STATIC ANALYSIS OF TENSEGRITY STRUCTURES

By

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To my mother for her infinite generosity.

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TABLE OF CONTENTS

Abstract of Thesis Presented to the Graduate School
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Tensegrity structures are three dimensional assemblages formed of rigid and elastic elements. They hold the promise of novel applications. However their behavior is not completely understood at this time. This research addresses the static analysis problem and determines the position assumed by the structure when external loads are applied. The derivation of the mathematical model for the equilibrium positions of the structure is based on the virtual work principle together with concepts related to geometry of lines. The solution for the resultant equations is performed using numerical methods. Several examples are presented to demonstrate this approach and all the results are carefully verified. A software that is able to generate and to solve the equilibrium equations is developed. The software also permits one to visualize different equilibrium positions for the analyzed structure and in this way to gain insight in the physics and the geometry of tensegrity systems.

CHAPTER 1
INTRODUCTION

Tensegrity structures are spatial structures formed by a combination of rigid elements (the struts) and elastic elements (the ties). No pair of struts touch and the end of each strut is connected to three non-coplanar ties [1].

The struts are always in compression and the ties in tension. The entire configuration stands by itself and maintains its form solely because of the internal arrangement of the ties and the struts [2].

Tensegrity is an abbreviation of tension and integrity. Figure 1.1 shows a number of anti-prism tensegrity structures formed with 3, 4 and 5 struts respectively.

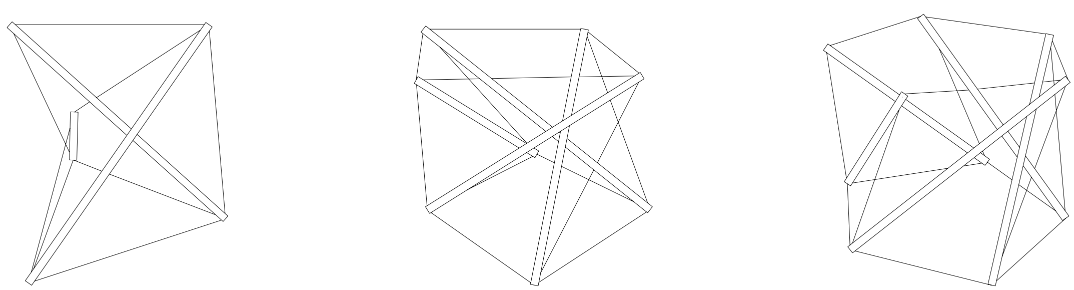


Figure 1.1. Tensegrity structures conformed by 3, 4 and 5 struts.

The development of tensegrity structures is relatively new and the works related have only existed for the 25 years. Kenner [3] establishes the relation between the rotation of the top and bottom ties. Tobie [2] presents procedures for the generation of tensile structures by physical and graphical means. Yin [1] obtains Kenner's results using energy consideration and finds the equilibrium position for the unloaded tensegrity prisms. Stern [4] develops generic design equations to find the lengths of the struts and elastic ties needed to create a desired geometry. Since no external forces are considered his results are referred to the unload position of the structure. Knight [5] addresses the problem of stability of tensegrity structures for the design of a deployable antenna.

The problem of the determination of the equilibrium position of a tensegrity structure when external forces and external moments act on the structure has not been studied previously. This is the focus of this research.

It is known that when the systems can store potential energy, as in the case of the elastic ties of a tensegrity structure, the energy methods are applicable. For this reason the virtual work formulation was selected from several possible approaches to solve the current problem.

Despite their complexity, anti-prism tensegrity structures exhibit a pattern in their configuration. This fact is used to develop a consistent nomenclature valid for any structure and with this basis to develop the equilibrium equations. To simplify the derivation of a mathematical model some assumptions are included. Those simplifications are related basically with the absence of internal dissipative forces and with the number and fashion that external loads are applied to the struts of the structure.

Even for the simplest case the resultant equations are lengthy and highly coupled. Numerical methods offer an alternative to solve the equations. Parallel to the research presented here a software in Matlab was developed. Basically the software is able to develop the equilibrium equations for a given tensegrity structure and to solve them when the external loads are given. The software uses the well known Newton Raphson method which is implemented by the function fsolve of Matlab. To avoid the limitations of numerical methods to converge to an answer, the proper selection of the initial conditions was considered carefully together with the guidance of the solution through small increments of the external loads.

Once the equations are solved, the output data consists of basically listing of the various coordinates of the ends of the structure expressed in a global coordinate system for the equilibrium position. When dealing with three dimensional systems, the numerical results by themselves are not sufficient to understand the behavior of the systems. To assist to the comprehension of the results the software developed provides graphic outputs. In this way the complex equilibrium equations are connected in an easy way to the physical situation.

One important question that arose was the validity of the numerical results. This point is specially important when one considers the complexity of the equations. An independent validation of the results was realized using Newton's Third Law.

This thesis is basically as follows: Chapter 2 introduces the basic concepts related to the tools required to develop the mathematical model for a tensegrity structure, Chapter 3 develops a systematic nomenclature for the elements of a tensegrity structure and presents the mathematical model. Chapter 4 provides examples to illustrate the application of the model.

CHAPTER 2
BASIC CONCEPTS

The main objective of this research is to find the final equilibrium position of a general anti-prism tensegrity structure after an arbitrary load and or moment has been applied. In this chapter the main concepts involved in the derivations of the equations that govern the statics of the structure are presented.

2.1 The Principle of Virtual Work

The principle of virtual work for a system of rigid bodies for which there is no energy absorption at the points of interconnection establishes that the system will be in equilibrium if [6]

$$\delta W = \sum_{i=1}^{N} \underline{F}_i \cdot \delta \underline{r}_i = 0 \tag{2.1}$$

where

δW : virtual work.

$\underline{F}_i$: force applied to the system at point i.

$\delta \underline{r}_i$: virtual displacement of the vector $\underline{r}_i$.

N : number of applied forces.

The virtual displacement represents imaginary infinitesimal changes $\delta \underline{r}_i$ of the position vector $\underline{r}_i$ that are consistent with the constraints of the system but otherwise arbitrary [7]. The symbol δ is used to emphasize the virtual character of the instantaneous variations. The virtual displacements obey the rules of differential calculus.

If the system has p degrees of freedom there are p generalized coordinates $q_k, \;\; k=1, \; 2, \; . \; . \; . \; , \; q_p$, then the variation of $\underline{r}_i$ must be evaluated with respect to each generalized coordinate.

$$\underline{r}_i = \underline{r}_i \; (q_1, \; q_2, \; . \; . \; . \; , \; q_p)$$

$$\delta \underline{r}_i = \frac{\partial \underline{r}_i}{\partial q_1} \delta q_1 + \frac{\partial \underline{r}_i}{\partial q_2} \delta q_2 + . \; . \; . + \frac{\partial \underline{r}_i}{\partial q_p} \delta q_p$$

$$\delta \underline{r}_i = \sum_{k=1}^{p} \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k \tag{2.2}$$

The principle of virtual work can be modified to allow for the inclusion of internal conservative forces in terms of potential functions [6]. In general the virtual work includes the contribution of both conservative and non-conservative forces

$$\delta W = \delta W_{nc} + \delta W_c \tag{2.3}$$

where the subscripts nc and c denote conservative and non-conservative virtual work respectively.

The virtual work performed by the non-conservative forces can be expressed as

$$\delta W_{nc} = \sum_{i=1}^{n} \underline{F}_{nci} \; . \; \delta \underline{r}_i \tag{2.4}$$

where $\underline{F}_{nci}$ is the non-conservative force i and n is the number of non-conservative forces. Substituting (2.2) into (2.4) yields,

$$\delta W_{nc} = \sum_{k=1}^{p} \sum_{i=1}^{n} \underline{F}_{nci} \cdot \frac{\partial \underline{r}_i}{\partial q_k} \delta q_k \tag{2.5}$$

The virtual work performed by the conservative force j can be expressed in the form [7]

$$\delta W_{cj} = -\delta V_j \tag{2.6}$$

where $V_j = V_j(q_1, \ q_2, \ . \ . \ . \ , q_p)$ is the potential energy associated with the conservative force j. Therefore

$$\delta W_{cj} = -\left(\frac{\partial V_j}{\partial q_1} \delta q_1 + \frac{\partial V_j}{\partial q_2} \delta q_2 + \ . \ . \ . \ + \frac{\partial V_j}{\partial q_p} \delta q_p \right) \tag{2.7}$$

And the total virtual work performed by the conservative forces is given by

$$\delta W_c = -\left(\sum_{k=1}^{p} \sum_{j=1}^{m} \frac{\partial V_j}{\partial q_k} \delta q_k \right) \tag{2.8}$$

where m is the number of conservative forces.

With the aid of (2.5) and (2.8), equation (2.3) can be rewritten in the form

$$\delta W = \left(\sum_{k=1}^{p} \sum_{i=1}^{n} \underline{F}_{nci} \cdot \frac{\partial \underline{r}_i}{\partial q_k} - \sum_{k=1}^{p} \sum_{j=1}^{m} \frac{\partial V_j}{\partial q_k} \right) \delta q_k$$

$$= \sum_{k=1}^{p} \left(\sum_{i=1}^{n} \underline{F}_{nci} \cdot \frac{\partial \underline{r}_i}{\partial q_k} - \sum_{j=1}^{m} \frac{\partial V_j}{\partial q_k} \right) \delta q_k \tag{2.9}$$

The principle of virtual work requires that the preceding expression vanishes for the equilibrium. Because the generalized virtual displacements δq_k are all independent and hence entirely arbitrary, (2.9) can be satisfied [7], if and only if

$$\sum_{i=1}^{n} \underline{F}_{nci} \cdot \frac{\partial \underline{r}_i}{\partial q_k} - \sum_{j=1}^{m} \frac{\partial V_j}{\partial q_k} = 0$$

$$Q_k - \sum_{j=1}^{m} \frac{\partial V_j}{\partial q_k} = 0, \quad k = 1, 2, \cdots, p \tag{2.10}$$

where

$$Q_k = \sum_{i=1}^{n} \underline{F}_{nci} \cdot \frac{\partial \underline{r}_i}{\partial q_k} \tag{2.11}$$

The term Q_k is known as the generalized forces and despite its name may include both the virtual work due to external non-conservative forces and the virtual work due to external non-conservative moments.

If the lower ends of the struts of a tensegrity system are constrained to move on the horizontal plane and also the rotation about the longitudinal axis of the strut is constrained, then each strut has 4 degrees of freedom and the whole system has

$$p = 4 * n\_struts \tag{2.12}$$

degrees of freedom where n_struts is the number of struts of the structure.

2.2 Plücker Coordinates

The coordinates of a line joining two finite points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) can be written as

$$\underline{\$} = \begin{bmatrix} \underline{S} \\ \underline{S}_o \end{bmatrix} \tag{2.13}$$

where $\underline{S}$ is in the direction along the line and $\underline{S}_0$ is the moment of the line about the origin O. $\underline{S}$ and $\underline{S}_0$ can be evaluated from the coordinates of the points as follows [8]

$$\underline{S} = \begin{bmatrix} L \\ M \\ N \end{bmatrix} \quad (2.14) \qquad \text{where} \qquad L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix} \tag{2.15}$$

$$M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix} \tag{2.16}$$

$$N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix} \tag{2.17}$$

and

$$\underline{S}_o = \begin{bmatrix} P \\ Q \\ R \end{bmatrix} \quad (2.18) \qquad \text{where} \qquad P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \tag{2.19}$$

$$Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} \tag{2.20}$$

$$R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \tag{2.21}$$

The numbers L, M, N, P, Q and R are called the Plücker line coordinates and they cannot be simultaneously equal to zero.

