^{B}$ is the position of P2, which is embedded in body B, measured with respect to a coordinate system embedded in body A which at this instant is coincident and aligned with the reference system attached to ground. It can also be decomposed by projecting it onto the orthonormal vectors $\underline{\mathbf{S}}$, $\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}$ which are defined in a similar way as Eqs. (3.3), (3.9), and (3.8). These three vectors correspond to the change of the spring length δl and the directional changes of the spring such as $l\sin\beta\,\delta\alpha$ and $l\,\delta\beta$ in terms of body A in a way that is analogous to that shown in Figure 3-3. Thus the variation of position of point P2 in body B in terms of body A can be written as

$$ \begin{aligned} {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B} &= \left({}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot\underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot {}^{A}\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha}\right){}^{A}\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + \left({}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot {}^{A}\frac{\partial\underline{\mathbf{S}}}{\partial\beta}\right){}^{A}\frac{\partial\underline{\mathbf{S}}}{\partial\beta} \\ &= \delta l\,\underline{\mathbf{S}} + l\sin\beta\,\delta\alpha\,{}^{A}\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + l\,\delta\beta\,{}^{A}\frac{\partial\underline{\mathbf{S}}}{\partial\beta} \end{aligned} \tag{3.31} $$

From Eqs. (2.24) and (2.25), δl in Eq. (2.23) can be obtained as

$$ \begin{aligned} \delta l = {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot\underline{\mathbf{S}} &= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B}\cdot\underline{\mathbf{S}} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\times{}^{A}\underline{\mathbf{r}}_{P2}^{B}\cdot\underline{\mathbf{S}} \\ &= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B}\cdot\underline{\mathbf{S}} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\cdot{}^{A}\underline{\mathbf{r}}_{P2}^{B}\times\underline{\mathbf{S}} \\ &= \underline{\$}^{T}\,{}^{A}\delta\underline{\mathbf{D}}^{B} \end{aligned} \tag{3.32} $$

In the same way, $l\sin\beta\,\delta\alpha$ and $l\,\delta\beta$ can be expressed as

$$ \begin{aligned} l\sin\beta\,\delta\alpha = {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} &= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\times{}^{A}\underline{\mathbf{r}}_{P2}^{B}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ &= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\cdot{}^{A}\underline{\mathbf{r}}_{P2}^{B}\times\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ &= \frac{{}^{A}\partial\underline{\$}''}{\partial\alpha}\,{}^{A}\delta\underline{\mathbf{D}}^{B} \end{aligned} \tag{3.33} $$

$$
\begin{aligned}
l\,\delta\beta = {}^{A}\delta\underline{\mathbf{r}}^{B}_{P2}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} &= {}^{A}\delta\underline{\mathbf{r}}^{B}_{o}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\times{}^{A}\underline{\mathbf{r}}^{B}_{P2}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \\
&= {}^{A}\delta\underline{\mathbf{r}}^{B}_{o}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}\cdot{}^{A}\underline{\mathbf{r}}^{B}_{P2}\times\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \\
&= \frac{{}^{A}\partial\underline{\$}''}{\partial\beta}\;{}^{A}\delta\underline{\mathbf{D}}^{B}
\end{aligned}
\tag{3.34}
$$

where

$$
{}^{A}\frac{\partial\underline{\$}''}{\partial\alpha} = \begin{bmatrix} {}^{A}\dfrac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ {}^{A}\underline{\mathbf{r}}^{B}_{P2}\times\;{}^{A}\dfrac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix}
\tag{3.35}
$$

$$
{}^{A}\frac{\partial\underline{\$}''}{\partial\beta} = \begin{bmatrix} {}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\beta} \\ {}^{A}\underline{\mathbf{r}}^{B}_{P2}\times\;{}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}.
\tag{3.36}
$$

Now in Eq. (2.23), only ${}^{E}\delta\underline{\$}$ is yet to be obtained. It is a derivative of the unit screw along the spring in terms of the inertial frame and may be written as

$$
{}^{E}\delta\underline{\$} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}^{A}_{P1}\times\underline{\mathbf{S}} + {}^{E}\underline{\mathbf{r}}^{A}_{P1}\times{}^{E}\delta\underline{\mathbf{S}} \end{bmatrix}.
\tag{3.37}
$$

Using an intermediate frame attached to body A, a derivative of the unit vector $\underline{\mathbf{S}}$ can be written by

$$
{}^{E}\delta\underline{\mathbf{S}} = {}^{A}\delta\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}.
\tag{3.38}
$$

Thus ${}^{E}\delta\underline{\$}$ may be decomposed into three screws as

$$
\begin{aligned}
{}^{E}\delta\underline{\$} &= \begin{bmatrix} {}^{E}\delta\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}} + {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{E}\delta\underline{\mathbf{S}} \end{bmatrix} \\
&= \begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}} + {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left({}^{A}\delta\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right) \end{bmatrix} \\
&= \begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{A}\delta\underline{\mathbf{S}} \end{bmatrix} + \begin{bmatrix} {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right) \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}} \end{bmatrix}
\end{aligned} \quad . \qquad (3.39)
$$

Since $\underline{\mathbf{S}}$ is a function of α and β from the vantage of body A and $l\sin\beta\,\delta\alpha$ and $l\,\delta\beta$ were already described in Eqs. (2.28) and (3.34), the first screw in Eq. (2.32) can be written as

$$
\begin{aligned}
\begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{A}\delta\underline{\mathbf{S}} \end{bmatrix} &= \begin{bmatrix} {}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha + {}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left({}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha + {}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta\right) \end{bmatrix} = \begin{bmatrix} {}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha \end{bmatrix} + \begin{bmatrix} {}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times{}^{A}\dfrac{\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta \end{bmatrix} \\
&= {}^{A}\frac{\partial\underline{\$}'}{\partial\alpha}\frac{1}{l}l\sin\beta\,\delta\alpha + {}^{A}\frac{\partial\underline{\$}}{\partial\beta}\frac{1}{l}l\,\delta\beta \\
&= \frac{1}{l}\left({}^{A}\frac{\partial\underline{\$}'}{\partial\alpha}\,{}^{A}\frac{\partial\underline{\$}''^{T}}{\partial\alpha} + {}^{A}\frac{\partial\underline{\$}}{\partial\beta}\,{}^{A}\frac{\partial\underline{\$}''^{T}}{\partial\beta}\right){}^{A}\delta\underline{\mathbf{D}}^{B}
\end{aligned} \quad .(3.40)
$$

As to the second screw in Eq. (2.32), ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}$ can be decomposed onto three orthonormal vectors along $\underline{\mathbf{S}}$, $\dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}$, respectively, as

$$
{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} = \left\{\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right)\cdot\underline{\mathbf{S}}\right\}\underline{\mathbf{S}} + \left\{\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right)\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\right\}\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} + \left\{\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right)\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\right\}\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}
$$

. (3.41)

From the fact that $\underline{\mathbf{S}}$, $\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}$ are unit vectors and perpendicular to each other (see Figure 3-3), each dot product of Eq. (3.41) can be expressed as

$$\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right)\cdot\underline{\mathbf{S}}=0 \tag{3.42}$$

$$\begin{aligned}\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right)\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}&={}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\left(\underline{\mathbf{S}}\times\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\right)=-{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\\&=-\begin{bmatrix}\underline{\mathbf{0}}\\\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}\begin{bmatrix}{}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\\{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\end{bmatrix}=-\begin{bmatrix}\underline{\mathbf{0}}\\\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}\end{aligned} \tag{3.43}$$

$$\begin{aligned}\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right)\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}&={}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\left(\underline{\mathbf{S}}\times\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\right)={}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\\&=\begin{bmatrix}\underline{\mathbf{0}}\\\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}\begin{bmatrix}{}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\\{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\end{bmatrix}=\begin{bmatrix}\underline{\mathbf{0}}\\\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}\end{aligned} \tag{3.44}$$

where $\underline{\mathbf{0}}=\begin{bmatrix}0&0&0\end{bmatrix}^{T}$.

Hence, ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}$ can be rewritten as

$${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}=-\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\begin{bmatrix}\underline{\mathbf{0}}\\\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}+\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\begin{bmatrix}\underline{\mathbf{0}}\\\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A} \tag{3.45}$$

and the second screw in Eq. (2.32) can be expressed as

$$\begin{bmatrix} {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}\right) \end{bmatrix} = \begin{bmatrix} -\dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A} + \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left\{-\dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A} + \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}\right\} \end{bmatrix}$$

$$= -\begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A} + \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}$$

$$= \left(-{}^{A}\frac{\partial\underline{\$}''}{\partial\alpha}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T} + {}^{A}\frac{\partial\underline{\$}''}{\partial\beta}\begin{bmatrix}\underline{\mathbf{0}} \\ \dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}\right){}^{E}\delta\underline{\mathbf{D}}^{A}$$

. (3.46)

As to the third screw in Eq. (2.32), ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$ can be decomposed onto three orthonormal vectors along $\underline{\mathbf{S}}$, $\dfrac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}$, respectively, as

$$\begin{aligned} {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A} \\ &= \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} + \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\cdot \end{aligned} \quad (3.47)$$

The first dot product in Eq. (3.47) can be expressed as

$$\begin{aligned} {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}} \\ &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A}\cdot\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}}. \\ &= \underline{\$}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \quad (3.48)$$

In the same way, the second and third dot products in Eq. (3.47) can be written as

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\
&= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \cdot \underline{\mathbf{r}}_{P1}^{A} \times \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\
&= \frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\alpha} \; {}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned}
\tag{3.49}
$$

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \\
&= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \cdot \underline{\mathbf{r}}_{P1}^{A} \times \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}. \\
&= \frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\beta} \; {}^{E}\delta\underline{\mathbf{D}}^{A}
\end{aligned}
\tag{3.50}
$$

Finally, ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}}$ of the third screw in Eq. (2.32) can be expressed as

$$
\begin{aligned}
{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} &= \left\{ \left(\underline{\mathbf{\$}}^{T}\, {}^{E}\delta\underline{\mathbf{D}}^{A} \right) \underline{\mathbf{S}} + \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\alpha} \; {}^{E}\delta\underline{\mathbf{D}}^{A} \right) \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} + \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\beta} \; {}^{E}\delta\underline{\mathbf{D}}^{A} \right) \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \right\} \times \underline{\mathbf{S}} \\
&= \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\alpha} \; {}^{E}\delta\underline{\mathbf{D}}^{A} \right) \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \times \underline{\mathbf{S}} + \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\beta} \; {}^{E}\delta\underline{\mathbf{D}}^{A} \right) \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \times \underline{\mathbf{S}} \\
&= \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\alpha} \; {}^{E}\delta\underline{\mathbf{D}}^{A} \right) \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} - \left(\frac{{}^{A}\partial\underline{\mathbf{\$}}''^{T}}{\partial\beta} \; {}^{E}\delta\underline{\mathbf{D}}^{A} \right) \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}
\end{aligned}
\tag{3.51}
$$

since $\underline{\mathbf{S}}$, $\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}$ are unit vectors and perpendicular to each other (see Figure 3-3).

Substituting Eq. (3.51) for ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}}$ of the third screw in Eq. (2.32) yields

$$\begin{bmatrix}\underline{\mathbf{0}}\\ {}^E\delta\underline{\mathbf{r}}^A_{P1}\times\underline{\mathbf{S}}\end{bmatrix} = \begin{bmatrix}\underline{\mathbf{0}}\\ \left(\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\alpha}\;{}^E\delta\underline{\mathbf{D}}^A\right)\frac{{}^A\partial\underline{\mathbf{S}}}{\partial\beta} - \left(\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\beta}\;{}^E\delta\underline{\mathbf{D}}^A\right)\frac{{}^A\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}$$

$$= \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}\left(\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\alpha}\;{}^E\delta\underline{\mathbf{D}}^A\right) - \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}\left(\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\beta}\;{}^E\delta\underline{\mathbf{D}}^A\right). \tag{3.52}$$

$$= \left(\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\alpha} - \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\beta}\right){}^E\delta\underline{\mathbf{D}}^A$$

By replacing δl and ${}^E\delta\underline{\underline{\$}}$ in Eq. (2.23) with Eqs. (2.27), (2.33), (2.35), and (2.40) and sorting it into the twists, the derivative of the spring wrench can be rewritten as

$$\begin{aligned} {}^E\delta\,\underline{\mathbf{w}} &= k\delta l\underline{\underline{\$}} + k(l-l_o)\,{}^E\delta\underline{\underline{\$}} \\ &= [K_F]\,{}^A\delta\underline{\mathbf{D}}^B + [K_M]\,{}^E\delta\underline{\mathbf{D}}^A \end{aligned} \tag{3.53}$$

where

$$[K_F] = k\underline{\underline{\$}}\,\underline{\underline{\$}}^T + k(1-\frac{l_o}{l})\left({}^A\frac{\partial\underline{\underline{\$}}'}{\partial\alpha}\,{}^A\frac{\partial\underline{\underline{\$}}''^T}{\partial\alpha} + {}^A\frac{\partial\underline{\underline{\$}}}{\partial\beta}\,{}^A\frac{\partial\underline{\underline{\$}}''^T}{\partial\beta}\right) \tag{3.54}$$

$$[K_M] = k(l-l_o)\left(\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\alpha} - {}^A\frac{\partial\underline{\underline{\$}}''}{\partial\alpha}\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^T + {}^A\frac{\partial\underline{\underline{\$}}''}{\partial\beta}\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^T - \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^A\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}\frac{{}^A\partial\underline{\underline{\$}}''^T}{\partial\beta}\right)$$

. (3.55)

It is important to note that $[K_M]$ is identical to the negative of the spring wrench expressed as a spatial cross product operator (see Featherstone 1985 and Ciblak and Lipkin 1994). To prove it, all terms in Eq. (2.43) are explicitly expressed in a polar coordinate system and ${}^E\underline{\mathbf{r}}^A_{P1} = \begin{bmatrix} p_x & p_y & p_z\end{bmatrix}^T$ to yield

$$\left[K_M\right]=\begin{bmatrix}[\mathbf{0}] & [\mathbf{K12}]\\ [\mathbf{K12}] & [\mathbf{K22}]\end{bmatrix} \tag{3.56}$$

where

$$[\mathbf{K12}]=k\left(l-l_o\right)\begin{bmatrix}0 & \mathrm{c}_\beta & -\mathrm{s}_\beta\,\mathrm{s}_\alpha\\ -\mathrm{c}_\beta & 0 & \mathrm{s}_\beta\,\mathrm{c}_\alpha\\ \mathrm{s}_\beta\,\mathrm{s}_\alpha & -\mathrm{s}_\beta\,\mathrm{c}_\alpha & 0\end{bmatrix} \tag{3.57}$$

$$[\mathbf{K22}]=k(l-l_o)\begin{bmatrix}0 & p_x\,\mathrm{s}_\beta\,\mathrm{s}_\alpha-p_y\,\mathrm{s}_\beta\,\mathrm{c}_\alpha & -p_z\,\mathrm{s}_\beta\,\mathrm{c}_\alpha+p_x\,\mathrm{c}_\beta\\ -p_x\,\mathrm{s}_\beta\,\mathrm{s}_\alpha+p_y\,\mathrm{s}_\beta\,\mathrm{c}_\alpha & 0 & p_y\,\mathrm{c}_\beta-p_z\,\mathrm{s}_\beta\,\mathrm{s}_\alpha\\ p_z\,\mathrm{s}_\beta\,\mathrm{c}_\alpha-p_x\,\mathrm{c}_\beta & -p_y\,\mathrm{c}_\beta+p_z\,\mathrm{s}_\beta\,\mathrm{s}_\alpha & 0\end{bmatrix} \tag{3.58}$$

and where $[\mathbf{0}]$ is 3×3 zero matrix, $c_\alpha=\cos(\alpha)$, and $s_\alpha=\sin(\alpha)$, etc.

In the same way the spring wrench can be explicitly written as

$$\begin{aligned}\underline{\mathbf{w}}&=k(l-l_o)\underline{\mathbf{\$}}=k(l-l_o)\begin{bmatrix}\underline{\mathbf{S}}\\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}}\end{bmatrix}\\ &=k(l-l_o)\begin{bmatrix}s_\beta c_\alpha\\ s_\beta s_\alpha\\ c_\beta\\ p_y c_\beta-p_z s_\beta s_\alpha\\ p_z s_\beta c_\alpha-p_x c_\beta\\ p_x s_\beta s_\alpha-p_y s_\beta c_\alpha\end{bmatrix}=\begin{bmatrix}f_x\\ f_y\\ f_z\\ m_x\\ m_y\\ m_z\end{bmatrix}=\begin{bmatrix}\underline{\mathbf{f}}\\ \underline{\mathbf{m}}\end{bmatrix}.\end{aligned} \tag{3.59}$$

By comparing Eqs. (3.57) and (3.58) with Eq. (3.59) it is obvious that

$$[\mathbf{K12}]=k\left(l-l_o\right)\begin{bmatrix}0 & f_z & -f_y\\ -f_z & 0 & f_x\\ f_y & -f_x & 0\end{bmatrix}=-\underline{\mathbf{f}}\times \tag{3.60}$$

$$[\mathbf{K22}]=\begin{bmatrix}0 & m_z & -m_y\\ -m_z & 0 & m_x\\ m_y & -m_x & 0\end{bmatrix}=-\underline{\mathbf{m}}\times \tag{3.61}$$

where $\underline{\mathbf{f}}\times$ and $\underline{\mathbf{m}}\times$ are skew-symmetric matrices representing vector multiplication.