The Plücker line coordinates can be expressed in unitized form by dividing the vectors $\underline{S}$ and $\underline{S}_0$ by $\sqrt{L^2+M^2+N^2}$ provided L, M and N are not all equal to zero.

$$\hat{\$} = \frac{1}{\sqrt{L^2+M^2+N^2}} \begin{bmatrix} \underline{S} \\ \underline{S}_o \end{bmatrix} = \begin{bmatrix} \underline{s} \\ \underline{s}_o \end{bmatrix} \tag{2.22}$$

A force $\underline{F}$ can be expressed as a scalar multiple of the unit vector $\underline{s}$ bound to the line. The moment of the force about a reference point O can be expressed as a scalar multiple of the moment vector $\underline{s}_0$ [9], therefore

$$\underline{\$}_F = f\ \hat{\$} = f \begin{bmatrix} \underline{s} \\ \underline{s}_o \end{bmatrix} \tag{2.23}$$

where f stands for the magnitude of the force $\underline{F}$.

If L, M and N are all equal to zero the unitized Plücker line coordinates have the form

$$\hat{\$} = \frac{1}{\sqrt{P^2+Q^2+R^2}} \begin{bmatrix} \underline{0} \\ \underline{S}_o \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{s}_o \end{bmatrix} \tag{2.24}$$

And the Plücker line coordinates of a pure moment are

$$\underline{\$}_M = m\ \hat{\$} = m \begin{bmatrix} \underline{0} \\ \underline{s}_o \end{bmatrix} \tag{2.25}$$

where m stands for the magnitude of the moment.

Consider two coordinates systems shown in Figure 2.1. The origin of system $X'Y'Z'$ is translated by (x, y, z) and rotated arbitrarily with respect to system XYZ. The Plücker coordinates of the line $\underline{\$}$ expressed in the system $X'Y'Z'$ can be transformed to the system XYZ using the following relation [9],

$$\underline{\$} = e \ \underline{\$}' \tag{2.26}$$

where

$\underline{\$}$: Plücker coordinates of the line expressed in the system XYZ

$\underline{\$}'$: Plücker coordinates of the line expressed in the system $X'Y'Z'$

and

$$e = \begin{bmatrix} {}^{A}_{B}R & O_3 \\ A_3 \, {}^{A}_{B}R & {}^{A}_{B}R \end{bmatrix} \tag{2.27}$$

where

${}^{A}_{B}R$: rotation matrix of the system $X'Y'Z'$ with respect to the system XYZ

O_3 : zeroes 3x3 matrix

$$A_3 = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{2.28}$$

Conversely if the Plücker coordinates of the line are given in the system $X'Y'Z'$ and it is desired to express them in the system XYZ, from (2.26)

$$\underline{\$}' = e^{-1} \ \underline{\$} \tag{2.29}$$

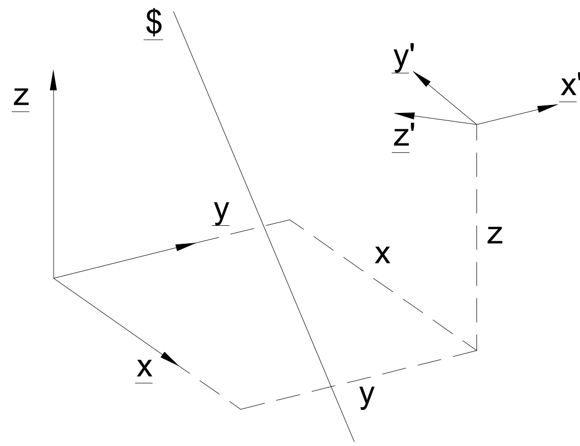


Figure 2.1. General change of a coordinate system.

where

$$e^{-1} = \begin{bmatrix} {}^A_B R^T & O_3 \\ {}^A_B R^T A_3^T & {}^A_B R^T \end{bmatrix} \tag{2.30}$$

2.3 Transformation Matrices

Figure 2.2 shows an arbitrary point $\underline{P}_2$ located on a strut of length L_s. In a reference system D whose z axis is along the axis of the strut and with its origin is located at the lower end of the strut, the coordinates of ${}^D\underline{P}_2$ are simply $(0,0,l)$. However frequently is more convenient for purposes of analysis to express the location of $\underline{P}_2$ in the global reference system A. This can be accomplished by a transformation matrix.

If the lower end of the strut is constrained to move on the horizontal plane $({}^A x\, {}^A y)$, and also the rotation about its longitudinal axis is constrained, the strut can be modeled by an universal joint. In this way the joint provides the 4 degrees of freedom associated with the strut.

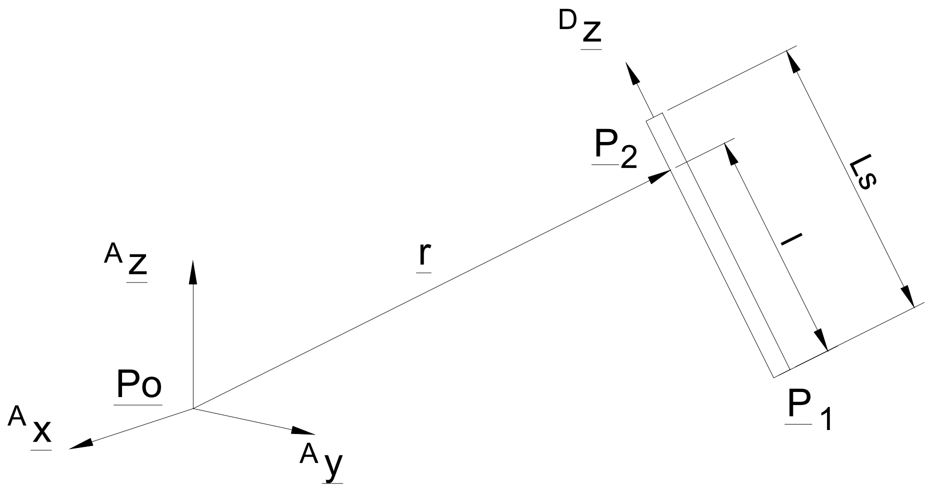


Figure 2.2. Strut in an arbitrary position.

The alignment of the z axis on the fixed system with the axis of the rod can be accomplished using the following three consecutive transformations [10] :

Translation, $\underline{t} = (a, b, 0)$, Figure 2.3. Note that the coordinate z is zero because of the restriction imposed to the movement of the lower end of the strut.

Rotation □, about the current x axis ($^{B}\underline{x}$), Figure 2.4.

Rotation □, about the current y axis ($^{C}\underline{y}$), Figure 2.5.

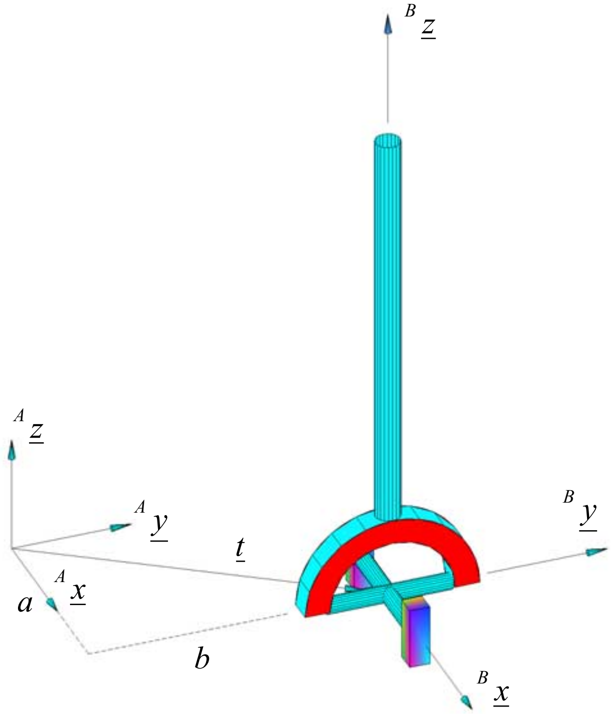


Figure 2.3. Translation $(a,b,0)$ in the system A.

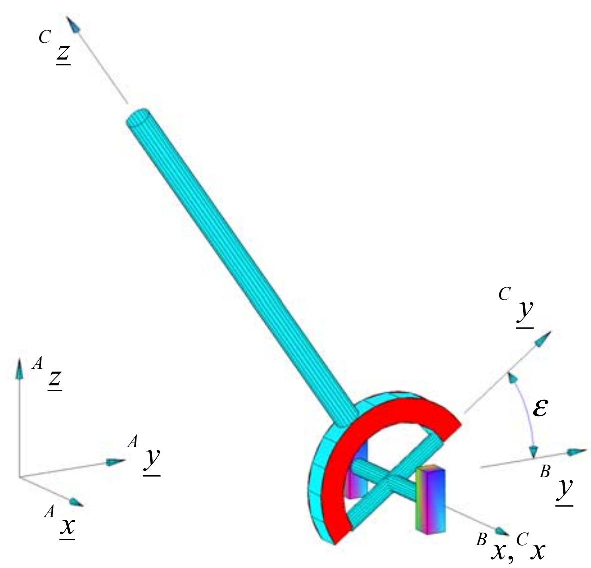


Figure 2.4. Rotation about x axis in the B system.

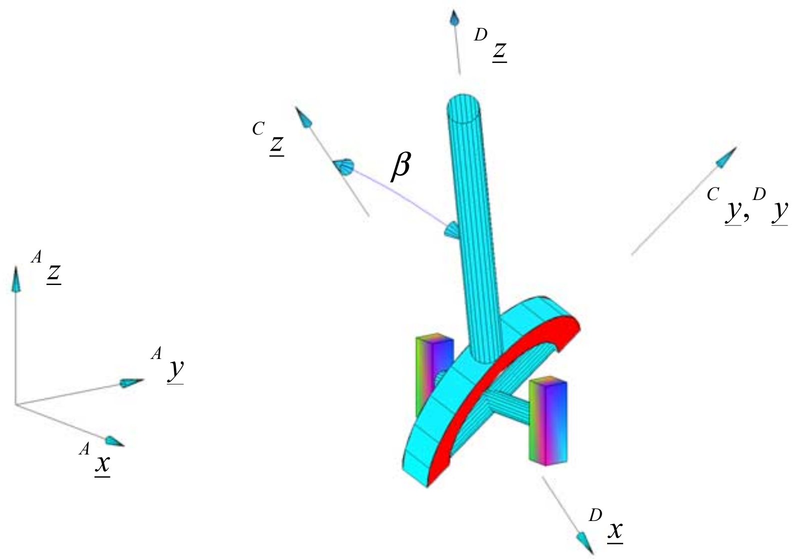


Figure 2.5. Rotation □ about current y axis in the C system.

The coordinates of $\underline{P}_2$ measured in the global reference system are

$$^{A}\underline{P}_2 = {}^{A}_{B}T_{a,b,0}\ {}^{B}_{C}T_{\varepsilon}\ {}^{C}_{D}T_{\beta}\ {}^{D}\underline{P}_2 \tag{2.31}$$

where:

$$^{A}_{B}T_{a,b,0} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.32}$$

$$^{B}_{C}T_{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon & 0 \\ 0 & \sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.33}$$

$$^{C}_{D}T_{\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.34}$$

$$^{D}\underline{P}_{2} = \begin{bmatrix} 0 \\ 0 \\ l \\ 1 \end{bmatrix} \tag{2.35}$$

Substituting the above previous expressions into (2.31) yields

$$^{A}\underline{P}_{2} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l\sin\beta + a \\ -l\sin\varepsilon\cos\beta + b \\ l\cos\varepsilon\cos\beta \\ 1 \end{bmatrix} \tag{2.36}$$

When the values of (x, y, z) are known, the angles □ and □ can be calculated from (2.36) and

$$\tan\varepsilon = \frac{b-y}{z} \tag{2.37}$$

$$\tan\beta = \frac{x-a}{\left(\dfrac{b-y}{\sin\varepsilon}\right)} \tag{2.38}$$

or

$$\tan\beta = \frac{x-a}{\left(\dfrac{z}{\cos\varepsilon}\right)} \tag{2.39}$$

As the signs are known for each numerator and denominator, equations (2.37) through (2.39) give unique values for □ and □.