Then $[K_M]$ can be expressed as

$$[K_M] = \begin{bmatrix} [\mathbf{0}] & -\underline{\mathbf{f}}\times \\ -\underline{\mathbf{f}}\times & -\underline{\mathbf{m}}\times \end{bmatrix} = -\underline{\mathbf{w}}\times \tag{3.62}$$

where $\underline{\mathbf{w}}\times$ is the spring wrench expressed as a spatial cross product operator (see Featherstone 1985).

Finally the derivative of the spring wrench can be written as

$${}^{E}\delta\underline{\mathbf{w}} = [K_F]\,{}^{A}\delta\underline{\mathbf{D}}^{B} - (\underline{\mathbf{w}}\times)\,{}^{E}\delta\underline{\mathbf{D}}^{A}\,. \tag{3.63}$$

As shown in Eq. (3.63), the derivative of the spring wrench joining two rigid bodies depends not only on a relative twist between two bodies but also on the twist of the intermediate body, in this case body A, in terms of the inertial frame unless the initial external wrench $\underline{\mathbf{w}}$ is zero. $[K_F]$ which maps a small twist of body B in terms of body A into the corresponding change of wrench upon body B is identical to the stiffness matrix of the spring assuming the body A is stationary.

3.3 Stiffness Mapping of Spatial Compliant Parallel Mechanisms in Series

The derivative of spring wrench derived in the previous section is applied to obtain the stiffness mapping of the compliant mechanism shown in Figure 3-5. Body A is connected to ground by six compliant couplings and body B is connected to body A in the same way. Each compliant coupling has a spherical joint at each end and a prismatic joint with a spring in the middle. It is assumed that an external wrench $\underline{\mathbf{w}}_{ext}$ is applied to body B and that both body B and body A are in static equilibrium. The poses of body A and body B and the spring constants and free lengths of all compliant couplings are known.

The stiffness matrix $[K]$ which maps a small twist of the moving body B in terms of the ground, ${}^{E}\delta\underline{\mathbf{D}}^{B}$ (written in axis coordinates), into the corresponding wrench variation, $\delta\,\underline{\mathbf{w}}_{ext}$ (written in ray coordinates), is desired to be derived and this relationship can be written as

$$\delta\,\underline{\mathbf{w}}_{ext} = [K]\,{}^{E}\delta\underline{\mathbf{D}}^{B}. \tag{3.64}$$

The stiffness matrix can be derived by taking a derivative of the static equilibrium equations of body A and body B which may be written as

$$\underline{\mathbf{w}}_{ext} = \sum_{i=1}^{6}\underline{\mathbf{w}}_{i} = \sum_{i=7}^{12}\underline{\mathbf{w}}_{i} \tag{3.65}$$

$$\delta\,\underline{\mathbf{w}}_{ext} = \sum_{i=1}^{6}\delta\,\underline{\mathbf{w}}_{i} = \sum_{i=7}^{12}\delta\,\underline{\mathbf{w}}_{i} \tag{3.66}$$

where $\underline{\mathbf{w}}_{i}$ is the wrench from i-th compliant coupling.

Since springs 1 to 6 join the two moving bodies and springs 7 to 12 connect body A to ground (see Figure 3-5), the derivatives of the spring wrenches can be written as

$$\begin{aligned}\sum_{i=1}^{6}\delta\underline{\mathbf{w}}_{i} &= \sum_{i=1}^{6}\left([K_F]_i\;{}^{A}\delta\underline{\mathbf{D}}^{B} - \left(\underline{\mathbf{w}}_i\times\right){}^{E}\delta\underline{\mathbf{D}}^{A}\right)\\ &= [K_F]_{R,U}\;{}^{A}\delta\underline{\mathbf{D}}^{B} - \left(\underline{\mathbf{w}}_{ext}\times\right){}^{E}\delta\underline{\mathbf{D}}^{A}\end{aligned} \tag{3.67}$$

$$\sum_{i=7}^{12}\delta\underline{\mathbf{w}}_{i} = \sum_{i=7}^{12}[K_F]_i\;{}^{A}\delta\underline{\mathbf{D}}_{i}^{B} = [K_F]_{R,L}\;{}^{E}\delta\underline{\mathbf{D}}^{A} \tag{3.68}$$

where

$$[K_F]_{R,L} = \sum_{i=7}^{12}[K_F]_i \tag{3.69}$$

$$[K_F]_{R,U} = \sum_{i=1}^{6}[K_F]_i \tag{3.70}$$

and where $\underline{\mathbf{w}}_{ext}\times$ is the external wrench expressed as a spatial cross product operator.

From Eqs. (3.67), (3.68), and (3.25), twist ${}^{E}\delta\underline{\mathbf{D}}^{A}$ can be written as Eq. (3.72).

$$\begin{aligned}[K_F]_{R,L}\ {}^{E}\delta\underline{\mathbf{D}}^{A} &= [K_F]_{R,U}\ {}^{A}\delta\underline{\mathbf{D}}^{B} - (\underline{\mathbf{w}}_{ext}\times)\ {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= [K_F]_{R,U}\left({}^{E}\delta\underline{\mathbf{D}}^{B} - {}^{E}\delta\underline{\mathbf{D}}^{A}\right) - (\underline{\mathbf{w}}_{ext}\times)\ {}^{E}\delta\underline{\mathbf{D}}^{A}\end{aligned} \tag{3.71}$$

$${}^{E}\delta\underline{\mathbf{D}}^{A} = \left([K_F]_{R,L} + [K_F]_{R,U} + (\underline{\mathbf{w}}_{ext}\times)\right)^{-1}[K_F]_{R,U}\ {}^{E}\delta\underline{\mathbf{D}}^{B}\ . \tag{3.72}$$

Substituting Eq. (3.72) for ${}^{E}\delta\underline{\mathbf{D}}^{A}$ in Eq. (3.68) and comparing it with Eq. (3.64) yield the stiffness matrix as Eq. (3.74).

$$\begin{aligned}[K]\ {}^{E}\delta\underline{\mathbf{D}}^{B} &= [K_F]_{R,L}\ {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= [K_F]_{R,L}\left([K_F]_{R,L} + [K_F]_{R,U} + (\underline{\mathbf{w}}_{ext}\times)\right)^{-1}[K_F]_{R,U}\ {}^{E}\delta\underline{\mathbf{D}}^{B}\end{aligned} \tag{3.73}$$

$$[K] = [K_F]_{R,L}\left([K_F]_{R,L} + [K_F]_{R,U} + (\underline{\mathbf{w}}_{ext}\times)\right)^{-1}[K_F]_{R,U}\ . \tag{3.74}$$

It was generally accepted that the resultant compliance, which is the inverse of the stiffness, of serially connected mechanisms is the summation of the compliances of all constituent mechanisms (see Griffis 1991). However, the stiffness matrix derived from this research shows a different result. Taking an inverse of the stiffness matrix Eq. (3.74) yields

$$[K]^{-1} = [K_F]_{R,L}^{\ \ -1} + [K_F]_{R,U}^{\ \ -1} + [K_F]_{R,U}^{\ \ -1}(\underline{\mathbf{w}}_{ext}\times)[K_F]_{R,L}^{\ \ -1}. \tag{3.75}$$

The third term in Eq. (3.75) is newly introduced in this research and it does not vanish unless the external wrench is zero.

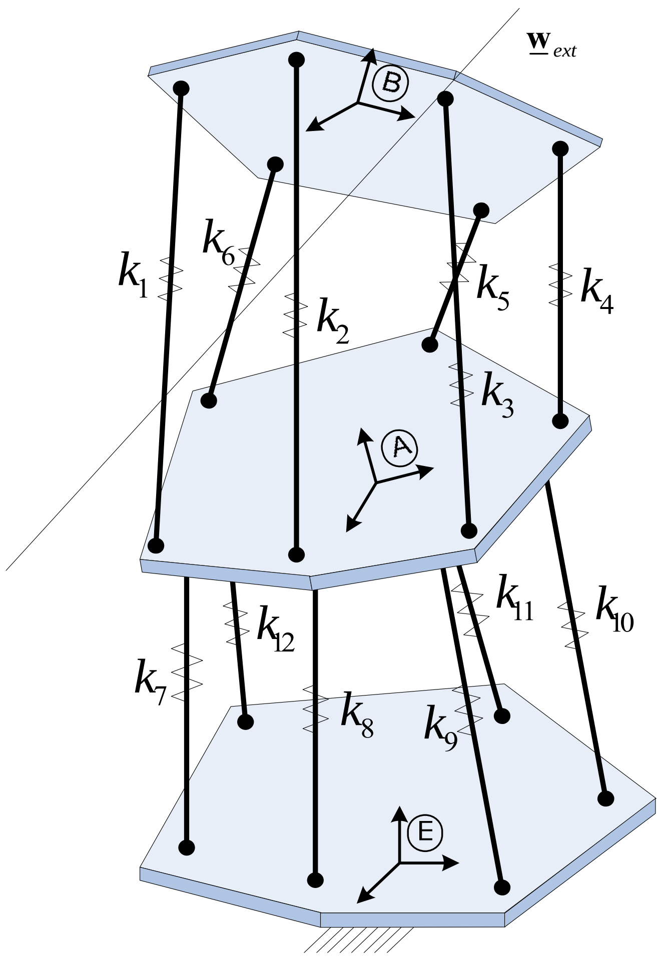

Figure 3-5. Mechanism having two compliant parallel mechanisms in series.[2]

A numerical example of the compliant mechanism depicted in Figure 3-5 is presented. The geometry information and spring properties of the mechanism shown in Figure 3-5 are presented in Tables 3-1 through 3-5. The external wrench $\underline{\mathbf{w}}_{ext}$ is given as

$$\underline{\mathbf{w}}_{ext} = \begin{bmatrix} -0.3 & 0.4 & 0.8 & -2.3 & -1.3 & 0.7 \end{bmatrix}^T . \text{ (unit:[N,N,N,Ncm,Ncm,Ncm])}$$

Table 3-1. Spring properties of the mechanism in Figure 3-5 (Unit: N/cm for k, cm for l_o).

Spring No.	1	2	3	4	5	6
Stiffness coefficient k_i	4.6	4.7	4.5	4.4	5.3	5.5

[2] The coordinate systems attached to bodies E, A, and B are for illustrative purposes only. In the analysis it is assumed that the three coordinate systems are at this instant coincident and aligned.

Free length l_{oi}	1.6305	1.0276	4.0098	1.8592	1.7591	3.8364

Table 3-1. Continued.

Spring No.	7	8	9	10	11	12
Stiffness coefficient k_i	4.4	4.9	4.7	4.5	5.1	4.8
Free length l_{oi}	4.4718	1.2760	5.2149	2.6780	2.2712	3.4244

Table 3-2. Positions of pivots in ground in Figure 3-5 (Unit: cm).

No.	1	2	3	4	5	6
X	0.0000	1.3000	0.6000	-0.7000	-1.1000	-0.5000
Y	0.0000	1.1000	2.7000	2.6000	1.8000	0.4000
Z	0.0000	0.2000	0.1000	-0.1000	0.3000	0.1000

Table 3-3. Positions of pivots in bottom side of body A in Figure 3-5 (Unit: cm).

No.	1	2	3	4	5	6
X	0.2000	1.1833	0.4616	-0.6575	-1.1452	-0.2189
Y	1.2000	2.1235	3.5111	3.3783	2.5652	1.6879
Z	3.2000	3.1843	3.3010	3.1013	3.0704	3.1196

Table 3-4. Positions of pivots in top side of body A in Figure 3-5 (Unit: cm).

No.	1	2	3	4	5	6
X	0.2086	1.4860	0.7553	-0.5501	-0.9278	-0.2945
Y	1.2033	2.3329	3.9187	3.7867	2.9797	1.5942
Z	3.2996	3.2514	3.3121	3.2792	3.3590	3.3804

Table 3-5. Positions of pivots in body B in Figure 3-5 (Unit: cm).

No.	1	2	3	4	5	6
X	-0.3000	0.9216	0.2183	-0.8385	-1.2525	-0.5589
Y	1.6000	2.7822	3.8980	3.9919	2.8972	2.0875
Z	5.5000	5.5000	5.4782	5.8447	5.8317	5.7745

Two stiffness matrices are calculated: $[K]_1$ from Eq. (3.74) and $[K]_2$ from the same equation but without the matrix $\underline{\mathbf{w}}\times$. The numerical results are

$$[K]_1 = \begin{bmatrix} 0.3429 & -0.0077 & -0.2661 & -0.7853 & 1.7378 & -0.4076 \\ -0.0077 & 0.5103 & 1.7122 & 1.2760 & 0.2157 & -0.2885 \\ -0.2661 & 1.7122 & 10.5103 & 20.0012 & 0.7518 & -0.2695 \\ -0.7853 & 2.0760 & 19.6012 & 54.3222 & 1.1348 & 1.2570 \\ 0.9378 & 0.2157 & 0.4518 & 0.4348 & 12.1329 & -3.8667 \\ -0.0076 & 0.0115 & -0.2695 & -0.0430 & -1.5667 & -0.0798 \end{bmatrix}$$

$$[K]_2 = \begin{bmatrix} 0.3039 & -0.0109 & -0.2641 & -0.7858 & 1.3134 & -0.2770 \\ -0.0504 & 0.4617 & 1.7122 & 1.9863 & -0.0875 & 0.0375 \\ -0.4364 & 1.6222 & 10.6633 & 21.7834 & -0.7144 & 0.5956 \\ -1.0788 & 1.9862 & 20.7574 & 59.4736 & -1.1874 & 2.5822 \\ 0.9754 & 0.0759 & -0.0901 & -0.1283 & 12.1852 & -3.0399 \\ -0.0319 & 0.0218 & -0.0576 & 0.6983 & -1.6440 & -0.1157 \end{bmatrix}$$

where, the units of upper left 3 × 3 sub matrix is N/cm, that of lower right 3 × 3 sub matrix is Ncm, and that of remainder is N.

The result is evaluated in the following way:

1. A small wrench $\delta \underline{\mathbf{w}}_T$ is applied in addition to $\underline{\mathbf{w}}_{ext}$ to body B and twists ${}^E\delta\underline{\mathbf{D}}_1^B$ and ${}^E\delta\underline{\mathbf{D}}_2^B$ are obtained by multiplying the inverse matrices of the stiffness matrices, $[K]_1$ and $[K]_2$, respectively, by $\delta \underline{\mathbf{w}}_T$ as of Eq. (3.64). Corresponding positions for body B are then determined, based on the calculated twists ${}^E\delta\underline{\mathbf{D}}_1^B$ and ${}^E\delta\underline{\mathbf{D}}_2^B$.

2. ${}^E\delta\underline{\mathbf{D}}^A$ is calculated by multiplying the inverse matrix of $[K_F]_{R,L}$ by $\delta \underline{\mathbf{w}}_T$ as of Eq. (3.68). The position of body A is then determined from this twist.

3. The wrench between body B and body A is calculated for the two cases based on knowledge of the positions of bodies A and B and the spring parameters. The change in wrench for the two cases is determined as the difference between the new equilibrium wrench and the original. The changes in the wrenches are named $\delta \underline{\mathbf{w}}_{AB,1}$ and $\delta \underline{\mathbf{w}}_{AB,2}$ which correspond to the matrices $[K]_1$ and $[K]_2$.

4. The given change in wrench $\delta \underline{\mathbf{w}}_T$ is compared to $\delta \underline{\mathbf{w}}_{AB,1}$ and $\delta \underline{\mathbf{w}}_{AB,2}$.

The given wrench $\delta \underline{\mathbf{w}}_T$ and the numerical results are presented as below.

$$\delta \underline{\mathbf{w}}_T = 10^{-4} \times [0.5 \quad -0.2 \quad 0.4 \quad 0.3 \quad -0.8 \quad 0.4]^T$$

$${}^E\delta\underline{\mathbf{D}}_1{}^B = 10^{-3} \times [0.3522 \quad -0.3081 \quad 0.0912 \quad -0.0137 \quad -0.0429 \quad -0.0367]^T$$

$${}^E\delta\underline{\mathbf{D}}_2{}^B = 10^{-3} \times [0.3354 \quad -0.2682 \quad 0.0845 \quad -0.0132 \quad -0.0404 \quad -0.0365]^T$$

$${}^E\delta\underline{\mathbf{D}}^A = 10^{-3} \times [0.1113 \quad -0.0100 \quad -0.0067 \quad 0.0081 \quad -0.0267 \quad -0.0650]^T$$

$$\delta \underline{\mathbf{w}}_{EA} = 10^{-4} \times [0.5000 \quad -0.1995 \quad 0.4017 \quad 0.3035 \quad -0.8000 \quad 0.4000]^T$$

$$\delta \underline{\mathbf{w}}_{AB,1} = 10^{-4} \times [0.4997 \quad -0.1998 \quad 0.4020 \quad 0.3041 \quad -0.8010 \quad 0.4011]^T$$

$$\delta \underline{\mathbf{w}}_{AB,2} = 10^{-4} \times [0.5462 \quad -0.0693 \quad 0.3534 \quad -0.7831 \quad -0.2319 \quad 0.0967]^T$$

where $\delta \underline{\mathbf{w}}_{EA}$ is the wrench between body A and ground. The unit for the wrenches is [N, N, N, Ncm, Ncm, Ncm]^T and that of the twists is [cm, cm, cm, rad, rad, rad]^T. The difference between $\delta \underline{\mathbf{w}}_{EA}$ and $\delta \underline{\mathbf{w}}_T$ is small and is due to the fact that the twist was not infinitesimal. The difference between $\delta \underline{\mathbf{w}}_{AB,1}$ and $\delta \underline{\mathbf{w}}_T$ is also small and is most likely attributed to the same fact. However, the difference between $\delta \underline{\mathbf{w}}_{AB,2}$ and $\delta \underline{\mathbf{w}}_T$ is not negligible. This indicates that the stiffness matrix formula derived in this research produces the proper result and that the term $\underline{\mathbf{w}}_{ext} \times$ cannot be neglected in Eq. (3.74).

CHAPTER 4
STIFFNESS MODULATION OF PLANAR COMPLIANT MECHANISMS

Planar mechanisms with variable compliance, specifically, compliant parallel mechanisms and mechanisms having two compliant parallel mechanisms in a serial arrangement are investigated in this chapter. The mechanisms consist of rigid bodies joined by adjustable compliant couplings. Each adjustable compliant coupling has a revolute joint at each end and a prismatic joint with an adjustable spring in the middle. The adjustable springs are assumed to be able to change their stiffness coefficient and free length and the mechanisms are in static equilibrium under an external wrench. It is desired to modulate the compliance of the mechanism while regulating the pose of the mechanism.

4.1 Parallel Mechanisms with Variable Compliance

4.1.1 Constraint on Stiffness Matrix

Figure 4-1 illustrates a compliant parallel mechanism having *N* number of compliant couplings. The mechanism is in static equilibrium under the external wrench $\underline{\mathbf{w}}_{ext}$ and it can be expressed as

$$\underline{\mathbf{w}}_{ext} = \sum_{i=1}^{N} \underline{\mathbf{f}}_i \tag{4.1}$$

where $\underline{\mathbf{f}}_i$ is the spring wrench of i-th compliant coupling. By using Eq. (2.15) a derivative of Eq. (4.1) may be written as

$$\begin{aligned}\delta \underline{\mathbf{w}}_{ext} &= \sum_{i=1}^{N} \delta \underline{\mathbf{f}}_i \\ &= \left(\sum_{i=1}^{N} \left[K_F \right]_i \right) {}^E\delta \underline{\mathbf{D}}^A \end{aligned} \tag{4.2}$$

where

$$\left[K_F \right]_i = k_i \underline{\$}_i \underline{\$}_i^T + k_i (1 - \frac{l_{oi}}{l_i}) \frac{\partial \underline{\$}_i}{\partial \theta_i} \frac{\partial \underline{\$}_i'^T}{\partial \theta_i} . \tag{4.3}$$

k_i , l_{oi} , l_i , and θ_i in Eq. (4.3) are the spring constant, the spring free length, the current spring length, and the rising angle of i-th compliant coupling, respectively (see Figure 2-4). In addition, $\underline{\$}_i$ represents the unitized Plücker coordinates of the line along the i^th compliant coupling as in Eq. (2-6) and may be written explicitly as

$$\underline{\$}_i = \begin{bmatrix} \underline{\mathbf{S}}_i \\ \underline{\mathbf{r}}_{P,i} \times \underline{\mathbf{S}}_i \end{bmatrix} = \begin{bmatrix} \cos \theta_i \\ \sin \theta_i \\ r_{x,i} \sin \theta_i - r_{y,i} \cos \theta_i \end{bmatrix} \tag{4.4}$$

where $r_{x,i}$ and $r_{y,i}$ are the pivot position of i-th compliant coupling in body E. Then Eq. (4.4) leads to

$$\frac{\partial \underline{\$}_i}{\partial \theta_i} = \begin{bmatrix} -\sin \theta_i \\ \cos \theta_i \\ r_{x,i} \cos \theta_i + r_{y,i} \sin \theta_i \end{bmatrix} . \tag{4.5}$$

The stiffness matrix of the mechanism $\left[K \right]$ can be written from Eq. (4.2) as

$$\left[K \right] = \sum_{i=1}^{N} \left[K_F \right]_i . \tag{4.6}$$

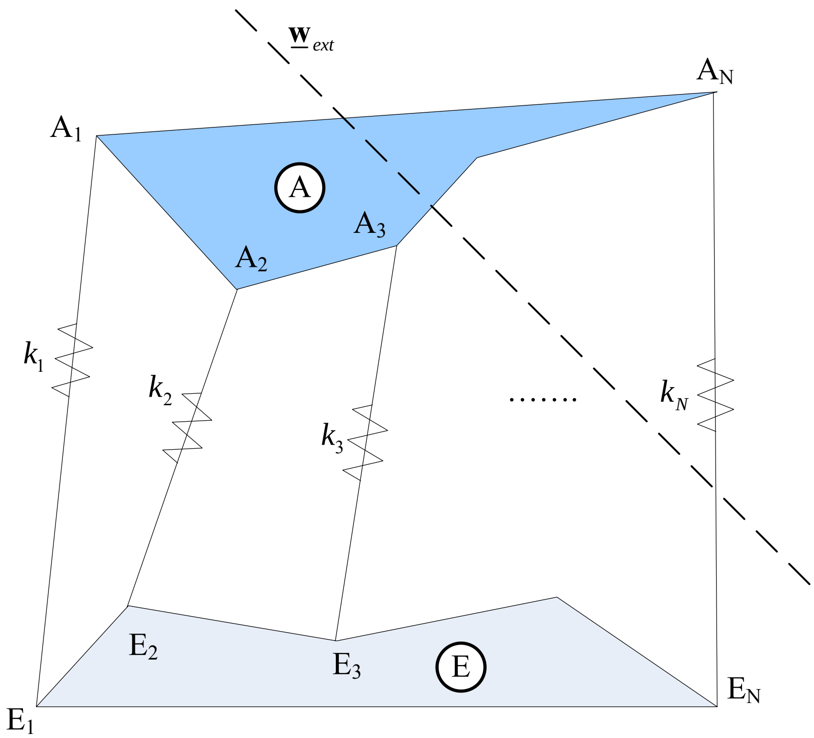

Figure 4-1. Compliant parallel mechanism with *N* number of couplings.

Ciblak and Lipkin (1994) showed that the stiffness matrix of compliant parallel mechanisms can be decomposed into a symmetric and a skew symmetric part and that the skew symmetric part is negative one-half the externally applied load expressed as a spatial cross product operator. For planar mechanisms, the skew symmetric part can be written as

$$\begin{aligned}[K] &= \frac{[K]+[K]^T}{2} + \frac{[K]-[K]^T}{2} \\ &= [K]_{Symmetric} + [K]_{Skew\ Symmetric}\end{aligned} \tag{4.7}$$

$$[K]_{Skew\ Symmetric} = \frac{[K]-[K]^T}{2} = -\frac{1}{2}\begin{bmatrix} 0 & 0 & f_y \\ 0 & 0 & -f_x \\ -f_y & f_x & 0 \end{bmatrix} \tag{4.8}$$

where $\underline{\mathbf{w}}_{ext} = \left[f_x, f_y, m_z \right]^T$ is the external wrench.

It is important to note that no matter how many compliant couplings are connected and no matter how the spring constants and the free lengths of the constituent compliant couplings are changed the stiffness matrix of a compliant parallel mechanism contains only six independent variables and the stiffness matrix may be rewritten as

$$[K]=\begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{32}+f_x \\ K_{13}+f_y & K_{32} & K_{33} \end{bmatrix}. \tag{4.9}$$

From Eqs. (4.3)-(4.6) the six independent elements of the stiffness matrix $[K]$ can be explicitly written as

$$K_{11}=\sum_{i=1}^{N}\left(k_i-k_i\frac{l_{oi}}{l_i}\sin^2\theta_i\right) \tag{4.10}$$

$$K_{12}=\sum_{i=1}^{N}\left(k_i\frac{l_{oi}}{l_i}\sin\theta_i\cos\theta_i\right) \tag{4.11}$$

$$K_{13}=\sum_{i=1}^{N}\left(-k_i(l_i\sin\theta_i+r_{y,i})+k_i\frac{l_{oi}}{l_i}(l_i\sin\theta_i+r_{x,i}\sin\theta_i\cos\theta_i+r_{y,i}\sin^2\theta_i)\right) \tag{4.12}$$

$$K_{22}=\sum_{i=1}^{N}\left(k_i-k_i\frac{l_{oi}}{l_i}\cos^2\theta_i\right) \tag{4.13}$$

$$K_{32}=\sum_{i=1}^{N}\left(k_ir_{x,i}-k_i\frac{l_{oi}}{l_i}(r_{x,i}\cos^2\theta_i+r_{y,i}\sin\theta_i\cos\theta_i)\right) \tag{4.14}$$

$$K_{33}=\sum_{i=1}^{N}\left(\begin{array}{l} k_ir_{x,i}(l_i\cos\theta_i+r_{x,i})+k_ir_{y,i}(l_i\sin\theta_i+r_{y,i}) \\ \qquad -k_i\frac{l_{oi}}{l_i}r_{x,i}(l_i\cos\theta_i+r_{y,i}\sin\theta_i\cos\theta_i+r_{x,i}\cos^2\theta_i) \\ \qquad -k_i\frac{l_{oi}}{l_i}r_{y,i}(l_i\sin\theta_i+r_{x,i}\sin\theta_i\cos\theta_i+r_{y,i}\sin^2\theta_i) \end{array}\right). \tag{4.15}$$

4.1.2 Stiffness Modulation by Varying Spring Parameters

In this case it is desired to find an appropriate set of spring constants and free lengths of the constituent compliant couplings of the mechanism shown in Figure 4-1 to implement a given stiffness matrix and to regulate the pose of body A under a given external wrench.

It is important to note that the stiffness matrix contains only six independent variables and the equations for the independent variables are linear in terms of k_i's and $k_i l_{oi}$'s as shown in Eqs. (4.10)-(4.15) since all geometrical terms are constant. In addition to the equations for the stiffness matrix, the system should satisfy static equilibrium equations to regulate the pose of the mechanism and from Eqs. (4.1) and (4.4) it can be written as

$$\underline{\mathbf{w}}_{ext} = \begin{bmatrix} f_x \\ f_y \\ m_z \end{bmatrix} = \sum_{i=1}^{N} \left(k_i \left(l_i - l_{oi} \right) \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ r_{x,i} \sin\theta_i - r_{y,i} \cos\theta_i \end{bmatrix} \right). \tag{4.16}$$

Eq. (4.16) consists of three equations which are also linear in terms of k_i's and $k_i l_{oi}$'s. Since there are nine linear equations to be fulfilled and each adjustable compliant coupling possesses two control variables such as spring constant and free length, at least five adjustable compliant couplings are required.

The nine equations may be written in matrix form as

$$[A]\underline{\mathbf{X}} = \underline{\mathbf{B}} \tag{4.17}$$

where

$$\underline{\mathbf{B}} = \left[K_{11}, K_{12}, K_{13}, K_{22}, K_{32}, K_{33}, f_x, f_y, m_z \right]^T \tag{4.18}$$

$$\underline{\mathbf{X}} = \left[k_1, k_2, \cdots, k_N, k_1 l_{o1}, k_2 l_{o2}, \cdots, k_N l_{oN} \right]^T \tag{4.19}$$

$$[A] = \begin{bmatrix} 1 & 1 & \cdots & 1 & G_{1,1} & G_{1,2} & \cdots & G_{1,N} \\ 0 & 0 & \cdots & 0 & G_{2,1} & G_{2,2} & \cdots & G_{2,N} \\ G_{3,1} & G_{3,2} & \cdots & G_{3,N} & G_{4,1} & G_{4,2} & \cdots & G_{4,N} \\ 1 & 1 & \cdots & 1 & G_{5,1} & G_{5,2} & \cdots & G_{5,N} \\ G_{6,1} & G_{6,2} & \cdots & G_{6,N} & G_{7,1} & G_{7,2} & \cdots & G_{7,N} \\ G_{8,1} & G_{8,2} & \cdots & G_{8,N} & G_{9,1} & G_{9,2} & \cdots & G_{9,N} \\ H_{1,1} & H_{1,2} & \cdots & H_{1,N} & H_{2,1} & H_{2,2} & \cdots & H_{2,N} \\ H_{3,1} & H_{3,2} & \cdots & H_{3,N} & H_{4,1} & H_{4,2} & \cdots & H_{4,N} \\ H_{5,1} & H_{5,2} & \cdots & H_{5,N} & H_{6,1} & H_{6,2} & \cdots & H_{6,N} \end{bmatrix} \tag{4.20}$$

and where

$$G_{1,i} = -\frac{\sin^2\theta_i}{l_i},\ G_{2,i} = \frac{\sin\theta_i \cos\theta_i}{l_i},\ G_{3,i} = -l_i \sin\theta_i - r_{y,i}$$

$$G_{4,i} = \sin\theta_i + \frac{r_{x,i}\sin\theta_i\cos\theta_i + r_{y,i}\sin^2\theta_i}{l_i},\ G_{5,i} = -\frac{\cos^2\theta_i}{l_i}$$

$$G_{6,i} = r_{x,i},\ \ G_{7,i} = \frac{-r_{x,i}\cos^2\theta_i - r_{y,i}\sin\theta_i\cos\theta_i}{l_i}$$

$$G_{8,i} = {r_{x,i}}^2 + {r_{y,i}}^2 + l_i(r_{x,i}\cos\theta_i + r_{y,i}\sin\theta_i)$$

$$G_{9,i} = -r_{x,i}\cos\theta_i - r_{y,i}\sin\theta_i - \frac{{r_{x,i}}^2\cos^2\theta_i + {r_{y,i}}^2\sin^2\theta_i + 2r_{x,i}r_{y,i}\sin\theta_i\cos\theta_i}{l_i}$$

$$H_{1,i} = l_i\cos\theta_i,\ H_{2,i} = -\cos\theta_i,\ H_{3,i} = l_i\sin\theta_i,\ H_{4,i} = -\sin\theta_i$$

$$H_{5,i} = l_i\left(r_{x,i}\sin\theta_i - r_{y,i}\cos\theta_i\right),\ H_{6,i} = -\left(r_{x,i}\sin\theta_i - r_{y,i}\cos\theta_i\right).$$

It is important to note that $[A]$, $\underline{\mathbf{X}}$, and $\underline{\mathbf{B}}$ are 9 × (2*N), (2*N) × 1, and 9 × 1 matrices, respectively, where N denotes the number of the adjustable compliant couplings.

It is required to solve Eq. (4.17) where the number of columns of matrix $[A]$ is in general greater than that of rows and the general solution $\underline{\mathbf{X}}_{sol}$ can be written as

$$\begin{aligned} \underline{\mathbf{X}}_{sol} &= \underline{\mathbf{X}}_p + \underline{\mathbf{X}}_h \\ &= \underline{\mathbf{X}}_p + [A_{Null}]\underline{\mathbf{C}} \end{aligned} \tag{4.21}$$

where $\underline{\mathbf{X}}_p$, $\underline{\mathbf{X}}_h$, $[A_{Null}]$, and $\underline{\mathbf{C}}$ are the particular solution, the homogeneous solution, the null space of matrix $[A]$, and the coefficient column matrix, respectively (see Strang 1988). Once a solution $\underline{\mathbf{X}}_{sol}$ is obtained, l_{oi}'s are calculated from k_i's and $k_i l_{oi}$'s in $\underline{\mathbf{X}}_{sol}$. It is important to note that $[A_{Null}]$ is (2*N) × (2*N-9) matrix and $\underline{\mathbf{C}}$ is (2*N-9) × 1 column matrix.

There might be many strategies to select the matrix $\underline{\mathbf{C}}$ which leads to a specific solution. For instance, if the norm of $\underline{\mathbf{X}}_{sol}$ is desired to be minimized, then by using projection matrix $[A_{Null-P}]$ (see Strang 1988), the solution can be obtained as

$$\underline{\mathbf{X}}_{Min.sol} = \underline{\mathbf{X}}_p + [A_{Null-P}]\left(-\underline{\mathbf{X}}_p\right) \tag{4.22}$$

where

$$[A_{Null-P}] = [A_{Null}]\left([A_{Null}]^T [A_{Null}]\right)^{-1} [A_{Null}]^T . \tag{4.23}$$

For another case, we might want the solution closest to a desired solution $\underline{\mathbf{X}}_d$ which may be constructed from operation ranges of adjustable compliant couplings, for instance, minimum and maximum spring constant and free length. Then, the solution can be obtained as

$$\underline{\mathbf{X}}_{d.sol} = \underline{\mathbf{X}}_p + [A_{Null-P}]\left(\underline{\mathbf{X}}_d - \underline{\mathbf{X}}_p\right). \tag{4.24}$$

Unfortunately, these methods involve mixed unit problems and do not guarantee a solution consisting of only positive spring constants and free lengths.