The generalized coordinates associated with the degrees of freedom of the strut are a, b, ε, and β; therefore the virtual displacement $\delta\underline{r}$ of $\underline{r}={}^{A}\underline{P}_{2}$ given by (2.36) can be evaluated using (2.2) as follows

$$\delta \underline{r} = \begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \frac{\partial \underline{r}}{\partial a} \delta a + \frac{\partial \underline{r}}{\partial b} \delta b + \frac{\partial \underline{r}}{\partial \varepsilon} \delta \varepsilon + \frac{\partial \underline{r}}{\partial \beta} \delta \beta$$

or,

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta a + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta b + \begin{bmatrix} 0 \\ -\ell \cos\varepsilon \cos\beta \\ -\ell \sin\varepsilon \cos\beta \end{bmatrix} \delta \varepsilon + \begin{bmatrix} \ell \cos\beta \\ \ell \sin\varepsilon \sin\beta \\ -\ell \cos\varepsilon \sin\beta \end{bmatrix} \delta \beta$$

and therefore,

$$\delta x = \delta a + l \cos \beta \delta \beta \tag{2.40}$$

$$\delta y = \delta b - l \cos \varepsilon \cos \beta \delta \varepsilon + l \sin \varepsilon \sin \beta \delta \beta \tag{2.41}$$

$$\delta z = - l \sin \varepsilon \cos \beta \delta \varepsilon - l \cos \varepsilon \sin \beta \delta \beta \tag{2.42}$$

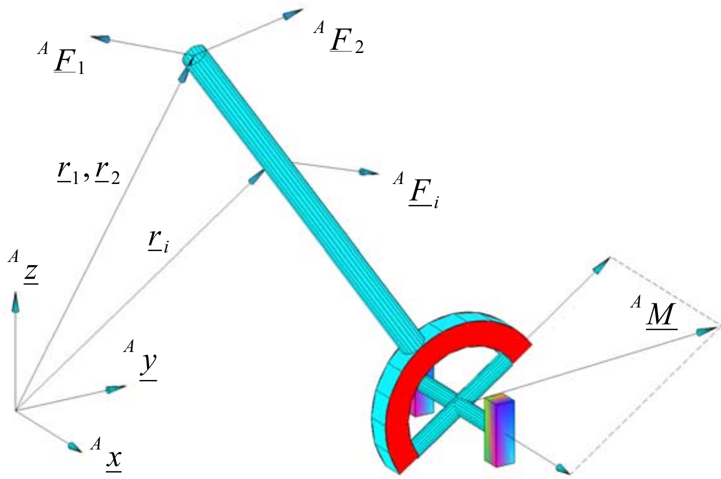
2.4 Reaction Forces and Reaction Moments

The virtual work approach does not yield the reaction forces and reaction moments. They are obtained using Newton's Third Law.

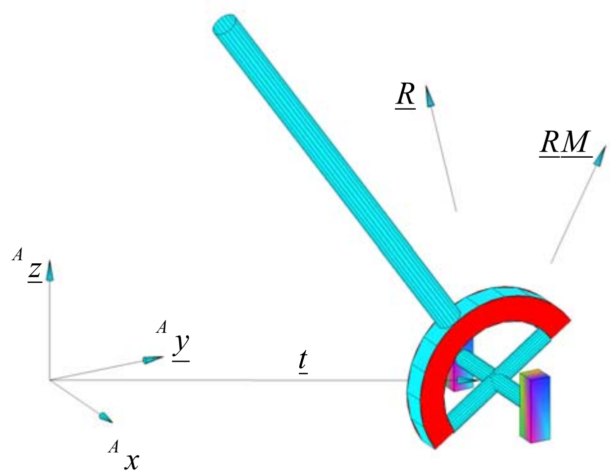
Several external forces have been applied at arbitrary points on the strut shown in Figure 2.6a together with an external moment which is the resultant of the external moments applied along the axis of the universal joint. Both external forces and external moment are expressed in the global reference system A. Figure 2.6b shows the reaction force and the reaction moment exerted by the support.

The equilibrium equation using Plücker coordinates expressed in the global reference system A is

$$\sum_{i=1}^{n} {}^{A}\underline{\$}_{F_i} + {}^{A}\underline{\$}_{M} + {}^{A}\underline{\$}_{R} + {}^{A}\underline{\$}_{RM} = \underline{0} \tag{2.43}$$



(a)



(b)

Figure 2.6. Static analysis of a strut.
a) External loads; b) Reactions

where:

$^{A}\underline{\$}_{Fi}$: Plücker coordinates of the external force i.

$^{A}\underline{\$}_{M}$: Plücker coordinates of the external moment.

$^{A}\underline{\$}_{R}$: Plücker coordinates of the reaction force.

${}^{A}\underline{\$}_{RM}$: Plücker coordinates of the reaction moment.

n : number of external forces

Since ${}^{A}\underline{\$}_{M}$ and ${}^{A}\underline{\$}_{RM}$ are pure moments (2.43) can be rewritten in the form

$$\sum_{i=1}^{n}\begin{bmatrix} {}^{A}\underline{F}_i \\ \\ {}^{A}\underline{r}_i \times {}^{A}\underline{F}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ {}^{A}Mx \\ {}^{A}My \\ {}^{A}Mz \end{bmatrix} + \begin{bmatrix} {}^{A}\underline{R} \\ \\ {}^{A}\underline{t} \times {}^{A}\underline{R} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ {}^{A}RMx \\ {}^{A}RMy \\ {}^{A}RMz \end{bmatrix} = \underline{0} \qquad (2.44)$$

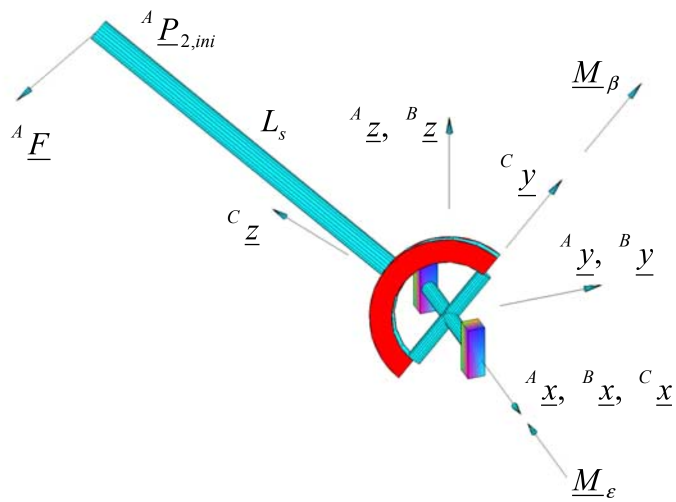
Usually the first and second terms together with the position vector $\underline{t}$ in the third term of (2.44) are known because they correspond to known data or as a result of the virtual work analysis. Hence the reaction force ${}^{A}\underline{R}$ and the reaction moment ${}^{A}\underline{RM}$ can be solved easily from (2.44).

2.5 Numerical Example

The following example helps to clarify the concepts discussed so far and also introduces to the numerical techniques employed to solve the resultant equations.

Figure 2.7 shows a massless strut of length L_s joined to the horizontal plane by an universal joint without friction in its moving parts. The support of one of the axis of the universal joint is firmly attached to the ground therefore the joint cannot perform any longitudinal displacement.

The strut is initially in equilibrium and the coordinates of the upper end, $\underline{P}_{2,ini}$, in the A system are known for the initial position. Then a constant force



$$L_S = 0.25 \quad m \qquad {}^{A}\underline{P}_{2,ini} = \begin{bmatrix} -0.148 \\ -0.150 \\ 0.134 \end{bmatrix} m \qquad M_{\varepsilon} = 0.15 \quad N \cdot m$$

$$M_{\beta} = 0.30 \quad N \cdot m$$

$${}^{A}\underline{F} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} N$$

Figure 2.7. Data for the static analysis of a strut.

and two constant moments along the axis of the universal joint are applied as it is shown in Figure 2.7. The force ${}^{A}\underline{F}$ is expressed in a global reference system whose origin is located at the intersection of the axes of the universal joint. Since the coordinates systems A and B are coincident, the vector $\underline{t}$ which represents the location of the origin of the B system with respect to the A system is $\underline{0}$.

It is required to determine the final equilibrium position of the strut and the reaction force and the reaction moment in the support of the strut. The numerical values for $\underline{P}_{2,ini}$, ${}^{A}\underline{F}$, and the magnitudes of the moments $\underline{M}_{\varepsilon}$ and $\underline{M}_{\beta}$ are illustrated in Figure 2.7.

Four coordinates systems are defined following the guidelines presented on Figures 2. 3 through 2.5.

System A: global reference coordinate system.

System B: obtained after a translation $(0,0,0)$ of system A.

System C: obtained after a rotation ε about ${}^{B}\underline{x}$.

System D: obtained after a rotation β about ${}^{C}\underline{y}$.

Systems A, B and C are shown in Figure 2. 7. With this notation $\underline{M}_{\varepsilon}$ and $\underline{M}_{\beta}$ expressed in the C system are

$$ {}^{C}\underline{M}_{\varepsilon} = M_{\varepsilon} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = 0.15 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} N \cdot m \tag{2.45} $$

$$ {}^{C}\underline{M}_{\beta} = M_{\beta} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0.30 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} N \cdot m \tag{2.46} $$

The strut has 2 degrees of freedom given by the rotations of the universal joint. The solution of the problem consists on finding the value of that rotations, ie ε and β.

The final position of the upper end of the strut can be found with the aid of (2.36) noting that $\underline{r}={}^{A}\underline{P}_{2,fin}$, $l = L_s$, $a = 0$ and $b = 0$.

$$ \underline{r} = {}^{A}\underline{P}_{2}, fin = \begin{bmatrix} L_s \sin\beta \\ -L_s \sin\varepsilon \cos\beta \\ L_s \cos\varepsilon \cos\beta \end{bmatrix} \tag{2.47} $$

where $\underline{r}$ has been expressed in rectangular coordinates instead of homogeneous coordinates. The virtual displacement $\delta\underline{r}$ is obtained from (2.2) noting that $\underline{r} = \underline{r}(\varepsilon, \beta)$.

$$\delta\underline{r} = \frac{\partial\underline{r}}{\partial\varepsilon}\,\delta\varepsilon + \frac{\partial\underline{r}}{\partial\beta}\,\delta\beta$$

From (2.47)

$$\delta\underline{r} = \begin{bmatrix} 0 \\ -L_s\cos\varepsilon\cos\beta \\ -L_s\sin\varepsilon\cos\beta \end{bmatrix}\delta\varepsilon + \begin{bmatrix} L_s\cos\beta \\ L_s\sin\varepsilon\sin\beta \\ -L_s\cos\varepsilon\sin\beta \end{bmatrix}\delta\beta \qquad (2.48)$$

Noting that the external force has no y component, the virtual work δW_F performed by the external force $\underline{F}$ is given by

$$\delta W_F = \underline{F}.\delta\underline{r} = \begin{bmatrix} Fx \\ 0 \\ Fz \end{bmatrix} \cdot \left\{ \begin{bmatrix} 0 \\ -L_s\cos\varepsilon\cos\beta \\ -L_s\sin\varepsilon\cos\beta \end{bmatrix}\delta\varepsilon + \begin{bmatrix} L_s\cos\beta \\ L_s\sin\varepsilon\sin\beta \\ -L_s\cos\varepsilon\sin\beta \end{bmatrix}\delta\beta \right\}$$

And after simplifying

$$\delta W_F = F_x L_S\cos\beta\delta\beta - F_z L_S\sin\varepsilon\cos\beta\delta\varepsilon - F_z L_S\cos\varepsilon\sin\beta\delta\beta \qquad (2.49)$$

The virtual work due to the external moments δW_M is given by

$$\delta W_M = \underline{M}_{\varepsilon}\cdot\delta\underline{\varepsilon} + \underline{M}_{\beta}\cdot\delta\underline{\beta} \qquad (2.50)$$

As the scalar or dot product is invariant under coordinate transformation the last expression can be evaluated easily if the terms on the right side are expressed in the C system. Since

$${}^{C}\underline{\varepsilon} = \varepsilon \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad {}^{C}\underline{\beta} = \beta \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

then

$${}^{C}\underline{\delta\varepsilon} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta\varepsilon \tag{2.51}$$

and

$${}^{C}\delta\underline{\beta} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta\beta \tag{2.52}$$

Substituting (2.45), (2.46), (2.51) and (2.52) into (2.50) the virtual work due to the external moments is simply

$$\delta W_M = M_{\varepsilon}\delta\varepsilon + M_{\beta}\delta\beta \tag{2.53}$$

The total virtual work is given by the sum of (2.49) and (2.53) and in the equilibrium must be zero, then

$$F_x L_S \cos\beta\delta\beta - F_z L_S \sin\varepsilon\cos\beta\delta\varepsilon - F_z L_S \cos\varepsilon\sin\beta\delta\beta + M_{\varepsilon}\delta\varepsilon + M_{\beta}\delta\beta = 0$$

And re-grouping

$$\left(-F_z L_S \sin\varepsilon\cos\beta + M_{\varepsilon}\right)\delta\varepsilon + \left(F_x L_S \cos\beta - F_z L_S \cos\varepsilon\sin\beta + M_{\beta}\right) \delta\beta = 0 \tag{2.54}$$

Since equation (2.54) is valid for all values of $\delta\varepsilon$ and $\delta\beta$ which are not in general equal to zero then

$$-F_z L_S \sin\varepsilon \cos\beta + M_\varepsilon = 0 \tag{2.55}$$

and

$$F_x L_S \cos\beta - F_z L_S \cos\varepsilon \sin\beta + M_\beta = 0 \tag{2.56}$$

For this example the resultant equations (2.55) and (2.56) are not strongly coupled and it is possible to obtain a solution in closed form, however in the most general problems this is not the case and it will be shown that numerical solutions are easier to implement.