A numerical example is presented. The external wrench $\underline{\mathbf{w}}_{ext}$ and the desired stiffness matrix $[K]$ are given as

$$\underline{\mathbf{w}}_{ext} = [-1.8832\,N \quad -2.8805\,N \quad 3.2851\,Ncm]$$

$$[K] = \begin{bmatrix} 0.0216 \quad N/cm & 2.2483 \quad N/cm & -2.2750 \quad N \\ 2.2483 \quad N/cm & 25.3914 \quad N/cm & 60.9800 \quad N \\ -5.1555 \quad N & 62.8632 \quad N & 270.4409 \quad Ncm \end{bmatrix}.$$

The geometry information of the mechanism shown in Figure 4-1 is given in Tables 4-1 and 4-2. The mechanism is assumed to have five compliant couplings.

Table 4-1. Positions of pivot points in body E for numerical example in 4.1.2.

Pivot points	E1	E2	E3	E4	E5
X	0.0000	0.6000	2.5000	3.9000	5.3000
Y	0.0000	0.8000	0.3000	0.9000	0.0000

(Unit: cm)

Table 4-2. Positions of pivot points in body A for numerical example in 4.1.2.

Pivot points	A1	A2	A3	A4	A5
X	0.6000	1.4055	2.6736	3.3368	4.7284
Y	4.5000	2.7447	3.3209	3.9614	4.1442

(Unit: cm)

The spring parameters which have the minimum norm and satisfy the given conditions can be obtained by using Eq. (4.22) and these values are shown in Table 4-3.

Table 4-3. Spring parameters with minimum norm for numerical example 4.1.2.

Spring No.	1	2	3	4	5
Stiffness constant k	4.6674	7.2485	3.5188	5.0243	6.3280
Free length l_o	4.1678	2.1490	6.3995	1.9322	3.9104

(Unit: N/cm for k, cm for l_o)

The spring parameters which are closest to the given spring parameters as shown in Table 4-4 can be obtained by applying Eq. (4.24) and it is shown in Table 4-5.

Table 4-4. Given optimal spring parameters for numerical example 4.1.2

Spring No.	1	2	3	4	5
Stiffness constant k	5.0	5.0	5.0	5.0	5.0
Free length l_o	3.0	3.0	3.0	3.0	3.0

(Unit: N/cm for k, cm for l_o)

Table 4-5. Spring parameters closest to given spring parameters for numerical example 4.1.2.

Spring No.	1	2	3	4	5
Stiffness constant k	4.8664	6.8783	3.8968	4.8990	6.2974
Free length l_o	4.3386	2.3374	5.0230	2.1667	4.0492

(Unit: N/cm for k, cm for l_o)

Two sets of the spring parameters, one in Table 4-3 and the other in Table 4-5, implement the given wrench and stiffness matrix.

4.1.3 Stiffness Modulation by Varying Spring Parameters and Displacement of the Mechanism

In this case, different from the previous section, the pose of body A is not constrained as fixed. A change of the pose of body A, which is considered to be in contact with the environment, may be compensated by attaching body E to the end of a robot system and by controlling the position of the robot end effector in a similar manner as described in the Theory of Kinestatic Control proposed by Griffis (1991). As presented in the previous section there are nine values to be fulfilled: six from the stiffness matrix and three from the wrench equations. A typical planar parallel mechanism which has three couplings is investigated since the mechanism has nine control input variables which is same with that of values to be fulfilled: six from adjustable compliant couplings and three from the planar displacement between body A and body E. The target variables may be expressed in matrix form as $\underline{\mathbf{B}}$ in Eq. (4.18) and control input variables $\underline{\mathbf{U}}$ may be written in matrix form as

$$\underline{\mathbf{U}} = [k_1, k_2, k_3, l_{o1}, l_{o2}, l_{o3}, x_o, y_o, \phi]^T \tag{4.25}$$

where k_i's and l_{oi}'s are the spring constant and free length of ith compliant coupling, respectively. In addition, x_o and y_o are the position of point O in body A, which is coincident with the origin of the inertial frame E, and ϕ is the rotation angle of body A with respect to ground.

The stiffness matrix equations and wrench equations are highly nonlinear in terms of the displacement of the bodies as shown in Eqs. (4.9)-(4.16). In this section a derivative of the target variables $\underline{\mathbf{B}}$ with respect to input variables $\underline{\mathbf{U}}$ is investigated and the derivative is used to obtain the small change of input variables for the desired small change of target values and it may be written as

$$\delta\underline{\mathbf{B}} = \frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}\delta\underline{\mathbf{U}} \tag{4.26}$$

$$\delta\underline{\mathbf{U}} = \left(\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}\right)^{-1}\delta\underline{\mathbf{B}} \tag{4.27}$$

where

$$\delta\underline{\mathbf{B}} = [\delta K_{11}, \delta K_{12}, \delta K_{13}, \delta K_{22}, \delta K_{32}, \delta K_{33}, \delta f_x, \delta f_y, \delta m_z]^T \tag{4.28}$$

$$\delta\underline{\mathbf{U}} = [\delta k_1, \delta k_2, \delta k_3, \delta l_{o1}, \delta l_{o2}, \delta l_{o3}, \delta x_o, \delta y_o, \delta\phi]^T\,. \tag{4.29}$$

$$\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}} = \begin{bmatrix} \frac{\partial B_1}{\partial U_1} & \cdots & \frac{\partial B_1}{\partial U_9} \\ \vdots & \ddots & \vdots \\ \frac{\partial B_9}{\partial U_1} & \cdots & \frac{\partial B_9}{\partial U_9} \end{bmatrix}. \tag{4.30}$$

For instance, δB_1 can be written as

$$\delta B_1 = \frac{d\,B_1}{d\,\underline{\mathbf{U}}}\delta\underline{\mathbf{U}}$$
$$= \frac{\partial K_{11}}{\partial k_1}\delta k_1 + \frac{\partial K_{11}}{\partial k_2}\delta k_2 + \frac{\partial K_{11}}{\partial k_3}\delta k_3 + \frac{\partial K_{11}}{\partial l_{o1}}\delta l_{o1} + \frac{\partial K_{11}}{\partial l_{o2}}\delta l_{o2} + \frac{\partial K_{11}}{\partial l_{o3}}\delta l_{o3}\,. \quad (4.31)$$
$$+ \frac{\partial K_{11}}{\partial x_o}\delta x_o + \frac{\partial K_{11}}{\partial y_o}\delta y_o + \frac{\partial K_{11}}{\partial \phi}\delta\phi$$

In Eqs. (4.9)-(4.16) all elements of $\underline{\mathbf{B}}$ were presented as functions of not $\underline{\mathbf{U}}$ but $\underline{\mathbf{U}}_p$ which is defined as

$$\underline{\mathbf{U}}_p = \left[k_1, k_2, k_3, l_{o1}, l_{o2}, l_{o3}, l_1, l_2, l_3, \theta_1, \theta_2, \theta_3\right]^T . \quad (4.32)$$

Hence among Eq. (4.31) $\frac{\partial K_{11}}{\partial x_o}$, $\frac{\partial K_{11}}{\partial y_o}$, and $\frac{\partial K_{11}}{\partial \phi}$ are not obtained from simple differentiation.

Since $\underline{\mathbf{B}}$ is a function of $\underline{\mathbf{U}}_p$, δB_1 can also be written as

$$\delta B_1 = \frac{d\,B_1}{d\,\underline{\mathbf{U}}_p}\delta\underline{\mathbf{U}}_p$$
$$= \frac{\partial K_{11}}{\partial k_1}\delta k_1 + \frac{\partial K_{11}}{\partial k_2}\delta k_2 + \frac{\partial K_{11}}{\partial k_3}\delta k_3 + \frac{\partial K_{11}}{\partial l_{o1}}\delta l_{o1} + \frac{\partial K_{11}}{\partial l_{o2}}\delta l_{o2} + \frac{\partial K_{11}}{\partial l_{o3}}\delta l_{o3}\,. \quad (4.33)$$
$$+ \frac{\partial K_{11}}{\partial l_1}\delta l_1 + \frac{\partial K_{11}}{\partial l_2}\delta l_2 + \frac{\partial K_{11}}{\partial l_3}\delta l_3 + \frac{\partial K_{11}}{\partial \theta_1}\delta\theta_1 + \frac{\partial K_{11}}{\partial \theta_2}\delta\theta_2 + \frac{\partial K_{11}}{\partial \theta_3}\delta\theta_3$$

In addition, δl_i 's and $\delta\theta_i$'s in Eq. (4.33) can be substituted with Eqs. (2.12) and (2.13) which is restated here as

$$\delta l_i = \underline{\$}_i^T \delta\underline{\mathbf{D}} = \underline{\$}_i^T \begin{bmatrix} \delta x_o \\ \delta y_o \\ \delta\phi \end{bmatrix} \quad (2.12)$$

$$\delta\theta_i = \frac{1}{l_i}\frac{\partial \underline{\$}_i^{'\,T}}{\partial\theta_i}\,\delta\underline{\mathbf{D}} = \frac{1}{l_i}\frac{\partial \underline{\$}_i^{'\,T}}{\partial\theta_i}\begin{bmatrix} \delta x_o \\ \delta y_o \\ \delta\phi \end{bmatrix}. \quad (2.13)$$

By the above substitution in Eq. (4.33) and collecting the coefficients according to each term of $\delta\underline{\mathbf{U}}$, δB_1 can be written as

$$\delta B_1 = \begin{bmatrix} \frac{\partial K_{11}}{\partial k_1} & \frac{\partial K_{11}}{\partial k_2} & \frac{\partial K_{11}}{\partial k_3} & \frac{\partial K_{11}}{\partial l_{o1}} & \frac{\partial K_{11}}{\partial l_{o2}} & \frac{\partial K_{11}}{\partial l_{o3}} & \frac{\partial K_{11}}{\partial x_o} & \frac{\partial K_{11}}{\partial y_o} & \frac{\partial K_{11}}{\partial \phi} \end{bmatrix} \begin{bmatrix} \delta k_1 \\ \delta k_2 \\ \delta k_3 \\ \delta l_{o1} \\ \delta l_{o2} \\ \delta l_{o3} \\ \delta x_o \\ \delta y_o \\ \delta \phi \end{bmatrix} . \quad (4.34)$$

$$= \frac{d\, B_1}{d\, \underline{\mathbf{U}}} \delta \underline{\mathbf{U}}$$

All terms of $\delta\underline{\mathbf{B}}$ may be obtained in the same way in which δB_1 was obtained and $\frac{d\,\mathbf{B}}{d\,\underline{\mathbf{U}}}$ can be derived by combining all $\frac{d\,B_i}{d\,\underline{\mathbf{U}}}$'s.

A numerical example is presented. The mechanism shown in Figure 4-1 is in static equilibrium under the external wrench $\underline{\mathbf{w}}_{ext}$ and the geometry information and the spring parameters are given below. The mechanism is assumed to have three compliant couplings.

Table 4-6. Positions of pivot points for numerical example 4.1.3.

Pivot points	E1	E2	E3	A1	A2	A3
X	0.0000	0.6000	2.5000	0.6000	1.4055	2.6736
Y	0.0000	0.8000	0.2000	4.5000	2.7447	3.3209

(Unit: cm)

Table 4-7. Initial spring parameters for numerical example 4.1.3.

Spring No.	1	2	3
Stiffness constant k	5.5	5.7	5.1
Free length l_o	4.8	3.1	2.0

(Unit: N/cm for k, cm for l_o)

The external wrench $\underline{\mathbf{w}}_{ext}$ and the initial stiffness matrix $[K]_I$ are calculated from the geometry of the mechanism and the spring parameters.

$$\underline{\mathbf{w}}_{ext} = [-2.0409\ N \quad -0.9263\ N \quad 12.8594\ Ncm]$$

$$[K]_I = \begin{bmatrix} 0.1679 \quad N/cm & 3.9107 \quad N/cm & 3.9623 \quad N \\ 3.9107 \quad N/cm & 14.9590 \quad N/cm & 10.9558 \quad N \\ 3.0360 \quad N & 12.9966 \quad N & 25.9764 \quad Ncm \end{bmatrix}.$$

The desired stiffness matrix $[K]_D$ is given below.

$$[K]_D = \begin{bmatrix} 0.6679 \quad N/cm & 4.3107 \quad N/cm & 4.1823 \quad N \\ 4.3107 \quad N/cm & 15.4290 \quad N/cm & 10.8358 \quad N \\ 3.2560 \quad N & 12.8766 \quad N & 26.3764 \quad Ncm \end{bmatrix}.$$

Since the difference between the desired stiffness matrix and the initial stiffness matrix is not small enough, the difference is divided into a number of small $\delta\underline{\mathbf{B}}$'s and Eq. (4.27) is applied repeatedly to obtain the spring parameters and the displacement of body B which implement the desired stiffness matrix and the given wrench.

The calculated spring parameters and the pose of body A are shown in Table 4-8 and Table 4-9 and the initial and final pose of the mechanism is shown in Figure 4-2.

Table 4-8. Calculated spring parameters for numerical example 4.1.3.

Spring No.	1	2	3
Stiffness constant k	6.2563	5.5311	5.1492
Free length l_o	5.4810	4.3584	3.1954

(Unit: N/cm for k, cm for l_o)

Table 4-9. Positions of pivot points in body A for numerical example 4.1.3.

Pivot points	A1	A2	A3
X	0.8201	1.9165	3.0661
Y	5.2909	3.7010	4.4874

(Unit: cm)

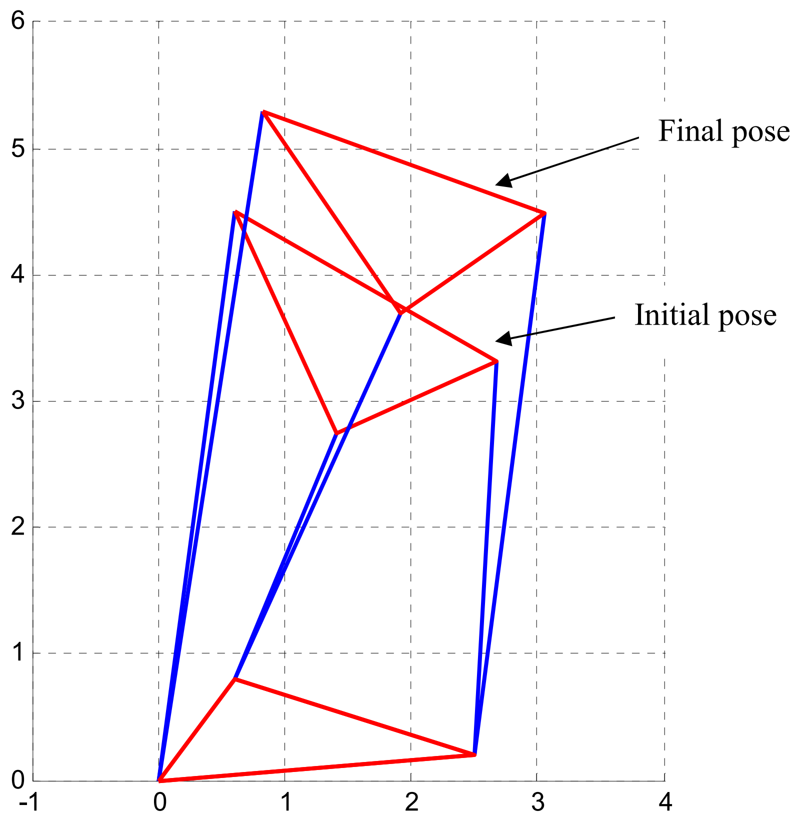

Figure 4-2. Poses of the compliant parallel mechanism for numerical example 4.1.3.

4.2 Variable Compliant Mechanisms with Two Parallel Mechanisms in Series

In this section mechanisms having two planar compliant parallel mechanisms that are serially arranged as shown in Figure 2-6 are investigated.

4.2.1 Constraints on Stiffness Matrix

The stiffness matrix of the mechanism was derived in Chapter two and restated as

$$[K]=[K_F]_{R,L}\left([K_F]_{R,L}+[K_F]_{R,U}-[K_M]_{R,U}\right)^{-1}[K_F]_{R,U}. \tag{2.54}$$

Applying the constraint presented by Ciblak and Lipkin (1994), $[K_F]_{R,L}$ and $[K_F]_{R,U}$ may be written as

$$[K_F]_{R,L}=\begin{bmatrix} K_{11}^L & K_{12}^L & K_{13}^L \\ K_{12}^L & K_{22}^L & K_{32}^L+f_x \\ K_{13}^L+f_y & K_{32}^L & K_{33}^L \end{bmatrix} \tag{4.35}$$

$$[K_F]_{R,U} = \begin{bmatrix} K_{11}^U & K_{12}^U & K_{13}^U \\ K_{12}^U & K_{22}^U & K_{32}^U + f_x \\ K_{13}^U + f_y & K_{32}^U & K_{33}^U \end{bmatrix} \tag{4.36}$$

where $\underline{\mathbf{w}}_{ext} = \left[f_x, f_y, m_z \right]^T$ is the external wrench. In addition, $[K_M]_{R,U}$ is a function of only the external wrench as shown in Eq. (2.46) which is restated as

$$[K_M]_{R,U} = \begin{bmatrix} 0 & 0 & -f_y \\ 0 & 0 & f_x \\ f_y & -f_x & 0 \end{bmatrix}. \tag{2.46}$$

Plugging in Eqs. (4.35), (4.36), and (2.46) into Eq. (2.54) and carrying out a symbolic operation using Maple software shows

$$[K]-[K]^T = \begin{bmatrix} 0 & 0 & -f_y \\ 0 & 0 & f_x \\ f_y & -f_x & 0 \end{bmatrix}.$$

which is the same with Ciblak and Lipkin (1994)'s statement for compliant parallel mechanisms in planar cases. This result indicates that mechanisms having two planar compliant parallel mechanisms in a serial arrangement also contain only six independent variables.

4.2.2 Stiffness Modulation by using a Derivative of Stiffness Matrix and Wrench

Since the stiffness matrix of the mechanism shown in Figure 2-6 is complicated and nonlinear in term of the spring parameters and the displacement of the rigid bodies, a derivative of the stiffness matrix and the static equilibrium equation is derived and applied for stiffness modulation of the compliant mechanism.

The stiffness matrix elements and the wrenches may be written in matrix form as

$$\underline{\mathbf{B}} = \left[K_{11}, K_{12}, K_{13}, K_{22}, K_{32}, K_{33}, f_x^A, f_y^A, m_z^A, f_x^B, f_y^B, m_z^B \right]^T \tag{4.37}$$

where $\underline{\mathbf{w}}^A = \left[f_x{}^A, f_y{}^A, m_z{}^A \right]^T$ and $\underline{\mathbf{w}}^B = \left[f_x{}^B, f_x{}^B, m_z{}^B \right]^T$ are the wrenches from the compliant couplings connecting body A to ground and from the couplings connecting body B to body A, respectively.

The spring parameters and the displacements of the rigid bodies may be written as

$$\underline{\mathbf{U}} = [k_1, k_2, k_3, k_4, k_5, k_6, l_{o1}, l_{o2}, l_{o3}, l_{o4}, l_{o5}, l_{o6}, x_o{}^A, y_o{}^A, \phi^A, x_o{}^B, y_o{}^B, \phi^B]^T \quad (4.38)$$

where k_i's and l_{oi}'s are the spring constant and free length of ith compliant coupling, respectively. In addition, $x_o{}^A$ and $y_o{}^A$ are the position of point O in body A which is coincident with the origin of the inertial frame E and ϕ^A is the rotation angle of body A with respect to ground. $x_o{}^B$, $y_o{}^B$, and ϕ^B are defined in the same way in terms of the inertial frame E.

In chapter two the stiffness matrix and the wrenches were presented as functions of $\underline{\mathbf{U}}_p$ which is defined as

$$\begin{aligned}\underline{\mathbf{U}}_p = [&k_1, k_2, k_3, k_4, k_5, k_6, l_{o1}, l_{o2}, l_{o3}, l_{o4}, l_{o5}, l_{o6}, l_1, l_2, l_3, l_4, l_5, l_6, \\ &\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, r_{x,1}, r_{x,2}, r_{x,3}, r_{y,1}, r_{y,2}, r_{y,3}]^T\end{aligned} \quad (4.39)$$

where l_i's and θ_i's are the current spring length and rising angle of i-th compliant coupling. In addition $r_{x,i}$ and $r_{y,i}$ are the pivot positions of i-th compliant coupling in body A.

A similar approach taken in the previous section is applied to get a derivative of the stiffness matrix and the wrenches $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$: each element of $\underline{\mathbf{B}}$ is differentiated with respect to $\underline{\mathbf{U}}_p$ and the terms not belonging to $\delta\underline{\mathbf{U}}$ are substituted in the terms of $\delta\underline{\mathbf{U}}$. In other words, δl_i's, $\delta\theta_i$'s, $\delta r_{x,i}$'s, and $\delta r_{y,i}$'s are expressed in terms of the twists of the bodies.

The coefficient of each term of $\delta\underline{\mathbf{U}}$ corresponds to an element of the derivative matrix in an analogous way to Eq. (4.34).

Since springs 4, 5, and 6 connect body A and ground, δl_i and $\delta\theta_i$ for i=4, 5, 6 can be written as Eqs. (2.12) and (2.13) which are

$$\delta l_i = \underline{\mathbf{\$}}_i^{T}\, {}^{E}\delta\underline{\mathbf{D}}^{A} = \underline{\mathbf{\$}}_i^{T} \begin{bmatrix} \delta x_o^{\ A} \\ \delta y_o^{\ A} \\ \delta\phi^{A} \end{bmatrix} \tag{2.12}$$

$$\delta\theta_i = \frac{1}{l_i}\frac{\partial \underline{\mathbf{\$}}_i^{'}}{\partial\theta_i}^{T}\, {}^{E}\delta\underline{\mathbf{D}}^{A} = \frac{1}{l_i}\frac{\partial \underline{\mathbf{\$}}_i^{'}}{\partial\theta_i}^{T} \begin{bmatrix} \delta x_o^{\ A} \\ \delta y_o^{\ A} \\ \delta\phi^{A} \end{bmatrix}. \tag{2.13}$$

Springs 1, 2, and 3 join body B and body A and thus δl_i and $\delta\theta_i$ for i=1, 2, 3 may be expressed as

$$\delta l_i = \underline{\mathbf{\$}}_i^{T}\, {}^{A}\delta\underline{\mathbf{D}}^{B} = \underline{\mathbf{\$}}_i^{T} \begin{bmatrix} \delta x_o^{\ B} - \delta x_o^{\ A} \\ \delta y_o^{\ B} - \delta y_o^{\ A} \\ \delta\phi^{B} - \delta\phi^{A} \end{bmatrix} \tag{4.40}$$

$$\delta\theta_i = \frac{1}{l_i}\frac{\partial \underline{\mathbf{\$}}_i^{'}}{\partial\theta_i}^{T}\, {}^{A}\delta\underline{\mathbf{D}}^{B} + \delta\phi^{A} = \frac{1}{l_i}\frac{\partial \underline{\mathbf{\$}}_i^{'}}{\partial\theta_i}^{T} \begin{bmatrix} \delta x_o^{\ B} - \delta x_o^{\ A} \\ \delta y_o^{\ B} - \delta y_o^{\ A} \\ \delta\phi^{B} - \delta\phi^{A} \end{bmatrix} + \delta\phi^{A}. \tag{4.41}$$

Lastly $\delta r_{x,i}$'s, and $\delta r_{y,i}$'s are the positions of the pivot point in body A and by using the twist equation it can be written as

$$\begin{bmatrix} \delta r_{x,i} \\ \delta r_{y,i} \\ 0 \end{bmatrix} = \begin{bmatrix} \delta x_o^{\ A} \\ \delta y_o^{\ A} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta\phi^{A} \end{bmatrix} \times \begin{bmatrix} r_{x,i} \\ r_{y,i} \\ 0 \end{bmatrix}. \tag{4.42}$$

Now all terms in the differential of $\underline{\mathbf{B}}$ are expressed in terms of $\delta\underline{\mathbf{U}}$ and by writing it in matrix form gives

$$\delta\underline{\mathbf{B}} = \frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}\delta\underline{\mathbf{U}} \tag{4.43}$$

where $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ is 12 × 18 matrix.

It is required to obtain the small change of input values $\delta\underline{\mathbf{U}}$ corresponding to a small change of stiffness matrix and wrenches $\delta\underline{\mathbf{B}}$ and since the number of columns of the matrix is greater than that of rows it is a redundant system. There are in general an infinite number of solutions and a variety of constraints may be imposed on the system.

Since $\delta\underline{\mathbf{U}}$ is the change from the current values, minimizing the norm of $\delta\underline{\mathbf{U}}$ may be one of reasonable options. Then in a similar way to Eq. (4.22) $\delta\underline{\mathbf{U}}_{\min}$ may be obtained as

$$\delta\underline{\mathbf{U}}_{\min} = \delta\underline{\mathbf{U}}_{p.sol} + \left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]\left(\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]^T\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]\right)^{-1}\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]^T\left(-\delta\underline{\mathbf{U}}_{p.sol}\right) \tag{4.44}$$

where $\delta\underline{\mathbf{U}}_{p.sol}$ is a particular solution of Eq. (4.43) and $\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]$ is the null space of matrix $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ (see Strang 1988).

Body B is considered to be in contact with the environment and it may be required to preserve the pose of body B. It indicates that the twist of body B is equal to zero and can be written as

$$\delta x_o{}^B = 0,\ \delta y_o{}^B = 0,\ \delta\theta^B = 0.$$

We can implement it by removing the last three columns of $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ and the last three rows of $\delta\underline{\mathbf{U}}$ and solving the problems in a similar way to that of the previous problem since the system is still redundant.

If both of the bodies are required to be stationary then the twists of the bodies should be zero and it may be written as

$$\delta x_o{}^A = 0,\ \delta y_o{}^A = 0,\ \delta\theta^A = 0,$$

$$\delta x_o{}^B = 0,\ \delta y_o{}^B = 0,\ \delta\theta^B = 0.$$

This can be implemented by removing the last six columns of $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ and the last six rows of $\delta\underline{\mathbf{U}}$, and by solving the problem which is not redundant.

4.2.3 Numerical Example

The geometry information and spring parameters of the mechanism shown in Figure 2-6 and the external wrench $\underline{\mathbf{w}}_{ext}$ are given below.

$$\underline{\mathbf{w}}_{ext} = [-1.7\,N \quad 2.5\,N \quad 12.7\,N]$$

Table 4-10. Spring parameters of the compliant couplings for numerical example 4.2.3.

Spring No.	1	2	3	4	5	6
Stiffness constant k	5.0	5.0	5.0	5.0	5.0	5.0
Free length l_o	3.0614	0.6791	2.3608	2.8657	0.7258	1.2732

(Unit: N/cm for k, cm for l_o)

Table 4-11. Positions of pivot points for numerical example 4.2.3.

Pivot points	E1	E2	E3	B1	B2	B3
X	0.0000	1.0000	3.0000	-0.8000	0.4453	1.1965
Y	0.0000	0.8000	0.0000	4.5000	3.7726	4.6186

(Unit: cm)

Table 4-11. Continued.

A1	A2	A3	A4
0.2000	1.1261	2.1646	1.2760
2.3000	1.8179	1.9252	2.6038

The initial stiffness matrix $[K]_I$ is calculated from the geometry of the mechanism and the spring parameters.

$$[K]_I = \begin{bmatrix} 1.5992 \quad N/cm & -1.1571 \quad N/cm & -6.0650 \quad N \\ -1.1571 \quad N/cm & 5.7047 \quad N/cm & 7.7521 \quad N \\ -3.5650 \quad N & 9.4521 \quad N & 22.6794 \quad Ncm \end{bmatrix}$$

The desired stiffness matrix $[K]_D$ is given below.

$$[K]_D = \begin{bmatrix} 1.6442 \quad N/cm & -1.1921 \quad N/cm & -6.0230 \quad N \\ -1.1921 \quad N/cm & 5.7417 \quad N/cm & 7.7931 \quad N \\ -3.5230 \quad N & 9.4931 \quad N & 22.6384 \quad Ncm \end{bmatrix}$$

Since the difference between the desired stiffness matrix and the initial stiffness matrix is not small enough, the difference is divided into a number of small δB's and the problem is solved repeatedly to obtain the spring parameters and the displacements of body B and body A which implement the desired stiffness matrix. Three sets of spring parameters are obtained: one with no constraint on the displacements of body A and body B, another with body A fixed, and the other with body A and body B fixed. The calculated spring parameters are presented in Table 4-12, Table 4-13, and Table 4-14, respectively. In addition, the initial and final poses of the mechanism are shown in Figures 4-3 and 4-4.

Table 4-12. Spring parameters with no constraint for numerical example 4.2.3

Spring No.	1	2	3	4	5	6
Stiffness constant k	5.5786	4.8852	5.5513	5.1865	5.1506	5.1505
Free length l_o	2.5502	0.5084	1.9188	2.7939	0.8039	1.3746

(Unit: N/cm for k, cm for l_o)

Table 4-13. Spring parameters with body B fixed for numerical example 4.2.3

Spring No.	1	2	3	4	5	6
Stiffness constant k	6.3556	4.4317	5.9605	5.3306	5.5194	4.4334
Free length l_o	3.0234	0.6588	2.6016	2.8192	0.6220	1.0270

(Unit: N/cm for k, cm for l_o)

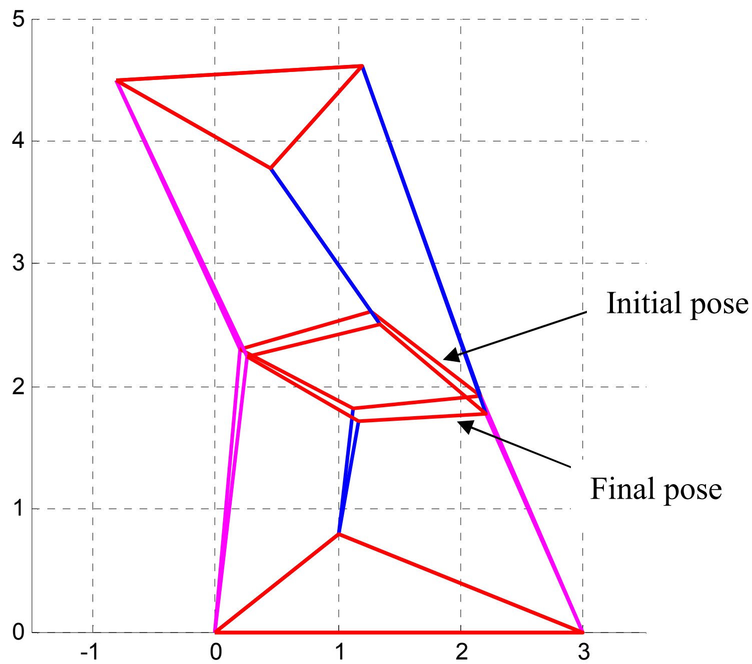

Figure 4-3. Poses of the compliant mechanism with body B fixed.

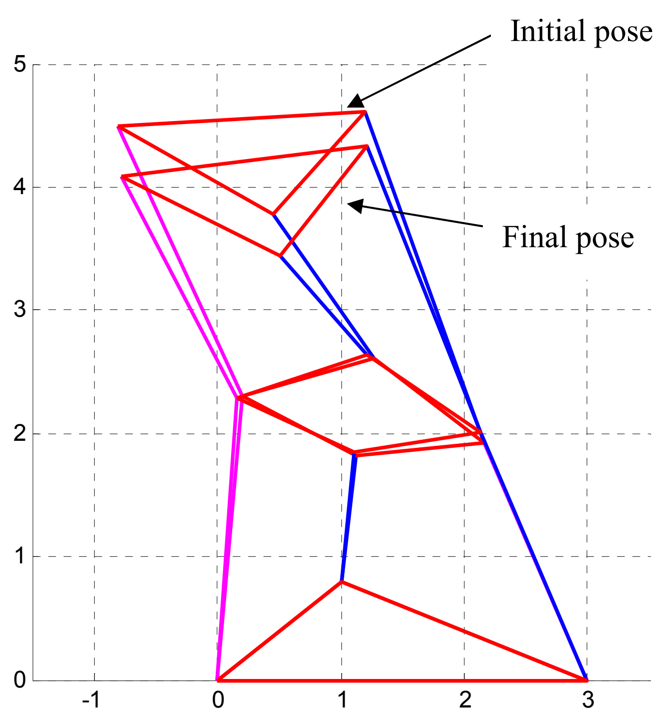

Figure 4-4. Poses of the compliant mechanism with no constraint.

Table 4-14. Spring parameters with body A and body B fixed for numerical example 4.2.3.

Spring No.	1	2	3	4	5	6
Stiffness constant k	12.3070	10.5719	6.9847	1.9411	4.9528	3.8741
Free length l_o	2.6786	1.0769	2.5033	3.7435	0.7229	1.0333

(Unit: N/cm for k, cm for l_o)

The results indicate that there are greater changes of the spring parameters with more constraints imposed on the bodies. These control methods all require the inverse of $\frac{d\underline{\mathbf{B}}}{d\underline{\mathbf{U}}}$ or $\left[\frac{d\underline{\mathbf{B}}}{d\underline{\mathbf{U}}_{Null}}\right]^T\left[\frac{d\underline{\mathbf{B}}}{d\underline{\mathbf{U}}_{Null}}\right]$ depending on the constraint and it may cause a singularity problem. With more constraints the mechanism is more vulnerable to the singularity problem.

CHAPTER 5
CONCLUSIONS

The compliance of mechanisms containing rigid bodies which are connected to each other by line springs was studied. A derivative of the planar spring wrench connecting two moving bodies was obtained and then, through a similar approach, a derivative of the spatial spring wrench which is more general was obtained. It is obvious from Eq. (3.36) that the derivative of the spring wrench joining two rigid bodies depends not only on a relative twist between two bodies but also on the twist of the intermediate body in terms of the inertial frame unless the initial external wrench is zero.