A very well known numerical technique is the Newton-Raphson method. The function fsolve of Matlab is used to implement the Newton-Raphson algorithm. In order to use it is necessary to specify the set of equations to be solved, for instance (2.55) and (2.56) in the current example, together with the initial values of ε and β.

The initial values of ε and β, (ε_0 and β_0) can be calculated from (2.37) and (2.38) noting that $a = b = 0$.

$$\tan\varepsilon_o = \frac{-y}{z} = \frac{0.150}{0.134} \qquad \therefore \qquad \varepsilon_o = 48.2°$$

$$\tan\beta_o = \frac{x}{\left(\dfrac{-y}{\sin\varepsilon}\right)} = \frac{-0.148}{\left(\dfrac{0.15}{\sin 48.2°}\right)} \qquad \therefore \qquad \beta_o = -36.3°$$

With these initial conditions the results given by the software are

$$\varepsilon = 72.5^o \quad \text{and} \quad \beta = -71.7^o \tag{2.57}$$

Substituting these values and the value of L_s into (2.47) yields

$$ {}^{A}\underline{P}_{2}, fin \;=\; \begin{bmatrix} -0.237 \\ -0.080 \\ -0.024 \end{bmatrix} \; m \tag{2.58} $$

The result is illustrated in Figure 2.8.

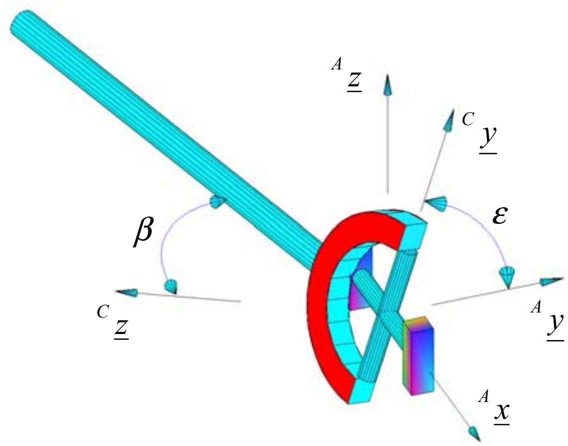


Figure 2.8. Final equilibrium position of the strut.

A solution by numerical methods is highly sensitive to a correct selection of the initial values. For this example the location of ${}^{A}\underline{P}_{2,ini}$ was given explicitly and this fact permitted to evaluate ε_0 and β_0, but in the analysis of tensegrity structures it is necessary to find them using another approach. This topic will be discussed in detail in Section 3.4.

Table 2.1 shows the results obtained when arbitrarily another set of angles ε_0 and β_0 are chosen as initial guesses. Although the Newton-Raphson algorithm still yields numerical results and that results are equilibrium positions, the solutions listed in Table 2.1 are not compatible with the initial conditions of this exercise. In general if the initial values are not correct the algorithm will not converge to a solution or to find answers that cannot be realized practically.

Table 2.1. Numerical solutions for different initial conditions.

ε_0	β_0	□	□
35º	-20º	18.7º	20.7º
125º	30º	107.5º	71.7º
135º	15º	161.3º	-20.7º

Another important consideration to assure the quality of the numerical solutions is to avoid large increments in the input values. It is always possible to increase gradually the value of the external moments and forces, for the static case. In this way the numerical solution is guided without difficulty.

Once the equilibrium position is solved the next step is to evaluate the reaction force and the reaction moment. For this example there is only a single external force and it is applied at the upper end of the strut, and due to the fact systems A and B are coincident, the vector $\underline{t}$ is zero, as shown in Figure 2.7. For the final equilibrium position of the strut, (2.44) becomes

$$\begin{bmatrix} {}^{A}\underline{F} \\ \\ {}^{A}\underline{P}_{2,fin} \times {}^{A}\underline{F} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ {}^{A}M_x \\ {}^{A}M_y \\ {}^{A}M_z \end{bmatrix} + \begin{bmatrix} {}^{A}R_x \\ {}^{A}R_y \\ {}^{A}R_z \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ {}^{A}RM_x \\ {}^{A}RM_y \\ {}^{A}RM_z \end{bmatrix} = \underline{0} \tag{2.59}$$

${}^{A}\underline{P}_{2,fin}$ in (2.59) is given by (2.47). Using this result the first term of (2.59) can be expanded as

$$\begin{bmatrix} {}^{A}\underline{F} \\ {}^{A}\underline{P}_{2,fin} \times {}^{A}\underline{F} \end{bmatrix} = \begin{bmatrix} {}^{A}F_x \\ {}^{A}F_y \\ {}^{A}F_z \\ -Ls\cos\beta\left({}^{A}F_y\cos\varepsilon + {}^{A}F_z\sin\varepsilon\right) \\ Ls\left({}^{A}F_x\cos\varepsilon\cos\beta - {}^{A}F_z\sin\beta\right) \\ Ls\left({}^{A}F_x\sin\varepsilon\cos\beta + {}^{A}F_y\sin\beta\right) \end{bmatrix} \tag{2.60}$$

The external moment is generated by the external moments $\underline{M}_{\varepsilon}$ and $\underline{M}_{\beta}$. However they were expressed in the C system, (see Figure 2.7). However (2.59) requires them to be expressed in system A. It is not difficult to establish the geometric relationships between systems C and A. Here the use of the general relations (2.26) to (2.28) is preferred because they are more useful in more complex situations.

As ${}^{A}\underline{\$}_M$ is the resultant of $\underline{M}_{\varepsilon}$ and $\underline{M}_{\beta}$ both expressed in the A system

$${}^{A}\underline{\$}_M = e\ \left({}^{C}\underline{\$}_e + {}^{C}\underline{\$}_{\beta} \right) \tag{2.61}$$

where

e: matrix that transforms a line expressed in the C system into the A system.

${}^{C}\underline{\$}_{\varepsilon}$: Plücker coordinates of ${}^{C}\underline{M}_{\varepsilon}$ in the C system.

${}^{C}\underline{\$}_{\beta}$: Plücker coordinates of ${}^{C}\underline{M}_{\beta}$ in the C system.

Matrix e is obtained using (2.27) and for this case

$$e = \begin{bmatrix} {}^{A}_{C}R & 0_3 \\ A_3\,{}^{A}_{C}R & {}^{A}_{C}R \end{bmatrix} \tag{2.62}$$

Since the origins of systems A and C are coincident then A_3 (see (2.28)) is given by

$$A_3 \quad = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.63}$$

The rotation matrix ${}^{A}_{C}R$ is obtained from the following transformation

$${}^{A}_{C}R \ = {}^{A}_{B}R \ \ {}^{B}_{C}R_{x,\varepsilon} \tag{2.64}$$

From Figure 2.7 is apparent that systems A and B are parallel, then

$${}^{A}_{B}R \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.65}$$

From Figures 2.4 and 2.7 is clear that system B is obtained after a rotation ε about ${}^{B}\underline{x}$, then

$${}^{B}_{C}R \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \tag{2.66}$$

From (2.65) and (2.66) is apparent that

$${}^{A}_{C}R \quad = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\ \varepsilon & -\sin\ \varepsilon \\ 0 & \sin\ \varepsilon & \cos\ \varepsilon \end{bmatrix} \tag{2.67}$$

Substituting (2.67) together with (2.63) into (2.62) yields

$$e = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & \cos\varepsilon & -\sin\varepsilon & & \underline{0} & \\ 0 & \sin\varepsilon & \cos\varepsilon & & & \\ & & & 1 & 0 & 0 \\ & \underline{0} & & 0 & \cos\varepsilon & -\sin\varepsilon \\ & & & 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \tag{2.68}$$

The Plücker coordinates of ${}^{C}\underline{M}_{\varepsilon}$ given by (2.45) are

$${}^{C}\underline{\$}_{\varepsilon} = M_{\varepsilon} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} \tag{2.69}$$

The Plücker coordinates of ${}^{C}\underline{M}_{\beta}$ given by (2.46) are

$${}^{C}\underline{\$}_{\beta} = M_{\beta} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{2.70}$$

Substituting (2.68), (2.69) and (2.70) into (2.61) yields

$${}^{A}\underline{\$}_{M} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ {}^{A}M_{x} \\ {}^{A}M_{y} \\ {}^{A}M_{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -M_{\varepsilon} \\ M_{\beta}\cos\varepsilon \\ M_{\beta}\sin\varepsilon \end{bmatrix} \tag{2.71}$$

Substituting (2.60) and (2.71) into (2.59) and solving for unknowns

$$
\begin{aligned}
{}^{A}R_x &= -{}^{A}F_x \\
{}^{A}R_y &= -{}^{A}F_y \\
{}^{A}R_z &= -{}^{A}F_z
\end{aligned}
$$

$$
\begin{aligned}
{}^{A}RM_x &= L_S \cos\beta\left({}^{A}F_y \cos\varepsilon + {}^{A}F_z \sin\varepsilon\right) + M_\varepsilon \qquad (2.72) \\
{}^{A}RM_x &= -L_S\left({}^{A}F_x \cos\varepsilon \cos\beta - {}^{A}F_z \sin\beta\right) - M_\beta \cos\varepsilon \\
{}^{A}RM_y &= -L_S\left({}^{A}F_x \sin\varepsilon \cos\beta + {}^{A}F_y \sin\beta\right) - M_\beta \sin\varepsilon
\end{aligned}
$$

Recalling the data provided by Figure 2.7 and the results obtained in (2.57)

$$
{}^{A}\underline{F} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} N \qquad L_S = 0.25m \qquad
\begin{aligned}
M_\varepsilon &= 0.15 N \cdot m \qquad \varepsilon = 72.5^\circ \\
M_\beta &= 0.30 N \cdot m \qquad \beta = -71.7^\circ
\end{aligned}
$$

The reaction force and reaction moment can be obtained from (2.72). Their numerical values are

$${}^{A}R_x = 2N$$

$${}^{A}R_y = 0N$$

$${}^{A}R_z = 2N$$

$${}^{A}RM_x = 0N \cdot m$$

$${}^{A}RM_y = 0.432N \cdot m$$

$${}^{A}RM_z = -0.136N \cdot m$$

2.6 Verification of the Numerical Results

As it will be shown in the next chapter the analysis of tensegrity structures involves very complex and lengthy equations. If there is an error in the derivation of the equation the numerical methods still give an answer. However the answer does not of course correspond to the real situation.

It is desirable to verify the validity of the answers obtained using the virtual work approach. Newton's Third Law assists the verification. Basically the idea is to state the equilibrium equation in such a way that some of the reactions vanish. The resultant equation depends only on the input data and on the generalized coordinates. If the numerical values of the generalized coordinates obtained using the virtual work approach are correct, they must satisfy the equilibrium equations obtained using the Newtonian approach. These concepts are demonstrated using the last example.