The derivative of the spring wrench was applied to obtain the resultant stiffness matrix of two compliant parallel mechanisms in a serial arrangement. The resultant stiffness matrix indicates that the resultant compliance, which is the inverse of the resultant stiffness matrix, is not the summation of the compliances of the constituent mechanisms unless the external wrench applied to the mechanism is zero which was generally accepted. The derivative of the spring wrench was also applied to acquire the stiffness matrix of planar springs in a hybrid arrangement and may be applied for mechanisms having an arbitrary number of parallel mechanisms in a serial arrangement.

Planar mechanisms with variable compliance were investigated with the knowledge of the stiffness model obtained in the research. Adjustable line springs which can change their spring constants and free lengths were employed to connect rigid bodies in the mechanisms. Ciblak and Lipkin (1994) showed that the stiffness matrix of compliant parallel mechanisms can be decomposed into a symmetric and a skew symmetric part and that the skew symmetric part is negative one-half the externally applied load expressed as a spatial cross product operator. It was shown through this research that the same statement is valid for the stiffness matrix of a mechanism having two compliant parallel mechanisms in a serial arrangement. In other words there are only six rather than nine independent variables in the stiffness matrix of planar compliant parallel mechanisms and in the stiffness matrix of a mechanism having two planar compliant parallel mechanisms in a serial arrangement.

Derivatives of the stiffness matrices of planar compliant mechanisms with respect to the spring parameters and the twists of the constituent rigid bodies were obtained. It was shown that these derivatives may be utilized to control and regulate the stiffness matrix and the pose of the mechanism respectively at the same time by adjusting the spring parameters of each constituent coupling with or without the change of the position of the robot where the compliant mechanism is attached.

Several future works are presented.

The singularity conditions associated with the resultant stiffness matrix of compliant parallel mechanisms in a serial arrangement needs to be studied. This study will identify under what condition the mechanism collapses even with a small change in the applied wrench.

It was required to solve a redundant system of linear equations to obtain the changes of the spring parameters and the twists of the bodies corresponding to a small change of the stiffness matrix and the resultant wrench of a compliant mechanism. There are in general an infinite number of solutions and the least square solution was chosen in this research. A better solution may be selected by considering the following considerations:

1. The inverse of matrix $\frac{d\underline{\underline{\mathbf{B}}}}{d\underline{\mathbf{U}}}$ or $\frac{d\underline{\underline{\mathbf{B}}}}{d\underline{\mathbf{U}}}^T \frac{d\underline{\underline{\mathbf{B}}}}{d\underline{\mathbf{U}}}$ as of Eq. (4.43) is required to solve the linear equations. Solutions close to singular cases of $\frac{d\underline{\underline{\mathbf{B}}}}{d\underline{\mathbf{U}}}$ or $\frac{d\underline{\underline{\mathbf{B}}}}{d\underline{\mathbf{U}}}^T \frac{d\underline{\underline{\mathbf{B}}}}{d\underline{\mathbf{U}}}$ should be avoided.
2. The operation ranges of the spring parameters should be taken into account.
3. Proper consideration should be given to singularities of the compliant mechanism.

APPENDIX A
MATLAB CODES FOR NUMERICAL EXAMPLES IN CHAPTER TWO AND THREE

Matlab codes for the numerical examples in chapter two and three are presented in this appendix. Table A-1 contains the function list and functions NuEx_21, NuEx_22, and NuEx_31 are main functions, and other functions are called inside these main functions. For instance, to get the result of numerical example 2-1, it is needed to call function NuEx_21 and other functions are placed in the same folder where function NuEx_21 is located.

Table A-1. Matlab function list.

Function Name	Description
NuEx_21	Main function for numerical example 2-1
NuEx_22	Main function for numerical example 2-2
NuEx_31	Main function for numerical example 2-3
StaticEq21	Static equilibrium equation for numerical example 2-1
StaticEq22	Static equilibrium equation for numerical example 2-2
StiffMatrix	Computes matrix [K_F]
GetKM	Computes matrix [K_M]
SpringWrench	Computes spring wrench
GetPLine	Computes Plucker line coordinates
GetOriginVel	Computes origin velocity from velocity of a point and angular velocity of a rigid body
GetVelP2D	Computes displacement of a point from twist
GetGlobalPos2D	Computes position of a point in terms of inertial frame in planar space
GetGlobalPos3D	Computes position of a point in terms of inertial frame in spatial space

function NuEx_21

```
% Numerical example 2-1
% Test Stiffness Matrix of planar parallel mechanisms in series
```

```
%==============================================================
%======    GIVEN VALUES
%==============================================================
%Wext : External wrench
%A : Coordinates of fixed pivot points
%lb: Local coordinates of vertices of middle platform
%    the first 3 columns are coordinates of pivot points connected to lower
%    parts and the second 3 columns for upper parts
%lt: Local coordinates of vertices of top triangle
%lo : Free lengths of springs
%k : Spring constants
%==============================================================

global Wext A lb lt lo k

%External wrench
Weq=[0.01, -0.02, 0.03]';

%Coordinates of fixed points
A = [ 0.0, 1.5, 3.0
     0.0, 1.2, 0.5 ];

%Local coordinates of vertices of triangles
lb = [0,  1.0,    2.0, 0.0,  1.0, 2.0
     0, -1.7321, 0.0, 0.0,  0.5, 0.0];

lt = [0,  1.0,    2.0
     0, -1.7321, 0.0];

%Free lengths of springs
lo = [5.0040, 2.2860, 4.9458, 5.5145, 3.1573, 5.2568]';

%Spring constants
k = [0.2, 0.3, 0.4, 0.5, 0.6, 0.7]';

%Initial guess of positions and orientatins of the local coordinate systems
Xo=zeros(6,1);
Bo=[0.2, 5.0, 10.8*pi/180]';
To=[-0.2, 10.5, 3.4*pi/180]';
Xo=[Bo;To];

%==============================================================
% Find Xo
%==============================================================
options=optimset('fsolve');
optionsnew=optimset(options, 'MaxFunEvals',1256);   % Option to display output
```

```
Wext=Weq;
[X error] = fsolve(@StaticEq_21, Xo, optionsnew);
Xo=X;
%Calculate the position of vertices of triangles
Bo = X(1:3);
To = X(4:6);
B = GetGlobalPos2D(Bo, lb);
T = GetGlobalPos2D(To, lt);

%Calculate initial wrenches
W_L_I = zeros(3,1);
W_L_I = W_L_I + SpringWrench(A(:,1), B(:,1), k(1), lo(1));
W_L_I = W_L_I + SpringWrench(A(:,2), B(:,2), k(2), lo(2));
W_L_I = W_L_I + SpringWrench(A(:,3), B(:,3), k(3), lo(3));

W_U_I = zeros(3,1);
W_U_I = W_U_I + SpringWrench(B(:,4), T(:,1), k(4), lo(4));
W_U_I = W_U_I + SpringWrench(B(:,5), T(:,2), k(5), lo(5));
W_U_I = W_U_I + SpringWrench(B(:,6), T(:,3), k(6), lo(6));

%=====================================================================
% Calculate stiffness matrix
%=====================================================================
k1=StiffMatrix(A(:,1), B(:,1), k(1), lo(1));
k2=StiffMatrix(A(:,2), B(:,2), k(2), lo(2));
k3=StiffMatrix(A(:,3), B(:,3), k(3), lo(3));
KF_L = k1+k2+k3;

k4=StiffMatrix(B(:,4), T(:,1), k(4), lo(4));
k5=StiffMatrix(B(:,5), T(:,2), k(5), lo(5));
k6=StiffMatrix(B(:,6), T(:,3), k(6), lo(6));
KF_U = k4+k5+k6;

k4_2 = GetKM(B(:,4), T(:,1), k(4), lo(4));
k5_2 = GetKM(B(:,5), T(:,2), k(5), lo(5));
k6_2 = GetKM(B(:,6), T(:,3), k(6), lo(6));
KM_U = k4_2+k5_2+k6_2;

K1 = KF_L*inv(KF_L+KF_U-KM_U)*KF_U;
K2 = KF_L*inv(KF_L+KF_U)*KF_U;
%=====================================================================

%=====================================================================
% Verify the results
%=====================================================================
dW = [0.000005,0.000002,0.000004]';
```

```
% 1.Calculate twists
D_EB_1 = inv(K1)*dW;
D_EB_2 = inv(K2)*dW;
D_EA = inv(KF_L)*dW;

% 2.Get the position of the bodies
Bo_W = GetVelP2D(D_EA, Bo(1:2))+ Bo(1:2);
Bo_W(3)= Bo(3)+D_EA(3);
B_F = GetGlobalPos2D(Bo_W, lb);

To_W_1 = GetVelP2D(D_EB_1, To(1:2))+ To(1:2);
To_W_1(3)= To(3)+D_EB_1(3);
T_F_1 = GetGlobalPos2D(To_W_1, lt);

To_W_2 = GetVelP2D(D_EB_2, To(1:2))+ To(1:2);
To_W_2(3)= To(3)+D_EB_2(3);
T_F_2 = GetGlobalPos2D(To_W_2, lt);

% 3.Calculate wrenches
W_L_F = zeros(3,1);
W_L_F = W_L_F + SpringWrench(A(:,1), B_F(:,1), k(1), lo(1));
W_L_F = W_L_F + SpringWrench(A(:,2), B_F(:,2), k(2), lo(2));
W_L_F = W_L_F + SpringWrench(A(:,3), B_F(:,3), k(3), lo(3));
dW_L_F = W_L_F - Weq;

W_U_F_1 = zeros(3,1);
W_U_F_1 = W_U_F_1 + SpringWrench(B_F(:,4), T_F_1(:,1), k(4), lo(4));
W_U_F_1 = W_U_F_1 + SpringWrench(B_F(:,5), T_F_1(:,2), k(5), lo(5));
W_U_F_1 = W_U_F_1 + SpringWrench(B_F(:,6), T_F_1(:,3), k(6), lo(6));
dW_U_F_1 = W_U_F_1 - Weq;

W_U_F_2 = zeros(3,1);
W_U_F_2 = W_U_F_2 + SpringWrench(B_F(:,4), T_F_2(:,1), k(4), lo(4));
W_U_F_2 = W_U_F_2 + SpringWrench(B_F(:,5), T_F_2(:,2), k(5), lo(5));
W_U_F_2 = W_U_F_2 + SpringWrench(B_F(:,6), T_F_2(:,3), k(6), lo(6));
dW_U_F_2 = W_U_F_2 - Weq;
%=====================================================================

%=====================================================================
% Display the result
%=====================================================================
K1, K2,  D_EB_1, D_EB_2, D_EA, dW, dW_L_F, dW_U_F_1, dW_U_F_2
%=====================================================================
```

function NuEx_22

```
% Numerical Example 2-2
% Test Stiffness Matrix of planar parallel mechanisms in hybird

%===========================================================
%======    GIVEN VALUES
%===========================================================
%Wext : External wrench
%A : Coordinates of fixed pivot points
%lb, lc, ld : Local coordinates of vertices of intermediate triangle
%lt: Local coordinates of vertices of top triangle
%lo : Free lengths of springs
%k : Spring constants
%===========================================================

global Wext A lb lc ld lt lo k

%External wrench
Weq=[0.1, 0.1, 0.2]';

%Coordinates of fixed points
A = [ 1.67,   4.46, 13.3449, 14.6731, 8.23, 4.94
    4.4333, 1.3964, 3.25, 6.84, 14.1400, 13.4943 ];

%Local coordinates of vertices of triangles
lb = [0, 2, 1
    0, 0, 1.7321];
lc = lb;
ld = lb;
lt = lb;

%Spring constants
k = [0.40       0.43    0.46    0.49    0.52    0.55   0.58   0.61   0.64];

%Spring free lengths
lo = [2.2547,2.4014,2.3924,1.5910,1.8450,2.2200,1.7077,2.2695,1.8711]';

%Initial guess of positions and orientatins of the local coordinate systems
Xo=zeros(12,1);
Bo=[3.8, 4.8,-39.8*pi/180]';
Co=[12.4, 4.5,76.5*pi/180]';
Do=[8.1, 12.4,202.4*pi/180]';
To=[7.0, 7.5,-23.4*pi/180]';
Xo=[Bo;Co;Do;To];
```

```
%=======================================================================
% Find Xo
%=======================================================================
Wext = Weq;
options=optimset('fsolve');
optionsnew=optimset(options, 'MaxFunEvals',1256);   % Option to display output
[X error] = fsolve(@StaticEq_22, Xo, optionsnew);
Xo=X;
%Calculate the position of vertices of triangles
Bo = X(1:3)
Co = X(4:6)
Do = X(7:9)
To = X(10:12)

B = GetGlobalPos2D(Bo, lb);
C = GetGlobalPos2D(Co, lc);
D = GetGlobalPos2D(Do, ld);
T = GetGlobalPos2D(To, lt);

%=======================================================================
%Calculate stiffness matrix using stiffness equation
%=======================================================================
k1=StiffMatrix(A(:,1), B(:,1), k(1), lo(1));
k2=StiffMatrix(A(:,2), B(:,2), k(2), lo(2));
k3=StiffMatrix(B(:,3), T(:,1), k(3), lo(3));
k3_2 = GetKM(B(:,3), T(:,1), k(3), lo(3));
KK1_1 = (k1+k2)*inv(k1+k2+k3-k3_2)*k3;
KK1_2 = (k1+k2)*inv(k1+k2+k3)*k3;

k4=StiffMatrix(A(:,3), C(:,1), k(4), lo(4));
k5=StiffMatrix(A(:,4), C(:,2), k(5), lo(5));
k6=StiffMatrix(C(:,3), T(:,2), k(6), lo(6));
k6_2 = GetKM(C(:,3), T(:,2), k(6), lo(6));
KK2_1 = (k4+k5)*inv(k4+k5+k6-k6_2)*k6;
KK2_2 = (k4+k5)*inv(k4+k5+k6)*k6;

k7=StiffMatrix(A(:,5), D(:,1), k(7), lo(7));
k8=StiffMatrix(A(:,6), D(:,2), k(8), lo(8));
k9=StiffMatrix(D(:,3), T(:,3), k(9), lo(9));
k9_2 = GetKM(D(:,3), T(:,3), k(9), lo(9));
KK3_1 = (k7+k8)*inv(k7+k8+k9-k9_2)*k9;
KK3_2 = (k7+k8)*inv(k7+k8+k9)*k9;

K1 = KK1_1+KK2_1+KK3_1;
K2 = KK1_2+KK2_2+KK3_2;
%=======================================================================
```

```
%==============================================================
% Verify the result
%==============================================================
dD = zeros(3,1);
dW = [0.00005,0.00002,0.00003]';
Wext=Weq+dW;
[X error] = fsolve(@StaticEq_22, Xo, optionsnew);
dDP = X(10:12)-To;
dD(1:2) = GetOriginVel(dDP, To(1:2));
dD(3)=dDP(3);

K1_dD = K1*dD;
K2_dD = K2*dD;
%==============================================================

%==============================================================
% Display the result
%==============================================================
K1, K2, dW, dD, K1_dD, K2_dD
%==============================================================
```