The equilibrium equation (2.43) in the C system for the strut of Section 2.5 is

$$^{C}\underline{\$}_{F} + {}^{C}\underline{\$}_{M} + {}^{C}\underline{\$}_{R} + {}^{C}\underline{\$}_{RM} = \underline{0} \tag{2.74}$$

$^{C}\underline{\$}_{F}$ is obtained expressing $^{A}\underline{\$}_{F}$ in the C system using (2.29) and (2.30) and noting that the term corresponding to the translation displacement is zero

$$^{C}\underline{\$}_{F} = e^{-1}\ {}^{A}\underline{\$}_{F} \tag{2.75}$$

where

$$e^{-1} = \begin{bmatrix} {}^{A}_{C}R^{T} & O_{3} \\ O_{3} & {}^{A}_{C}R^{T} \end{bmatrix} \tag{2.76}$$

${}^{A}_{C}R$ was obtained in (2.67). Substituting the transpose of (2.67) into (2.76) yields

$$e^{-1} = \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & \cos\varepsilon & \sin\varepsilon & & \underline{0} & \\ 0 & -\sin\varepsilon & \cos\varepsilon & & & \\ & & & 1 & 0 & 0 \\ & \underline{0} & & 0 & \cos\varepsilon & \sin\varepsilon \\ & & & 0 & -\sin\varepsilon & \cos\varepsilon \end{bmatrix} \tag{2.77}$$

${}^{A}\underline{\$}_{F}$ is given by (2.60). Substituting (2.77) and (2.60) into (2.75) yields

$${}^{C}\underline{\$}_{F} = \begin{bmatrix} {}^{A}F_{x} \\ {}^{A}F_{y}\cos\varepsilon + {}^{A}F_{z}\sin\varepsilon \\ -{}^{A}F_{y}\sin\varepsilon + {}^{A}F_{z}\cos\varepsilon \\ -L_{S}\cos\beta\ \left({}^{A}F_{y}\cos\varepsilon + {}^{A}F_{z}\sin\varepsilon\right) \\ L_{S}\left({}^{A}F_{x}\cos\beta + {}^{A}F_{y}\sin\varepsilon\sin\beta - {}^{A}F_{z}\cos\varepsilon\sin\beta\right) \\ L_{S}\sin\beta\ \left({}^{A}F_{y}\cos\varepsilon + {}^{A}F_{z}\sin\varepsilon\right) \end{bmatrix} \tag{2.78}$$

${}^{C}\underline{\$}_{M}$ is given by the Plücker coordinates of $\underline{M}_{\varepsilon}$ and $\underline{M}_{\beta}$, equations (2.69) and (2.70)

$${}^{C}\underline{\$}_{M} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -M_{\varepsilon} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ M_{\beta} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -M_{\varepsilon} \\ M_{\beta} \\ 0 \end{bmatrix} \tag{2.79}$$

${}^{C}\underline{\$}_{R}$ is given by the Plücker coordinates of a force passing through the origin of the C system, therefore it always has the form

$$
{}^{C}\underline{\$}_{R} = \begin{bmatrix} {}^{C}Rx \\ {}^{C}Ry \\ {}^{C}Rz \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.80}
$$

Finally in the system C the universal joint cannot provide moment reactions along its moving axes, then ${}^{C}\underline{\$}_{RM}$ has the form

$$
{}^{C}\underline{\$}_{RM} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ {}^{C}RMz \end{bmatrix} \tag{2.81}
$$

Substituting (2.78), (2.79), (2.80) and (2.81) into (2.74) yields

$$
\begin{bmatrix} {}^{A}F_x \\ {}^{A}F_y \cos\varepsilon + {}^{A}F_z \sin\varepsilon \\ -{}^{A}F_y \sin\varepsilon + {}^{A}F_z \cos\varepsilon \\ -L_S \cos\beta \left({}^{A}F_y \cos\varepsilon + {}^{A}F_z \sin\varepsilon\right) \\ L_S\left({}^{A}F_x \cos\beta + {}^{A}F_y \sin\varepsilon \sin\beta - {}^{A}F_z \cos\varepsilon \sin\beta\right) \\ L_S \sin\beta \left({}^{A}F_y \cos\varepsilon + {}^{A}F_z \sin\varepsilon\right) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -M_\varepsilon \\ M_\beta \\ 0 \end{bmatrix} + \begin{bmatrix} {}^{C}R_x \\ {}^{C}R_y \\ {}^{C}R_z \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ {}^{C}RM_z \end{bmatrix} = \underline{0} \tag{2.82}
$$

From the forth and fifth rows in (2.82) is possible to define g_1 and g_2 as

$$
g_1 = -L_S \cos\beta \left({}^{A}F_y \cos\varepsilon + {}^{A}F_z \sin\varepsilon\right) - M_\varepsilon \tag{2.83}
$$

$$
g_2 = L_S \left({}^{A}F_x \cos\beta + {}^{A}F_y \sin\varepsilon \sin\beta - {}^{A}F_z \cos\varepsilon \sin\beta\right) + M_\beta \tag{2.84}
$$

Equations (2.83) and (2.84) involve only the input data and the generalized coordinates □ and □ whose values are known from the virtual work approach. After substituting □ and □ and the input data into (2.83) and (2.84), g_1 and g_2 must be zero if the values of □ and □ correspond to an equilibrium position.

Substituting back the values for L_S, ${}^{A}F_x$, ${}^{A}F_y$, ${}^{A}F_z$, □ and □ given by Figure 2. 7 and (2.57) into the last expressions yields

$$g_1 = -0.25\cos(-71.7^\circ)(-2)\sin(72.5^\circ) - 0.15 = 0$$

$$g_2 = 0.25(-2\cos(-71.7^\circ) - (-2)\cos(72.5^\circ)\sin(-71.7^\circ)) + 0.30 = 0$$

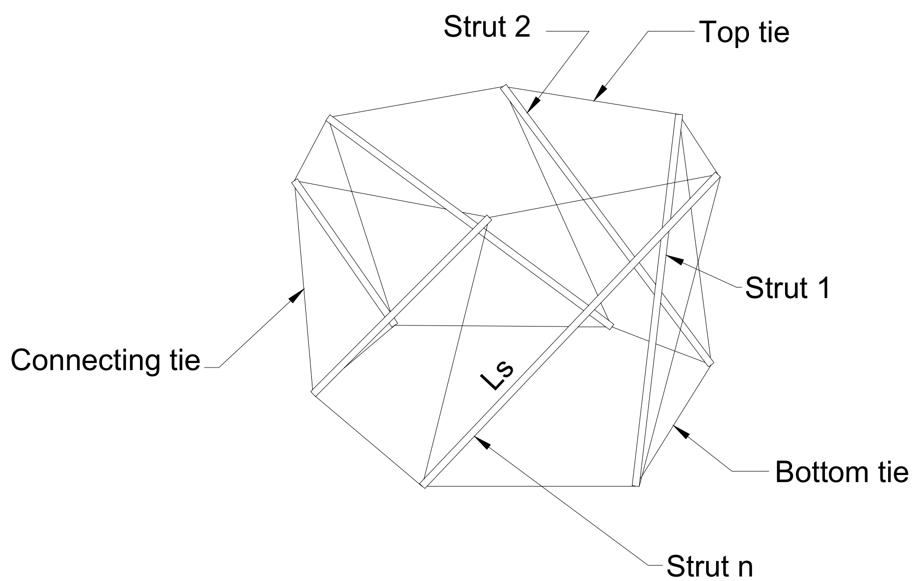
As both g_1 and g_2 vanish, the results obtained using the virtual work for calculating □ and □ correspond to an equilibrium position.

CHAPTER 3
GENERAL EQUATIONS FOR THE STATICS OF TENSEGRITY STRUCTURES

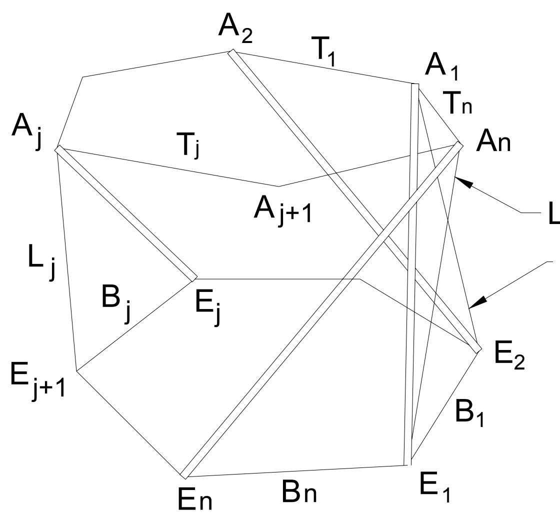
When an external wrench is applied to a tensegrity structure the ties are deformed and the struts go to a new equilibrium position. This new position would be perfectly defined using the coordinates of the lower and upper ends of the struts in a global reference system. However they are unknown. Equations are developed in this section using the principle of virtual work to solve this problem.

Tensegrity structures exhibit a pattern in their configuration and it is possible to take advantage of that situation to generate general equations for the static analysis. Before starting to implement the method it is necessary to establish the nomenclature for the system and some assumptions to simplify the problem.

Figure 3.1a shows a tensegrity structure conformed by n struts each one of length L_S. Figure 3.1b shows the same structure but with only some of its struts. The selection of the first strut is arbitrary but once it is chosen it should not be changed. The bottom ends of the strut are labeled consecutively as $E_1,\ E_2,\ \cdots\ ,E_j,\ \cdots\ ,E_n$ where 1 identifies the first strut and n stands for the last strut. Similarly the top ends of the struts are labeled as $A_1,\ A_2,\ \cdots\ ,A_j,\ \cdots\ ,A_n$, as shown in Figure 3.1 b.



(a)



(b)

Figure 3.1. Nomenclature for tensegrity structures.
a) Generic names; b) Specific nomenclature.

In every structure it is possible to identify the top ties, the bottom ties and the lateral or connecting ties, as shown in Figure 3.1a. The current length of the top, bottom and lateral ties are called T, B and L respectively.

The top tie T_j extends between the top ends $\underline{A}_j$ and $\underline{A}_{j+1}$ if $j < n$ and between $\underline{A}_n$ and $\underline{A}_1$ if $j = n$.

The bottom tie B_j extends between the bottom ends $\underline{E}_j$ and $\underline{E}_{j+1}$ if $j < n$ and between $\underline{E}_n$ and $\underline{E}_1$ if $j = n$.

The lateral tie L_j extends between the top end $\underline{A}_j$ and the bottom end $\underline{E}_{j+1}$ if $j < n$ and between $\underline{A}_n$ and $\underline{E}_1$ if $j = n$.

In Section 2.3 it was established that the motion of an arbitrary strut can be described by modeling its lower end with a universal joint constrained to move in the horizontal plane. The same model is used now for the derivation of the equilibrium equations for a general tensegrity structure. In addition the following assumptions are made without loss of generality:

- The external moments are applied along the axes of the universal joints.
- The struts are massless.
- All the struts have the same length.
- Only one external force is applied per strut.
- There are no dissipative forces acting on the system.
- All the ties are in tension at the equilibrium position; i.e., the current lengths of the ties are longer than their respective free lengths.
- The free lengths of the top ties are equal.
- The free lengths of the bottom ties are equal.

- The free lengths of the connecting ties are equal.
- There are no interferences between struts.
- The stiffness of all the top ties is the same.
- The stiffness of all the bottom ties is the same.
- The stiffness of all the connecting ties is the same.
- The bottom ends of the strut remain in the horizontal plane for all the positions of the structure.

3.1 Generalized Coordinates

Due to the fact the lower end of each strut is constrained to move in the horizontal plane and there is no motion along the longitudinal axis since it is constrained by a universal joint, each strut has four degrees of freedom and the total system has $4*n$ degrees of freedom which means there are $4*n$ generalized coordinates.

For each strut the generalized coordinates are the horizontal displacements a_j, b_j, as illustrated in Figure 3.2, of the lower end of the strut together with two rotations about the axes of the universal joint. The angular coordinates associated with the strut j are ε_j and β_j where ε_j corresponds to the rotation of the strut about the current ${}^{B}\underline{x}$ axis and β_j corresponds to the rotation about ${}^{C}\underline{y}$ axis, as it was shown in Figures 2.4 and 2.5. Table 3.1 shows the generalized coordinates associated with each strut.

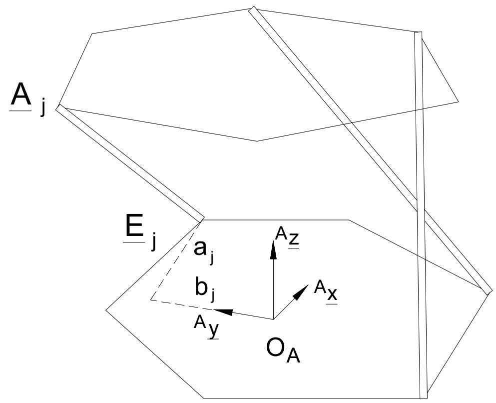


Figure 3.2. Coordinates of the ends of a strut in the global reference system A with reference point O_A.

Table 3.1. Generalized coordinates associated with each strut.

Strut	Generalized coordinates			
1	a_1	b_1	ε_1	β_1
2	a_2	b_2	ε_2	β_2
	$\vdots$	$\vdots$	$\vdots$	$\vdots$
j	a_j	b_j	ε_j	β_j
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	a_n	b_n	ε_n	β_n

3.2 The Principle of Virtual Work for Tensegrity Structures

Equations (2.10) and (2.11) of Section 2.1 established the conditions for the equilibrium of a system of rigid bodies. The notation used there assumes that the generalized coordinates are grouped in a vector $\underline{q}$ such that $\underline{q} = (q_1, q_2, ... , q_p)$ where p is the number of generalized coordinates. However, since the notation used for the tensegrity structures differs from Section 2.1, there is only one external force per strut and the moments act only along the axes of the universal joint it is more convenient to state the equilibrium equations using the current notation and taking in account the simplifications introduced here.

From (2.3)

$$\delta W = \delta W_{nc} + \delta W_c \tag{3.1}$$

where δW is the total virtual work, δW_{nc} is the virtual work performed for non-conservative forces and moments and δW_c is the virtual work performed by conservative forces. δW_{nc} can be represented as

$$\delta W_{nc} = \delta W_F + \delta W_M \tag{3.2}$$

where δW_F is the total virtual work performed by non-conservative forces and δW_M is the total virtual work performed by non-conservative moments.

In (2.6) was established that the virtual work performed by the conservative force j, δW_{cj} is $\delta W_{cj} = -\delta V_j$ where δV_j is the potential energy associated with the conservative force j, therefore the total contribution of the conservatives forces δW_c is

$$\delta W_c = -\delta V \tag{3.3}$$

where δV is the summation over all the δV_j present in the structure.

Substituting (3.2) and (3.3) into (3.1) yields

$$\delta W = \delta W_F + \delta W_M - \delta V \tag{3.4}$$

In equilibrium the virtual work described by (3.4) must be zero, then the equilibrium conditions can be deduced from

$$\delta W_F \;+\; \delta W_M - \;\delta V \;=\; 0 \tag{3.5}$$

In what follows each term in the expression (3.5) will be determined.

3.3 Coordinates of the Ends of the Struts

The coordinates of the lower ends can be expressed directly in the global reference system A. The linear displacements associated with the strut j are a_j and b_j, they correspond to the coordinates x, y measured in the ${}^A\underline{x}\,{}^A\underline{y}\,{}^A\underline{z}$ system. Therefore the coordinates of the lower end $\underline{E}_j$ expressed in the global reference system A, (see Figure 3.2), are simply

$${}^A\underline{E}_j \;=\; \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} \tag{3.6}$$

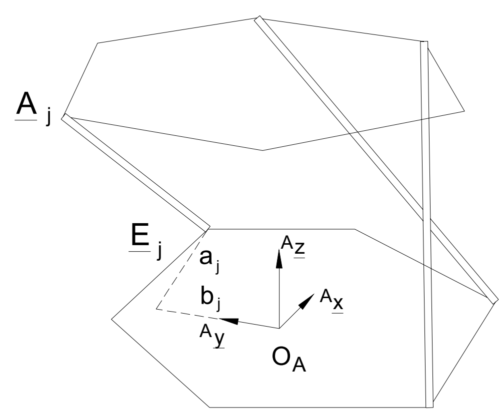


Figure 3.2. Coordinates of the ends of a strut in the global reference system A with reference point O_A.

The coordinates of the upper end of the strut are evaluated with the aid of equation (2.36),

$$ {}^{A}\underline{P}_{2} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l\sin\beta + a \\ -l\sin\varepsilon\cos\beta + b \\ l\cos\varepsilon\cos\beta \\ 1 \end{bmatrix} \tag{2.36} $$

When the angles □ and □ for the j-th strut are replaced by ε_j and β_j respectively and l is replaced by L_S, (2.36) yields

$$ {}^{A}\underline{A}_{j} = \begin{bmatrix} L_s \sin\beta_j + a_j \\ -L_s \sin\varepsilon_j \cos\beta_j + b_j \\ L_s \cos\varepsilon_j \cos\beta_j \end{bmatrix} \tag{3.7} $$

Now it is possible to obtain expressions for the lengths of the top, bottom and lateral ties.

The lengths of the top ties T are given by

$$ T_1 = \left((A_{2x} - A_{1x})^2 + (A_{2y} - A_{1y})^2 + (A_{2z} - A_{1z})^2 \right)^{1/2} $$

$$ T_2 = \left((A_{3x} - A_{2x})^2 + (A_{3y} - A_{2y})^2 + (A_{3z} - A_{2z})^2 \right)^{1/2} $$

$$ \vdots $$

$$ T_j = \left((A_{j+1,x} - A_{j,x})^2 + (A_{j+1,y} - A_{j,y})^2 + (A_{j+1,z} - A_{j,z})^2 \right)^{1/2} \tag{3.8} $$

if $j = n$ then $j + 1 = 1$

The lengths of the bottom ties B are given by

$$ B_1 = \left((E_{2x} - E_{1x})^2 + (E_{2y} - E_{1y})^2 + (E_{2z} - E_{1z})^2 \right)^{1/2} $$

$$ B_2 = \left((E_{3x} - E_{2x})^2 + (E_{3y} - E_{2y})^2 + (E_{3z} - E_{2z})^2 \right)^{1/2} $$

$$ \vdots $$

$$ B_j = \left((E_{j+1,x} - E_{j,x})^2 + (E_{j+1,y} - E_{j,y})^2 + (E_{j+1,z} - E_{j,z})^2 \right)^{1/2} \tag{3.9} $$

if $j = n$ then $j+1=1$

The lengths of the lateral ties L are given by

$$L_1 = \left((A_{1x}-E_{2x})^2+(A_{1y}-E_{2y})^2+(A_{1z}-E_{2z})^2\right)^{1/2}$$

$$L_2 = \left((A_{2x}-E_{3x})^2+(A_{2y}-E_{3y})^2+(A_{2z}-E_{3z})^2\right)^{1/2}$$

$$\vdots$$

$$L_j = \left((A_{j,x}-E_{j+1,x})^2+(A_{j,y}-E_{j+1,y})^2+(A_{j,z}-E_{j+1,z})^2\right)^{1/2} \tag{3.10}$$

if $j = n$ then $j+1=1$

3.4 Initial Conditions

In the example of Section 2.5 it was established that the numerical methods are highly sensitive to the selection of the initial values. The problem of the initial position of a tensegrity structure, this is the position of the structure in its unloaded position were addressed by Yin [1]. In this section his results are presented without proof and are adapted to the current nomenclature.

The free lengths of the top and bottom ties and the current lengths of the top and bottom ties satisfy the relations illustrated in Figure 3.3, therefore

$$R_{T0} = \frac{T_o}{2\sin\frac{\gamma}{2}} \tag{3.11}$$

$$R_{B0} = \frac{B_o}{2\sin\frac{\gamma}{2}} \tag{3.12}$$

$$T = 2R_T\sin\frac{\gamma}{2} \tag{3.13}$$

$$B = 2R_B\sin\frac{\gamma}{2} \tag{3.14}$$

where T_0 and B_0 are the free lengths of the top and bottom ties respectively and T and B are the current lengths of the top and bottom ties for the unloaded position. The angle γ depends on the number of struts and is given by

$$\gamma = \frac{2\pi}{n} \tag{3.15}$$

where n is the number of struts

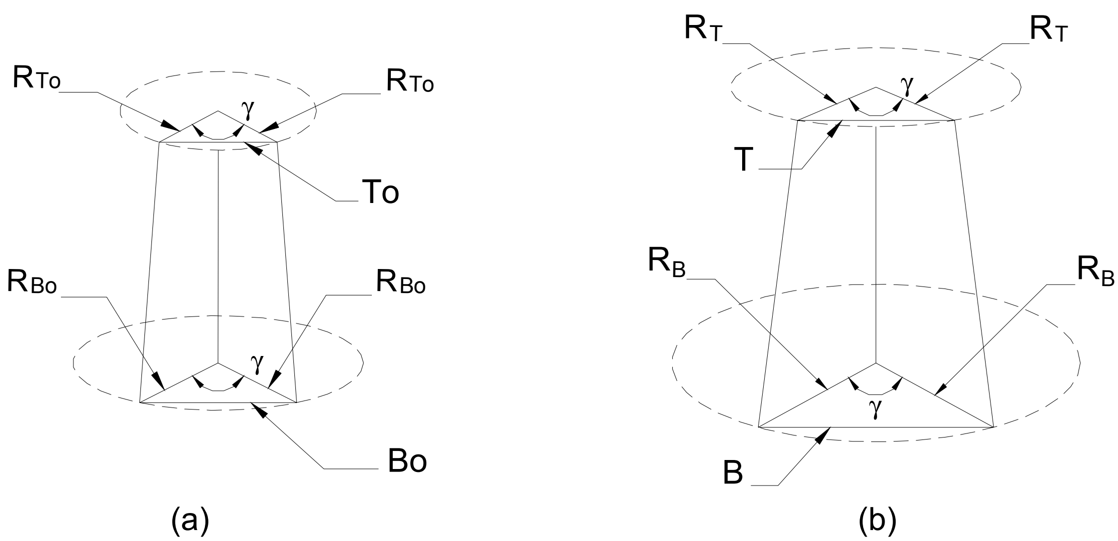


Figure 3.3. Relations for the top and bottom ties of a tensegrity structure.
a) Ties with their free lengths; b) Ties after elongation.

In the unloaded position the quantities R_T, R_B and the current length of the lateral ties L satisfy the following equations

$$k_L \left(1 - \frac{L_o}{L}\right) R_B - 2k_T \left(R_T - R_{To}\right) \sin\frac{\gamma}{2} = 0 \tag{3.16}$$

$$k_L \left(1 - \frac{L_o}{L}\right) R_T - 2k_B \left(R_B - R_{Bo}\right) \sin\frac{\gamma}{2} = 0 \tag{3.17}$$

$$L \; - \; \sqrt{L_s^2 \; + \; 2R_B R_T \; \left[\cos(\alpha+\gamma)-\cos\alpha\right]} \; = 0 \qquad (3.18)$$

where

k_T : stiffness of the top ties.

k_B : stiffness of the bottom ties.

k_L : stiffness of the lateral ties.

L_S : length of the struts.

L_0 : free length of the lateral ties.

α : angle related to the rotation of the polygon conformed by the top end with respect to the polygon conformed by the bottom ends of the struts and is given by

$$\alpha \; = \; \frac{\pi}{2} \; - \; \frac{\pi}{n} \qquad (3.19)$$

The solution of (3.16), (3.17) and (3.18) can be carried out numerically. Once R_B, R_T (and L) have been evaluated the values of T and B are calculated from (3.13) and (3.14).

Summarizing, when the free lengths of the top, bottom and lateral ties of a tensegrity structure are given, together with their stiffness, strut lengths and number of struts, equations (3.16), (3.17) and (3.18) yield the current values of the top, bottom and lateral ties in its unloaded position.