function NuEx31

```
% Numerical example 3.1
% Test Stiffness Matrix of spatial parallel mechanisms in serial

%==============================================================
%======     GIVEN VALUES
%==============================================================
%Wext : External wrench
%A : Coordinates of fixed pivot points
%lb_L, lb_U : Local coordinates of vertices of middle platform
%lt: Local coordinates of vertices of top triangle
%lo_U, lo_L : Free lengths of springs
%k_U, k_L : Spring constants
%==============================================================

%External wrench
Weq=[ -0.3, 0.4, 0.8, -2.3, -1.3, 0.7 ]';

%Coordinates of fixed points
A = [ 0.0, 1.3, 0.6, -0.7, -1.1, -0.5
      0.0, 1.1, 2.7,  2.6,  1.8,  0.4
      0.0, 0.2, 0.1, -0.1,  0.3,  0.1 ];
```

```
%Local coordinates of vertices of triangles
lb_L = [ 0.0, 1.0, 0.3, -0.8, -1.3,  -0.4
        0.0, 0.9, 2.3,  2.2,  1.4,   0.5
        0.0, 0.1, 0.2, -0.1,  -0.2, -0.1 ];

lb_U = [ 0.0, 1.3, 0.6, -0.7, -1.1,  -0.5
        0.0, 1.1, 2.7,  2.6,  1.8,   0.4
        0.1, 0.2, 0.25, 0.1,  0.12,  0.15 ];

lt = [ 0.0, 1.2, 0.5, -0.6, -1.0, -0.3
      0.0, 1.2, 2.3,  2.4,  1.3,  0.5
      0.0, 0.1, -0.1, 0.1,  0.1,  0.2 ];

%Spring constants
k_U = [4.6, 4.7, 4.5, 4.4, 5.3, 5.5]';
k_L = [4.4, 4.9, 4.7, 4.5, 5.1, 4.8]';

%Positions and orientatins of the local coordinate systems
%Rotation angles are Euler angles (3-2-1)
Xo=zeros(12,1);
B_Po=[0.2, 1.2, 3.2]';
B_R=[1.2*pi/180, 5.0*pi/180, -1.8*pi/180]';

T_Po=[-0.3, 1.6, 5.5]';
T_R=[-0.4*pi/180, 8.5*pi/180, 3.8*pi/180]';

Xo=[B_Po;B_R;T_Po;T_R];
Xo_I = Xo;

%Convert local coord. to global coord.
B_L = GetGlobalPos3D(B_Po, B_R, lb_L)
B_U = GetGlobalPos3D(B_Po, B_R, lb_U)
T = GetGlobalPos3D(T_Po, T_R, lt)

%==============================================================
% Plucker line coordinates(or Jacobian) and lengths of all springs
%==============================================================
JS_U=zeros(6,6);
JS_L=zeros(6,6);

l_U=zeros(6,1);
l_L=zeros(6,1);
lo_U=zeros(6,1);
lo_L=zeros(6,1);
F_U=zeros(6,1);
F_L=zeros(6,1);
```

```
for i=1:6
   [JS_U(:,i), l_U(i)] = GetPLine(B_U(:,i), T(:,i));
   [JS_L(:,i), l_L(i)] = GetPLine(A(:,i), B_L(:,i));
end

%==============================================================
% spring forces
F_U = inv(JS_U)*Weq;
F_L = inv(JS_L)*Weq;

%==============================================================
% spring free lengths for static equilibrium of the platform
for i=1:6
   lo_U(i)=l_U(i)-F_U(i)/k_U(i);
   lo_L(i)=l_L(i)-F_L(i)/k_L(i);
end

lo_I = [lo_U; lo_L]

%==============================================================
% Check wrench
%==============================================================
W_L_I = zeros(6,1);
W_U_I = zeros(6,1);
for i=1:6
   W_L_I = W_L_I + SpringWrench(A(:,i), B_L(:,i), k_L(i), lo_L(i));
   W_U_I = W_U_I + SpringWrench(B_U(:,i), T(:,i), k_U(i), lo_U(i));
end
%==============================================================

%==============================================================
%Calculate stiffness matrix using stiffness equation
%==============================================================
KF_U = zeros(6,6);
KF_L = zeros(6,6);
KM_U = zeros(6,6);

for i=1:6
   KF_U = KF_U+StiffMatrix(B_U(:,i), T(:,i), k_U(i), lo_U(i));
   KF_L = KF_L+StiffMatrix(A(:,i), B_L(:,i), k_L(i), lo_L(i));
   KM_U = KM_U+GetKM(B_U(:,i), T(:,i), k_U(i), lo_U(i));
end

K1 = KF_L*inv(KF_L+KF_U-KM_U)*KF_U;
K2 = KF_L*inv(KF_L+KF_U)*KF_U;
```

```
%==========================================================

%==========================================================
% Check the wrench
%==========================================================
dW = 10^(-5)*[5,-2,4,3,-8,4]';

% 1.Calculate twists
D_ET_1 = inv(K1)*dW;
D_ET_2 = inv(K2)*dW;
D_EB = inv(KF_L)*dW;

% 2.Get the position of the bodies
for i=1:6
   B_L_1(:,i) = B_L(:,i)+ D_EB(1:3) + cross(D_EB(4:6), B_L(:,i));
   B_U_1(:,i) = B_U(:,i)+ D_EB(1:3) + cross(D_EB(4:6), B_U(:,i));
   T_1(:,i) = T(:,i)+ D_ET_1(1:3) + cross(D_ET_1(4:6), T(:,i));
   T_2(:,i) = T(:,i)+ D_ET_2(1:3) + cross(D_ET_2(4:6), T(:,i));
end

% 3.Calculate wrenches
W_L_F = zeros(6,1);
W_U_F_1 = zeros(6,1);
W_U_F_2 = zeros(6,1);
for i=1:6
   W_L_F = W_L_F + SpringWrench(A(:,i), B_L_1(:,i), k_L(i), lo_L(i));
   W_U_F_1 = W_U_F_1 + SpringWrench(B_U_1(:,i), T_1(:,i), k_U(i), lo_U(i));
   W_U_F_2 = W_U_F_2 + SpringWrench(B_U_1(:,i), T_2(:,i), k_U(i), lo_U(i));
end

dW_L = W_L_F - Weq;
dW_U_1 = W_U_F_1 - Weq;
dW_U_2 = W_U_F_2 - Weq;
%==========================================================

%==========================================================
% Display the result
%==========================================================
K1, K2, D_EB, D_ET_1, D_ET_2, dW, dW_L, dW_U_1, dW_U_2
%==========================================================
```

function f = StaticEq21(x)

```
%==========================================================
% Global variables
```

```
%=====================================================================
%Wext : External wrench
%A : Coordinates of fixed pivot points
%lb,lt: Local coordinates of vertices of triangles
%lo : Free lengths of springs
%k : Spring constants
%=====================================================================
global Wext A lb lt lo k

%Bo,Co,Do,To : Postions and Orientations of local coordinate systems
Bo = x(1:3);
To = x(4:6);

%Get the position of pivot points
B = GetGlobalPos2D(Bo, lb);
T = GetGlobalPos2D(To, lt);

%Resultant wrench for the middle platform
RW_1 =zeros(3,1);
for i=1:3
   RW_1 = RW_1 + SpringWrench(A(:,i), B(:,i), k(i), lo(i));
end

%Resultant wrench for the middle platform
RW_2 =zeros(3,1);
for i=1:3
   RW_2 = RW_2 + SpringWrench(B(:,i+3), T(:,i), k(i+3), lo(i+3));
end

f = [ RW_1-RW_2; RW_2-Wext ]';
```

function f = StaticEq_22(x)

```
%=====================================================================
% Global variables
%=====================================================================
%Wext : External wrench
%A : Coordinates of fixed pivot points
%lb,lc,ld,lt : Local coordinates of vertices of triangles
%lo : Free lengths of springs
%k : Spring constants
%=====================================================================
global Wext A lb lc ld lt lo k

%Bo,Co,Do,To : Postions and Orientations of local coordinate systems
Bo = x(1:3);
```

```
Co = x(4:6);
Do = x(7:9);
To = x(10:12);

B = GetGlobalPos2D(Bo, lb);
C = GetGlobalPos2D(Co, lc);
D = GetGlobalPos2D(Do, ld);
T = GetGlobalPos2D(To, lt);

%spring forces
fA1B1 = SpringWrench(A(:,1), B(:,1), k(1), lo(1));
fA2B2 = SpringWrench(A(:,2), B(:,2), k(2), lo(2));
fB3T1 = SpringWrench(B(:,3), T(:,1), k(3), lo(3));

fA3C1 = SpringWrench(A(:,3), C(:,1), k(4), lo(4));
fA4C2 = SpringWrench(A(:,4), C(:,2), k(5), lo(5));
fC3T2 = SpringWrench(C(:,3), T(:,2), k(6), lo(6));

fA5D1 = SpringWrench(A(:,5), D(:,1), k(7), lo(7));
fA6D2 = SpringWrench(A(:,6), D(:,2), k(8), lo(8));
fD3T3 = SpringWrench(D(:,3), T(:,3), k(9), lo(9));

RW_1 = fA1B1+fA2B2-fB3T1;
RW_2 = fA3C1+fA4C2-fC3T2;
RW_3 = fA5D1+fA6D2-fD3T3;
RW_4 = fB3T1+fC3T2+fD3T3-Wext;

f = [ RW_1; RW_2; RW_3; RW_4 ];
```

function K = StiffMatrix(aa, bb, k, lo)

```
%Calculate Stiffness matrix for 2D and 3D
%Check whether it is planar or spatial case and make it spatial
a=zeros(3,1);
b=zeros(3,1);
if(size(aa,1)==2)
   a(1:2)=aa;
   b(1:2)=bb;
else
   a=aa;
   b=bb;
end

N = b - a; %non-unitized directional vector
l = norm(N,2); %length
rho = lo/l;
```

```
S = N/l; %unit direction vector
So = cross(a, S);
w = [S;So];
K1 = k*(w*w');

alpha = atan2(S(2),S(1));
beta = atan2(sqrt(S(1)^2+S(2)^2), S(3));

dSdB_S =[cos(beta)*cos(alpha), cos(beta)*sin(alpha), -sin(beta)]';
dSdB_So = cross(a,dSdB_S);
dSdB = [dSdB_S; dSdB_So];

dSdB_So_2 = cross(b,dSdB_S);
dSdB_2 = [dSdB_S; dSdB_So_2];
K2 = k*(1-rho)*(dSdB*dSdB_2');

udSdA_S = [-sin(alpha), cos(alpha), 0.0]';
udSdA_So = cross(a, udSdA_S);
dSdA = [udSdA_S; udSdA_So];

dSdA_So_2 = cross(b, udSdA_S);
dSdA_2 = [udSdA_S; dSdA_So_2];
K3 = k*(1-rho)*(dSdA*dSdA_2');

K = K1+K2+K3;
if(size(aa,1)==2)
   K(:,3:5)=[];
   K(3:5,:)=[];
end
```

function KM = GetKM(aa, bb, k, lo)

```
a=zeros(3,1);
b=zeros(3,1);
if(size(aa,1)==2)
   a(1:2)=aa;
   b(1:2)=bb;
else
   a=aa;
   b=bb;
end

N = b - a; %non-unitized directional vector
l = norm(N,2); %length
S = N/l; %unit direction vector
```

```
alpha = atan2(S(2),S(1));
beta = atan2(sqrt(S(1)^2+S(2)^2), S(3));

dSdB_S =[cos(beta)*cos(alpha), cos(beta)*sin(alpha), -sin(beta)]';
dSdB_So = cross(a,dSdB_S);
dSdB = [dSdB_S; dSdB_So];

dSdB_So_2 = cross(b,dSdB_S);
dSdB_2 = [dSdB_S; dSdB_So_2];

udSdA_S = [-sin(alpha), cos(alpha), 0.0]';
udSdA_So = cross(a, udSdA_S);
dSdA = [udSdA_S; udSdA_So];

dSdA_So_2 = cross(b, udSdA_S);
dSdA_2 = [udSdA_S; dSdA_So_2];

K1=[0;0;0;dSdB_S]*dSdA_2';
K2=dSdB_2*[0;0;0;udSdA_S]';

KM = k*(l-lo)*(K1+K2-(K1+K2)');

if(size(aa,1)==2)
   KM(:,3:5)=[];
   KM(3:5,:)=[];
End
```

function w = SpringWrench(lp1, lp2, k, lo)

```
%Calculate spring wrench

%p1,p2 : pivot points
%k : spring constant
%l0 : spring free length

if size(lp1,1) == 2
   p1=[lp1;0];
   p2=[lp2;0];
else
   p1=lp1;
   p2=lp2;
end

%Current spring lengths
l=norm(p2-p1, 2);
rho =lo/l;
```

```
w = k*(1-rho)*[p2-p1; cross(p1,p2-p1)];

if size(lp1,1) == 2
   w(3:5)=[];
end
```

function [w, l] = GetPLine(lp1, lp2)

```
% Get Plucker line coordinates and length of spring
%p1,p2 : pivot points

%Convert into spatial vector
if size(lp1,1) == 2
   p1=[lp1;0];
   p2=[lp2;0];
else
   p1=lp1;
   p2=lp2;
end

%Magnitude
l=norm(p2-p1, 2);

%Unitize
w = 1/l*[p2-p1; cross(p1,p2-p1)];

if size(lp1,1) == 2
   w(3:5)=[];
end
```

function Vo = GetOriginVel(lVp, lrp)

```
% Calculate origin velocity from velocity of a point and angular velocity
% of rigid body

Vp = zeros(3,1);
W = zeros(3,1);
rp = zeros(3,1);

if size(lVp,1)==3
   Vp(1:2) = lVp(1:2);
   W(3) = lVp(3);
   rp(1:2) = lrp;
else
   Vp=lVp(1:3);
   W=lVp(4:6);
```

```
    rp = lrp;
end

Vo = Vp - cross(W, rp);

if size(lVp,1)==3
    Vo(3)=[];
end
```

function [dRp] = GetVelP2D(dD, lRp)

```
%Calculate small displacement of point P from twist

RM = [ cos(dD(3)), -sin(dD(3))
       sin(dD(3)),  cos(dD(3)) ];

dRp= dD(1:2) + RM*lRp - lRp;
```

function GP = GetGlobalPos2D(Po, LP)

```
% Calculate Global coordinates for local coordinates
%   Po(1:2) : position of origin of local coord.
%   Po(3)   : rotation angle of local coord.
%   LP      : local coord.

num=size(LP, 2);
GP = zeros(2, num);

R_GL = [cos(Po(3)), -sin(Po(3))
        sin(Po(3)), cos(Po(3))];

for i=1:num
    GP(:,i) = Po(1:2) + R_GL*LP(:,i);
end
```

function GP = GetGlobalPos3D(Po, ER, LP)

```
%Convert coordinates using Po(Position of origin) and ER(Euler angles)

num=size(LP, 2);
GP = zeros(3, num);

gamma = ER(1);
beta = ER(2);
alpha = ER(3);
```

```
RM = [cos(gamma)*cos(beta), -
sin(gamma)*cos(alpha)+cos(gamma)*sin(beta)*sin(alpha),
sin(gamma)*sin(alpha)+cos(gamma)*sin(beta)*cos(alpha)
     sin(gamma)*cos(beta), cos(gamma)*cos(alpha)+sin(gamma)*sin(beta)*sin(alpha), -
cos(gamma)*sin(alpha)+sin(gamma)*sin(beta)*cos(alpha)
    -sin(beta), cos(beta)*sin(alpha), cos(beta)*cos(alpha)];

for i=1:num
   GP(:,i) = Po + RM*LP(:,i);
end
```

APPENDIX B
MAPLE CODE FOR DERIVATIVE OF STIFFNESS MATRIX IN CHAPTER FOUR

Maple code to compute the matrix $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ shown in Eq. (4.43) is presented in this appendix. This code creates a text file called "dBdU.map" and writes symbolic equations for the derivative in it. The computation of the derivative is quite complicated and thus the size of the created file exceeds two megabytes. Then this file may be converted to a Matlab file with a little modification. Since the matrix $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ is quite complicated, symbolic equations for matrices $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{P}}}$, $\frac{d\,\underline{\mathbf{P}}}{d\,\underline{\mathbf{Q}}}$, and $\frac{d\,\underline{\mathbf{Q}}}{d\,\underline{\mathbf{U}}}$ are obtained and column vectors $\underline{\mathbf{P}}$ and $\underline{\mathbf{Q}}$ are defined as

$$\begin{aligned}\underline{\mathbf{P}} = [&K_{11}^{U}, K_{12}^{U}, K_{13}^{U}, K_{22}^{U}, K_{32}^{U}, K_{33}^{U}, f_x^{U}, f_y^{U}, m_z^{U},\\ &K_{11}^{L}, K_{12}^{L}, K_{13}^{L}, K_{22}^{L}, K_{32}^{L}, K_{33}^{L}, f_x^{L}, f_y^{L}, m_z^{L}]^T\end{aligned}$$

$$\begin{aligned}\underline{\mathbf{Q}} = [&K_{11}^{1}, K_{11}^{2}, \ldots, K_{11}^{6}, K_{12}^{1}, K_{12}^{2}, \ldots, K_{12}^{6}, K_{13}^{1}, K_{13}^{2}, \ldots, K_{13}^{6},\\ &K_{22}^{1}, K_{22}^{2}, \ldots, K_{22}^{6}, K_{32}^{1}, K_{32}^{2}, \ldots, K_{32}^{6}, K_{33}^{1}, K_{33}^{2}, \ldots, K_{33}^{6},\\ &f_x^{1}, f_x^{2}, \ldots, f_x^{6}, f_y^{1}, f_y^{2}, \ldots, f_y^{6}, m_z^{1}, m_z^{2}, \ldots, m_z^{6}]^T\end{aligned}$$

Please reference chapter four for detailed description of each term.