Although in the work of Yin [1], the following relations are not established explicitly, it can be shown that if the global reference system A is oriented in such a way that its x axis passes through the bottom of one of the struts when the structure is in its unloaded position, then the coordinates of the top and lower ends of the strut for its initial position in a global reference system A, (see Figure 3.4), are

$$ {}^{A}\underline{E}_{j_1 o} = \begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos(\ (j-1)\ \gamma) \\ R_B \sin(\ (j-1)\ \gamma) \\ 0 \end{bmatrix} \quad , \ j = 1, \ 2, \ .... \ , \ n \tag{3.20} $$

$$ {}^{A}\underline{A}_{j_1 o} = \begin{bmatrix} R_T \cos(\ (j-1)\ \gamma+\alpha) \\ R_T \sin(\ (j-1)\ \gamma+\alpha) \\ H \end{bmatrix} \quad , \ j = 1, \ 2, \ .... \ , \ n \tag{3.21} $$

where if $j=1$ then $j-1=n$. Further,

$$ H = \sqrt{L_s^2 - R_B^2 - R_T^2 - 2R_B R_T \sin\frac{\gamma}{2}} \tag{3.22} $$

H represents the height between the platform defined by the lower ends of the struts and the platform defined by the upper ends of the struts.

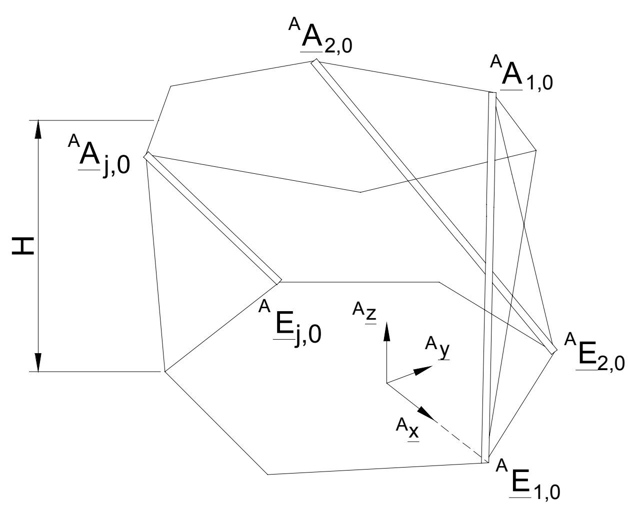


Figure 3.4. Initial position of a tensegrity structure.

Once the coordinates of ${}^{A}\underline{E}_{j,0}$ and ${}^{A}\underline{A}_{j,0}$ are obtained, the initial angles $\varepsilon_{j,0}$ and $\alpha_{j,0}$ corresponding to the rotation of each strut are given by (2.37), (2.38) or (2.39) and,

$$\tan\varepsilon \;=\; \frac{b-y}{z} \tag{2.37}$$

$$\tan\beta \;=\; \frac{x-a}{\left(\dfrac{b-y}{\sin\varepsilon}\right)} \tag{2.38}$$

or

$$\tan\beta \;=\; \frac{x-a}{\left(\dfrac{z}{\cos\varepsilon}\right)} \tag{2.39}$$

where $a,\ b$ are replaced by $a_{j,0},\ b_{j,0}$ given by (3.20) and $(x, y, z)^T$ is replaced by ${}^{A}\underline{A}_{j,0}$ given by (3.21), then

$$\tan\varepsilon_{j,0} \;=\; \frac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{H} \tag{3.23}$$

$$\tan\beta_{j,0} \;=\; \frac{R_T \cos(\ (j-1)\ \ \gamma + \alpha) - a_{j,0}}{\left(\dfrac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{\sin\varepsilon_{j,0}}\right)} \tag{3.24}$$

or

$$\tan\beta_{j,0} \;=\; \frac{R_T \cos(\ (j-1)\ \ \gamma + \alpha) - a_{j,0}}{\left(\dfrac{H}{\cos\varepsilon_{j,0}}\right)} \tag{3.25}$$

3.5 The Virtual Work Due to the External Forces

As it is assumed that there is only one external force acting on each strut, the virtual work δW_F performed by all the external forces is given by

$$\delta W_F = \sum_{j=1}^{n} \underline{F}_j \cdot \delta \underline{r}_j \qquad (3.26)$$

where $\underline{F}_j$ is the external force acting in the strut j, $\underline{r}_j$ is the vector to the point of application of the external force. In (3.26) both $\underline{F}_j$ and $\underline{r}_j$ must be expressed in the same coordinate system. If the system chosen is the global reference system ${}^A\underline{x}\,{}^A\underline{y}\,{}^A\underline{z}$ then the terms satisfying (3.26) have the form

$${}^A\underline{F}_j \cdot {}^A\delta\,\underline{r_j} = {}^AF_{j\ x}\,{}^A\delta r_{j\ x} + {}^AF_{j\ y}\,{}^A\delta r_{j\ y} + {}^AF_{j\ z}\,{}^A\delta r_{j\ z} \qquad (3.27)$$

If the distance between the point of application of the force and the lower end of the strut is L_f, see Figure 3.5, then an expression for $\underline{r}_j$ in the global system can be obtained from (2.36) where the angles □ and □ and the distances a, b, l and are substituted by ε_j, β_j, a_j, b_j and L_{Fj} respectively.

$${}^A\underline{r}_j = \begin{bmatrix} {}^Ar_{j\ x} \\ {}^Ar_{j\ y} \\ {}^Ar_{j\ z} \end{bmatrix} = \begin{bmatrix} L_{Fj}\sin\beta_j + a_j \\ -L_{Fj}\sin\varepsilon_j\cos\beta_j + b_j \\ L_{Fj}\cos\varepsilon_j\cos\beta_j \end{bmatrix} \qquad (3.28)$$

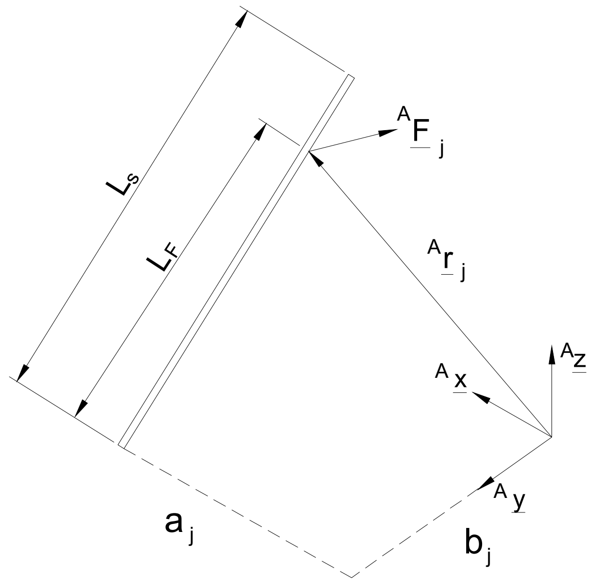


Figure 3.5. Location of the external force acting on the strut j.

The virtual displacements can be deduced from equation (3.28) where the generalized coordinates for the strut j are ε_j, β_j, a_j and b_j

$$\delta^A \underline{r}_j = \begin{bmatrix} ^A\delta r_{j\ x} \\ ^A\delta r_{j\ y} \\ ^A\delta r_{j\ z} \end{bmatrix} = \begin{bmatrix} \delta a_j + L_{Fj}\cos\beta_j\delta\beta_j \\ \delta b_j - L_{Fj}\cos\varepsilon_j\cos\beta_j\delta\varepsilon_j + L_{Fj}\sin\varepsilon_j\sin\beta_j\delta\beta_j \\ -L_{Fj}\sin\varepsilon_j\cos\beta_j\delta\varepsilon_j - L_{Fj}\cos\varepsilon_j\sin\beta_j\delta\beta_j \end{bmatrix} \qquad (3.29)$$

Substituting (3.29) into (3.27) regrouping terms, and substituting into (3.26), the general expression for the virtual work performed by external forces is given by

$$\delta W_F = \sum_{j=1}^{n} (L_{Fj}\left[-^AF_{j\ y}\cos\varepsilon_j\cos\beta_j - ^AF_{j\ z}\sin\varepsilon_j\cos\beta_j\right]\delta\varepsilon_j$$

$$+ L_{Fj}\left[^AF_{j\ x}\cos\beta_j + ^AF_{j\ y}\sin\varepsilon_j\sin\beta_j - ^AF_{j\ z}\cos\varepsilon_j\sin\beta_j\right]\delta\beta_j$$

$$+ ^AF_{j\ x}\delta a_j$$

$$+ ^AF_{j\ y}\delta b_j) \qquad (3.30)$$

3.6 The Virtual Work Due to the External Moments

Provided that in this model of the tensegrity structure the external moments can be exerted only along the axis of the universal joint, the virtual work performed by the external moments is given by

$$\delta W_M = \sum_{j=1}^{n} \underline{M}_{\varepsilon j} \cdot \delta \underline{\varepsilon}_j + \underline{M}_{\beta j} \cdot \delta \underline{\beta}_j \tag{3.31}$$

As before all the elements of equation (3.31) must be expressed in the same coordinate system. However as the scalar product is invariant under transformation of coordinates any convenient coordinate system may be selected. It was established at Section 2.5 that when (3.31) is expressed in a reference system C obtained translating the general reference system to the base of strut j and then rotating by ε_j about the current x axis, (see Figure 3.6) the terms in (3.31) have the form

$$^{C}\underline{M}_{\varepsilon} = M_{\varepsilon j} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \tag{3.32}$$

$$^{C}\underline{\varepsilon}_j = \varepsilon_j \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad then \quad ^{C}\delta \underline{\varepsilon}_j = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \delta\varepsilon_j \tag{3.33}$$

and

$$^{C}\underline{M}_{\beta} = M_{\beta j} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{3.34}$$

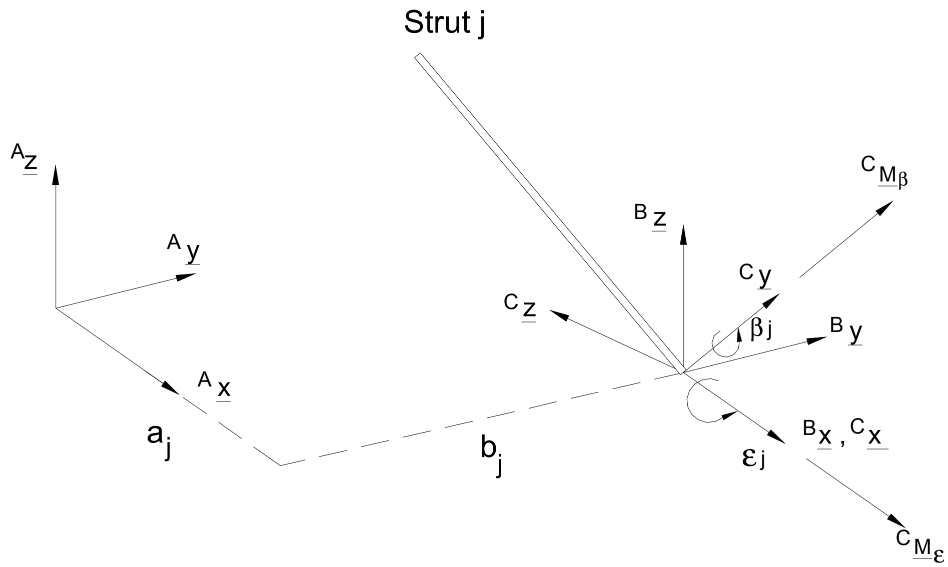


Figure 3.6. External moments and coordinate systems at the base of strut j.

$$ {}^{C}\underline{\beta}_j = \beta_j \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad then \quad {}^{C}\delta\,\underline{\beta}_j = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \delta\beta_j \tag{3.35} $$

Substituting (3.32), (3.33), (3.34) and (3.35) into (3.31) yields

$$ \delta W_M = \sum_{j=1}^{n} M_{\varepsilon\ j}\delta\varepsilon_j + M_{\beta\ j}\delta\beta_j \tag{3.36} $$

3.7 The Potential Energy

Provided that the struts are considered massless the term related to the potential energy in the principle of virtual work is the resultant of the elastic potential energy contributions given by the ties. The potential elastic energy for a general tie j is given by [6]

$$ V_j = \frac{1}{2}k\ (w_j - w_{j0})^2 \tag{3.37} $$

where

V_j : elastic potential energy for tie j

k : tie stiffness

w_j : current length of the tie j

w_{j0} : free length of the tie j

Therefore the differential potential energy for tie j is

$$\delta V_j = k \ (w_j - w_{j0}) \ \delta w_j \tag{3.38}$$

The differential of the potential energy for all the tensegrity structure, δV, is the resultant of the contributions of the top ties, the bottom ties and the lateral ties and can be expressed as

$$\delta V = \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \delta T_j + \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \delta B_j + \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \delta L_j \tag{3.39}$$

where k_T, k_B, k_L are the stiffness of the top, bottom and lateral ties respectively.