Then $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ can be computed as

$$\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}} = \frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{P}}}\frac{d\,\underline{\mathbf{P}}}{d\,\underline{\mathbf{Q}}}\frac{d\,\underline{\mathbf{Q}}}{d\,\underline{\mathbf{U}}}.$$

```
> restart;
> with(LinearAlgebra):
> fopen("dBdU.map",WRITE):
```

==

Get dXdP

==

```
>B:=[KR[1,1],KR[1,2],KR[1,3],KR[2,2],KR[3,2],KR[3,3],Fx_U,Fy_U,Mz_U,Fx_L,Fy_L,Mz_L];
```

$$B := [KR_{1,1}, KR_{1,2}, KR_{1,3}, KR_{2,2}, KR_{3,2}, KR_{3,3}, Fx\_U, Fy\_U, Mz\_U, Fx\_L, Fy\_L, Mz\_L]$$

```
>F_L:=<<k11_L,k12_L,k13_L+Fy_L>|<k12_L,k22_L,k32_L>|<k13_L,k32_L+Fx_L,k33_L>>;
```

$$KF\_L := \begin{bmatrix} k11\_L & k12\_L & k13\_L \\ k12\_L & k22\_L & k32\_L + Fx\_L \\ k13\_L + Fy\_L & k32\_L & k33\_L \end{bmatrix}$$

```
>KF_U:=<<k11_U,k12_U,k13_U+Fy_U>|<k12_U,k22_U,k32_U>|<k13_U,k32_U+Fx_U,k33_U>>;
```

$$KF\_U := \begin{bmatrix} k11\_U & k12\_U & k13\_U \\ k12\_U & k22\_U & k32\_U + Fx\_U \\ k13\_U + Fy\_U & k32\_U & k33\_U \end{bmatrix}$$

```
> KM_U:=<<0,0,Fy_U>|<0,0,-Fx_U>|<-Fy_U,Fx_U,0>>;
```

$$KM\_U := \begin{bmatrix} 0 & 0 & -Fy\_U \\ 0 & 0 & Fx\_U \\ Fy\_U & -Fx\_U & 0 \end{bmatrix}$$

```
> CM:=KF_L+KF_U-KM_U;
```

$$CM := \begin{bmatrix} k11\_L + k11\_U & k12\_L + k12\_U & k13\_L + k13\_U + Fy\_U \\ k12\_L + k12\_U & k22\_L + k22\_U & k32\_L + Fx\_L + k32\_U \\ k13\_L + Fy\_L + k13\_U & k32\_L + k32\_U + Fx\_U & k33\_L + k33\_U \end{bmatrix}$$

```
> KR:=KF_L.MatrixInverse(CM).KF_U:
> P:=[k11_U,k12_U,k13_U,k22_U,k32_U,k33_U,Fx_U,Fy_U,Mz_U,
    k11_L,k12_L,k13_L,k22_L,k32_L,k33_L,Fx_L,Fy_L,Mz_L];
```

$$P := [k11\_U, k12\_U, k13\_U, k22\_U, k32\_U, k33\_U, Fx\_U, Fy\_U, Mz\_U, k11\_L, k12\_L, k13\_L, k22\_L, k32\_L, k33\_L, Fx\_L, Fy\_L, Mz\_L]$$

```
> nrow:=nops(B);
```

$$nrow := 12$$

```
> ncol:=nops(P);
```

$$ncol := 18$$

```
> dBdP:=Matrix(nrow,ncol):
> for i from 1 to nrow do
```

```
  for j from 1 to ncol do
    dBdP[i,j]:=diff(B[i], P[j]);
  end do:
end do:
```

==

Write dBdP on file "dBdU.map"

==

```
> for i from 1 to nrow do
   for j from 1 to ncol do

      fprintf("dBdU.map","dBdP(%d,%d)=%s;\n\n",i,j,convert(d
      BdP[i,j],string));
   end do:
  end do:
> fprintf("dBdU.map","\n\n\n"):
```

==

Get dPdQ

==

```
> k11_U:=k11[1]+k11[2]+k11[3]:
> k12_U:=k12[1]+k12[2]+k12[3]:
> k13_U:=k13[1]+k13[2]+k13[3]:
> k22_U:=k22[1]+k22[2]+k22[3]:
> k32_U:=k32[1]+k32[2]+k32[3]:
> k33_U:=k33[1]+k33[2]+k33[3]:
> Fx_U:=fx[1]+fx[2]+fx[3]:
> Fy_U:=fy[1]+fy[2]+fy[3]:
> Mz_U:=mz[1]+mz[2]+mz[3]:
> k11_L:=k11[4]+k11[5]+k11[6]:
> k12_L:=k12[4]+k12[5]+k12[6]:
> k13_L:=k13[4]+k13[5]+k13[6]:
> k22_L:=k22[4]+k22[5]+k22[6]:
> k32_L:=k32[4]+k32[5]+k32[6]:
> k33_L:=k33[4]+k33[5]+k33[6]:
> Fx_L:=fx[4]+fx[5]+fx[6]:
> Fy_L:=fy[4]+fy[5]+fy[6]:
> Mz_L:=mz[4]+mz[5]+mz[6]:
>
> Q:=[seq(k11[i], i=1..6)]:
> Q:=[op(Q),seq(k12[i], i=1..6)]:
> Q:=[op(Q),seq(k13[i], i=1..6)]:
> Q:=[op(Q),seq(k22[i], i=1..6)]:
> Q:=[op(Q),seq(k32[i], i=1..6)]:
> Q:=[op(Q),seq(k33[i], i=1..6)]:
```

```
> Q:=[op(Q),seq(fx[i], i=1..6)]:
> Q:=[op(Q),seq(fy[i], i=1..6)]:
> Q:=[op(Q),seq(mz[i], i=1..6)];
```

$Q := [k11_1, k11_2, k11_3, k11_4, k11_5, k11_6, k12_1, k12_2, k12_3, k12_4, k12_5, k12_6, k13_1, k13_2, k13_3, k13_4, k13_5, k13_6, k22_1, k22_2, k22_3, k22_4, k22_5, k22_6, k32_1, k32_2, k32_3, k32_4, k32_5, k32_6, k33_1, k33_2, k33_3, k33_4, k33_5, k33_6, fx_1, fx_2, fx_3, fx_4, fx_5, fx_6, fy_1, fy_2, fy_3, fy_4, fy_5, fy_6, mz_1, mz_2, mz_3, mz_4, mz_5, mz_6]$

```
> nrow:=nops(P);
```

$$nrow := 18$$

```
> ncol:=nops(Q);
```

$$ncol := 54$$

```
> dPdQ:=Matrix(nrow,ncol):
> for i from 1 to nrow do
   for j from 1 to ncol do
     dPdQ[i,j]:=diff(P[i], Q[j]);
   end do:
  end do:
```

====================================

Write dPdQ on file "dBdU.map"

====================================

```
> for i from 1 to nrow do
   for j from 1 to ncol do

      fprintf("dBdU.map","dPdQ(%d,%d)=%s;\n\n",i,j,convert(d
      PdQ[i,j],string));
   end do:
  end do:
> fprintf("dBdU.map","\n\n\n"):
```

====================================

Get dQdU

====================================

```
> for i from 1 to 6 do
   k11[i]:=k[i]-k[i]*lo[i]/l[i]*sin(theta[i])^2;

   k12[i]:=k[i]*lo[i]/l[i]*sin(theta[i])*cos(theta[i]);

   k13[i]:=-k[i]*(l[i]*sin(theta[i])+y[i])
   +k[i]*lo[i]/l[i]*(l[i]*sin(theta[i])+x[i]*sin(theta[i])*c
   os(theta[i])+y[i]*sin(theta[i])^2);

   k22[i]:=k[i]-k[i]*lo[i]/l[i]*cos(theta[i])^2;
```

```
  k32[i]:=k[i]*x[i]-k[i]*lo[i]/l[i]*(x[i]*cos(theta[i])^2
  +y[i]*sin(theta[i])*cos(theta[i]));

  k33[i]:=k[i]*x[i]*(l[i]*cos(theta[i])+x[i])+k[i]*y[i]*(l[
  i]*sin(theta[i])+y[i])-k[i]*lo[i]/l[i]*x[i]
  *(l[i]*cos(theta[i])+y[i]*sin(theta[i])*cos(theta[i])
  +x[i]*cos(theta[i])^2)-k[i]*lo[i]/l[i]*y[i]
  *(l[i]*sin(theta[i])+x[i]*sin(theta[i])*cos(theta[i])
  +y[i]*sin(theta[i])^2);

  fx[i]:=k[i]*(l[i]-lo[i])*cos(theta[i]);

  fy[i]:=k[i]*(l[i]-lo[i])*sin(theta[i]);

  mz[i]:=k[i]*(l[i]-lo[i])*(x[i]*sin(theta[i])-
  y[i]*cos(theta[i]));
end do:
>
> Z:=[seq(k[i], i=1..6)]:
> Z:=[op(Z),seq(lo[i], i=1..6)]:
> Z:=[op(Z),seq(theta[i], i=1..6)]:
> Z:=[op(Z),seq(l[i], i=1..6)]:
> Z:=[op(Z),seq(x[i], i=1..3)]:
> Z:=[op(Z),seq(y[i], i=1..3)];
```

$$Z := [k_1, k_2, k_3, k_4, k_5, k_6, lo_1, lo_2, lo_3, lo_4, lo_5, lo_6, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, l_1, l_2, l_3, l_4, l_5, l_6, x_1, x_2, x_3, y_1, y_2, y_3]$$

```
> nrow:=nops(Q);
```

$$nrow := 54$$

```
> ncol:=nops(Z);
```

$$ncol := 30$$

```
> dQdZ:=Matrix(nrow,ncol):
> for i from 1 to nrow do
  for j from 1 to ncol do
    dQdZ[i,j]:=diff(Q[i], Z[j]);
  end do:
 end do:
>
> dQdU:=Matrix(nrow,18):
> for i from 1 to nrow do
  for j from 1 to 12 do
    dQdU[i,j]:=dQdZ[i,j];
  end do:
 end do:
```

```
>
> #column 13: dAxo, 14:dAyo, 15:dAt
> #column 16: dBxo, 17:dByo, 18:dBt
> for i from 1 to nrow do

   # For theta:1,2,3 and l:1,2,3
   for j from 1 to 3 do

     dQdZ[i,12+j]:=1/l[j]*( Lp[j]*(dxo_b-dxo_a)+Mp[j]
      *(dyo_b-dyo_a)+Rp[j]*(dth_b-dth_a) + l[j]*dth_a)
      *dQdZ[i,12+j];

     dQdZ[i,18+j]:=( L[j]*(dxo_b-dxo_a)+M[j]*(dyo_b-dyo_a)
      +R[j]*(dth_b-dth_a) )*dQdZ[i,18+j];

   end do;

   # For theta:4,5,6 and l:4,5,6
   for j from 4 to 6 do

     dQdZ[i,12+j]:=1/l[j]*(Lp[j]*dxo_a+Mp[j]*dyo_a
      +Rp[j]*dth_a)*dQdZ[i,12+j];

     dQdZ[i,18+j]:=(L[j]*dxo_a+M[j]*dyo_a+R[j]*dth_a)
      *dQdZ[i,18+j];

   end do;

   # For x:1,2,3 and y:1,2,3
   for j from 1 to 3 do
     dQdZ[i,24+j]:=(dxo_a-y[j]*dth_a)*dQdZ[i,24+j];
     dQdZ[i,27+j]:=(dyo_a+x[j]*dth_a)*dQdZ[i,27+j];
   end do;

   temp:=0;
   for j from 13 to 30 do
     temp:=temp+dQdZ[i,j];
   end do;
   dQdU[i,13]:=coeff(temp,dxo_a);
   dQdU[i,14]:=coeff(temp,dyo_a);
   dQdU[i,15]:=coeff(temp,dth_a);
   dQdU[i,16]:=coeff(temp,dxo_b);
   dQdU[i,17]:=coeff(temp,dyo_b);
   dQdU[i,18]:=coeff(temp,dth_b);
end do:
>
> for i from 1 to 3 do
```

```
  L[i]:=cos(theta[i]);
  M[i]:=sin(theta[i]);
  R[i]:=x[i]*sin(theta[i])-y[i]*cos(theta[i]);
  Lp[i]:=-sin(theta[i]);
  Mp[i]:=cos(theta[i]);
  Rp[i]:=x[i]*cos(theta[i])+y[i]*sin(theta[i])+l[i];
end do:
```

=================================

Write dQdU on file "dBdU.map"

=================================

```
> for i from 1 to nrow do
  for j from 1 to 18 do
    fprintf("dBdU.map","dQdU(%d,%d)=%s;\n\n",
     i,j,convert(dQdU[i,j],string));
  end do:
 end do:
> fprintf("dBdU.map","\n\n\n"):
>
> fclose("dBdU.map"):
```

LIST OF REFERENCES

Ball, R. S., 1900, *A Treatise on the Theory of Screws*, Cambridge University Press, Cambridge, UK.

Chen, S., and Kao, I., 2000, "Conservative Congruence Transformation for Joint and Cartesian Stiffness matrices of Robotic Hands and Fingers," *International Journal of Robotic Research,* Vol. 19, No. 9, pp. 835-847.

Ciblak, N., and Lipkin, H., 1994, "Asymmetric Cartesian Stiffness for the Modeling of Compliant Robotic Systems," *Proc. ASME 23rd Biennial Mech. Conf., Des. Eng. Div.,* Vol. 72, pp. 197-204, New York, NY.

Ciblak, N., and Lipkin, H., 1999, "Synthesis of Cartesian Stiffness for Robotic Applications," *Proceedings of the IEEE International Conference on Robotics and Automation,* pp. 2147–2152, Detroit, MI.

Crane, C. D., and Duffy, J., 1998, *Kinematic Analysis of Robot Manipulators,* Cambridge University Press, Cambridge, UK.

Crane, C. D., Rico, J. M., and Duffy, J., 2006, *Screw Theory and Its Application to Spatial Robot Manipulators,* Cambridge University Press, Cambridge, UK.

Craig, J. J., 1989, *Introduction to Robotics: Mechanics and Control,* Addison Wesley, Reading, MA.

Dimentberg, F. M., 1965, *The Screw Calculus and its Applications in Mechanics*. Foreign Technology Division, Wright-Patterson Air Force Base, Ohio. Document No. FTD-HT-23-1632-67.

Duffy, J., 1996, *Statics and Kinematics with Applications to Robotics,* Cambridge University Press, Cambridge, UK.

Featherstone, J., 1985, *Robot Dynamics Algorithms,* Kluwer Academic Publishers, Boston, MA.

Griffis, M., 1991, "A Novel Theory for Simultaneously Regulating Force and Displacement," Ph.D. dissertation, University of Florida, Gainesville, FL.

Henrie, A. M., 1997, "Variable Compliance via Magneto-Rheological Materials," M.S. Thesis, Brigham Young University, Provo, UT.

Huang, S., 1998, "The Analysis and Synthesis of Spatial Compliance," Ph.D. dissertation, Marquette University, Milwaukee, WI.

Huang, S., and Schimmels, J. M., 1998, "The Bounds and Realization of Spatial Stiffness Achieved with Simple Springs Connected in Parallel," *IEEE Transactions on Robotics and Automation*, Vol. 14, No. 3, pp. 466-475.

Hurst, J. W., Chestnutt, J., and Rizzi, A., 2004, "An Actuator with Physically Variable Stiffness for Highly Dynamic Legged Locomotion," *Proceedings of the 2004 International Conference on Robotics and Automation,* pp. 4662-4667, New Orleans, LA.

Kane, T. R., and Levinson, D. A., 1985, *Dynamics: Theory and Applications,* McGraw, New York.

Loncaric, J., 1985, "Geometrical Analysis of Compliant Mechanisms in Robotics," Ph.D. Dissertation, Harvard University, Cambridge, MA.

Loncaric, J., 1987, "Normal Forms of Stiffness and Compliance Matrices," *IEEE Journal of Robotics and Automation*, Vol. 3, No. 6, pp. 567–572.

McCarthy, J. M., 1990, *Introduction to Theoretical Kinematics,* MIT Press, Cambridge, MA.

McLachlan, S., and Hall, T., 1999, "Robust Forward Kinematic Solution for Parallel Topology Robotic Manipulator," *32nd ISATA Conference – Track : Simulation, Virtual Reality and Supercomputing Automotive Applications*, pp 381-388, Vienna, Austria.

Peshkin, M., 1990, "Programmed Compliance for Error Corrective Assembly," *IEEE Transactions on Robotics and Automation*, Vol. 6, No. 4, pp. 473-482.

Pigoski, T., 1993, "An Introductory Theoretical Analysis of Planar Compliant Couplings," M.S. Thesis, University of Florida, Gainesville, FL.

Ryan, M. W., Franchek, M.A., and Bernhard, R., 1994. Adaptive-Passive Vibration Control of Single Frequency Excitation Applied to Noise Control. *Noise-Con Proceedings,* pp. 461-466, Fort Lauderdale, FL.

Roberts, R. G., 1999, "Minimal Realization of a Spatial Stiffness Matrix with Simple Springs Connected in Parallel," *IEEE Transactions on Robotics and Automation*, Vol. 15, No. 5, pp. 953-958.

Salisbury, J. K., 1980, "Active Stiffness Control of a Manipulator in Cartesian Coordinates," *Proc. 19th IEEE Conference on Decision and Control*, pp. 87-97, Albuquerque, NM.

Simaan, N., and Shoham, M., 2002, "Stiffness Synthesis of a Variable Geometry Planar Robots," *the 8th International Symposium on Advances in Robot Kinematics (ARK 2002)*, Kluwer Academic Publisher, Caldes de Malavella, Spain.

Simaan, N., and Shoham, M., 2003, "Geometric Interpretation of the Derivatives of Parallel Robot's Jacobian Matrix with Application to Stiffness Control," *ASME Journal of Mechanical Design*, Vol. 125, pp. 33-42.

Strang, G., 1988, *Linear Algebra and Its Applications,* Harcourt Brace Jovanovich, New York.

Whitney, D. E., 1982, "Quasi-static Assembly of Compliantly Supported Rigid Parts," *ASME Journal of Dynamic Systems, Measurement, and Control*, Vol. 104, pp. 65-77.

BIOGRAPHICAL SKETCH

Hyun Kwon Jung was born in Chunju, South Korea, in 1971. He attended Hanyang University, Seoul, where he received Bachelor of Science and Master of Science degrees in precision mechanical engineering in 1994 and 1996 respectively. He then worked for five years at Samsung Electronics, Suwon, South Korea, and that was the complement to mandatory military service for every Korean man. In 2002 he enrolled in graduate school at the University of Florida for a Ph.D. degree in mechanical engineering and in 2003 he started working with Dr. Carl D. Crane III at the Center for Intelligent Machines and Robotics as a research assistant. His fields of interest include kinematics, control, and realtime programming.