The current lengths of the ties are functions of some sets of the generalized coordinates for the structure, shown in (3.40)

$$\begin{aligned} T_j &= T_j \ \left(a_1, \ b_1, \ \varepsilon_1, \ \beta_1, \ a_2, \ b_2, \ \varepsilon_2, \ \beta_2, \ ..... \ , \ a_n, \ b_n, \ \varepsilon_n, \ \beta_n\right) \\ B_j &= B_j \ \left(a_1, \ b_1, \ \varepsilon_1, \ \beta_1, \ a_2, \ b_2, \ \varepsilon_2, \ \beta_2, \ ..... \ , \ a_n, \ b_n, \ \varepsilon_n, \ \beta_n\right) \\ L_j &= L_j \ \left(a_1, \ b_1, \ \varepsilon_1, \ \beta_1, \ a_2, \ b_2, \ \varepsilon_2, \ \beta_2, \ ..... \ , \ a_n, \ b_n, \ \varepsilon_n, \ \beta_n\right) \end{aligned} \tag{3.40}$$

Therefore (3.39) can be expanded in the form

$$\delta V = \sum_{j=1}^{n} k_T\left(T_j - T_0\right) \; *$$
$$\left[\frac{T_j}{\partial a_1}\delta a_1 + \frac{T_j}{\partial b_1}\delta b_1 + \frac{T_j}{\partial \varepsilon_1}\delta \varepsilon_1 + \frac{T_j}{\partial \beta_1}\delta \beta_1 + \cdots + \frac{T_j}{\partial a_n}\delta a_n + \frac{T_j}{\partial b_n}\delta b_n + \frac{T_j}{\partial \varepsilon_n}\delta \varepsilon_n + \frac{T_j}{\partial \beta_n}\delta \beta_n\right]$$
$$+\sum_{j=1}^{n} k_B\left(B_j - B_0\right) \; *$$
$$\left[\frac{\partial B_j}{\partial a_1}\delta a_1 + \frac{\partial B_j}{\partial b_1}\delta b_1 + \frac{\partial B_j}{\partial \varepsilon_1}\delta \varepsilon_1 + \frac{\partial B_j}{\partial \beta_1}\delta \beta_1 + \cdots + \frac{\partial B_j}{\partial a_n}\delta a_n + \frac{\partial B_j}{\partial b_n}\delta b_n + \frac{\partial B_j}{\partial \varepsilon_n}\delta \varepsilon_n + \frac{\partial B_j}{\partial \beta_n}\delta \beta_n\right]$$
$$+\sum_{j=1}^{n} k_L\left(L_j - L_0\right) \; *$$
$$\left[\frac{\partial L_j}{\partial a_1}\delta a_1 + \frac{\partial L_j}{\partial b_1}\delta b_1 + \frac{\partial L_j}{\partial \varepsilon_1}\delta \varepsilon_1 + \frac{\partial L_j}{\partial \beta_1}\delta \beta_1 + \cdots + \frac{\partial L_j}{\partial a_n}\delta a_n + \frac{\partial L_j}{\partial b_n}\delta b_n + \frac{\partial L_j}{\partial \varepsilon_n}\delta \varepsilon_n + \frac{\partial L_j}{\partial \beta_n}\delta \beta_n\right]$$

(3.41)

3.8 The General Equations

Now that each one of the terms contributing to the virtual work has been evaluated, the equilibrium condition for the general tensegrity structure can be established. Substituting (3.30), (3.36) and (3.41) into (3.5) and re-grouping yields

$$f_1\delta a_1 \; + \; f_2\delta a_2 \; + \; ..... \; f_n\delta a_n$$
$$f_{n+1}\delta b_1 \; + \; f_{n+2}\delta b_2 \; + \; ..... \; f_{2n}\delta b_n$$
$$f_{2n+1}\delta \varepsilon_1 \; + \; f_{2n+2}\delta \varepsilon_2 \; + \; ..... \; f_{3n}\delta \varepsilon_n$$
$$f_{3n+1}\delta \beta_1 \; + \; f_{3n+2}\delta \beta_2 \; + \; ..... \; f_{4n}\delta \beta_n \qquad = 0 \qquad (3.42)$$

where

$$
\begin{aligned}
f_i \quad = \quad & {}^{A}F_{i\ x} \\
& - \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial a_i} \\
& - \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial a_i} \\
& - \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial a_i}
\end{aligned}
\tag{3.43}
$$

$i = 1,\ 2,\ .\ .\ .\ .\ ,\ n$

$$
\begin{aligned}
f_{n+i} \quad = \quad & {}^{A}F_{i\ y} \\
& - \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial b_i} \\
& - \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial b_i} \\
& - \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial b_i}
\end{aligned}
\tag{3.44}
$$

$i = 1,\ 2,\ .\ .\ .\ .\ ,\ n$

$$f_{2n+i} \;=\; L_{Fi}\;\left[-{}^{A}F_{i\;y}\cos\varepsilon_i\cos\beta_i - {}^{A}F_{i\;z}\sin\varepsilon_i\cos\beta_i\right]$$

$$+\;M_{\varepsilon\;i}$$

$$-\;\sum_{j=1}^{n}\;k_T\;\left(T_j - T_o\right)\frac{\partial T_j}{\partial \varepsilon_i}$$

$$-\;\sum_{j=1}^{n}\;k_B\;\left(B_j - B_o\right)\frac{\partial B_j}{\partial \varepsilon_i}$$

$$-\;\sum_{j=1}^{n}\;k_L\;\left(L_j - L_o\right)\frac{\partial L_j}{\partial \varepsilon_i} \qquad (3.45)$$

$i = 1,\;\; 2,\;\; .\;\; .\;\; .\;\; .\;\; ,\;\; n$

$$f_{3n+i} \;=\; L_{Fi}\;\left[{}^{A}F_{i\;x}\cos\beta_i + {}^{A}F_{i\;y}\sin\varepsilon_i\sin\beta_i - {}^{A}F_{i\;z}\cos\varepsilon_i\sin\beta_i\right]$$

$$+\;M_{\beta\;i}$$

$$-\;\sum_{j=1}^{n}\;k_T\;\left(T_j - T_o\right)\;\frac{\partial T_j}{\partial \beta_i}$$

$$-\;\sum_{j=1}^{n}\;k_B\;\left(B_j - B_o\right)\;\frac{\partial B_j}{\partial \beta_i}$$

$$-\;\sum_{j=1}^{n}\;k_L\;\left(L_j - L_o\right)\;\frac{\partial L_j}{\partial \beta_i} \qquad (3.46)$$

$i = 1,\;\; 2,\;\; .\;\; .\;\; .\;\; .\;\; ,\;\; n$

Equation (3.42) must be satisfied for all the values of the generalized coordinates which in general are different from zero, then

$$\begin{aligned} & f_1 = 0 \\ & f_2 = 0 \\ & \quad \vdots \\ & \quad \vdots \\ & f_{4n} = 0 \end{aligned} \tag{3.47}$$

where f_i is given by equations (3.43) to (3.46). Equations (3.47) represent a strongly coupled system of $4*n$ equations depending only on the $4*n$ generalized coordinates. The solution is obtained numerically. The initial conditions for $a_j,\ b_j,\ \varepsilon_j$ and β_j are given by (3.20), (3.23), (3.24) or (3.25).

The equilibrium position for a general tensegrity structure is obtained by solving the set (3.47) for $a_1,\ b_1,\ \varepsilon_1,\ \beta_1,\ \ldots\ldots\ldots,\ a_n,\ b_n,\ \varepsilon_n,\ \beta_n$. Equations (3.6) and (3.7) are explicit expressions for the coordinates of the ends of the struts in the global coordinate system.

CHAPTER 4
NUMERICAL RESULTS

This chapter presents the methodology to find the equilibrium position for tensegrity structures. Three numerical examples are provided to illustrate the concepts discussed in the previous sections. Tensegrity structures with different number of struts and different external loads are analyzed. Each example is developed in detail until to obtain the numerical solutions. In addition to the numerical results, the graphics of the structures in their equilibrium positions are also provided.

The static analysis is performed in two steps: initially the equilibrium position of the structure in its unloaded position is evaluated, then the external loads are considered and the new equilibrium position is found.

The numerical results are obtained here by evaluating some of the equations what where derived in detail in Chapter 3. The author has repeated some of these equations in the present chapter for convenience in order to minimize repeated reference to the pages of Chapter 3.

4.1 Analysis of Tensegrity Structures in their Unloaded Positions.

When there are no external loads applied, the equilibrium position can be determined using Yin's results. Numerical values are given in Section 4.3. In order to determine the unloaded equilibrium position the lengths of the struts are specified, L_S, which are assumed to be all the same, together with the stiffness of the top ties k_T (assumed equal), bottom ties k_B (assumed equal), connecting ties k_L (assumed equal) and the free lengths of the top ties T_0 (assumed equal), bottom ties B_0 (assumed equal) and connecting ties L_0 (assumed equal).

In order to find the equilibrium position of the structure in its unloaded position, the coordinates of the ends of all the struts measured in a global reference system are determined. This is accomplished first by computing the three unknowns R_B, R_T and the length of the connecting ties L in the following equations given in Chapter 3 (see also Figures 3.3 (a) and (b)).

$$k_L \left(1 - \frac{L_o}{L}\right) R_B - 2k_T \left(R_T - R_{To}\right) \sin\frac{\gamma}{2} = 0 \tag{3.16}$$

$$k_L \left(1 - \frac{L_o}{L}\right) R_T - 2k_B \left(R_B - R_{Bo}\right) \sin\frac{\gamma}{2} = 0 \tag{3.17}$$

$$L - \sqrt{L_s^2 + 2R_B R_T \left[\cos(\alpha+\gamma) - \cos\alpha\right]} = 0 \tag{3.18}$$

where

$$R_{T0} = \frac{T_o}{2\sin\frac{\gamma}{2}} \tag{3.11}$$

$$R_{B0} = \frac{B_o}{2\sin\frac{\gamma}{2}} \tag{3.12}$$

And the angles γ and α are given by

$$\gamma = \frac{2\pi}{n} \tag{3.15}$$

$$\alpha = \frac{\pi}{2} - \frac{\pi}{n} \tag{3.19}$$

where n is the number of struts.

The values of R_B and R_T are then substituted into the equations (3.20), (3.21) and (3.22) which yield the coordinates ${}^{A}\underline{E}_{j,o}$ and ${}^{A}\underline{A}_{j,o}$, this is the coordinates of the lower and the upper ends of the struts in the global reference system A respectively. Note that the sub-index 0 indicates the unloaded position.

$${}^{A}\underline{E}_{j,0} = \begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos(\ (j-1)\ \gamma) \\ R_B \sin(\ (j-1)\ \gamma) \\ 0 \end{bmatrix} \quad , \ j = 1,\ 2,\ ....\ ,\ n \tag{3.20}$$

$${}^{A}\underline{A}_{j,0} = \begin{bmatrix} R_T \cos(\ (j-1)\ \gamma+\alpha) \\ R_T \sin(\ (j-1)\ \gamma+\alpha) \\ H \end{bmatrix} \quad , \ j = 1,\ 2,\ ....\ ,\ n \tag{3.21}$$

where if $j=1$ then $j-1=n$, and

$$H = \sqrt{L_s^2 - R_B^2 - R_T^2 - 2R_B R_T \sin\frac{\gamma}{2}} \tag{3.22}$$

4.2 Analysis of Loaded Tensegrity Structures

The external loads acting on a tensegrity structure may be external forces and external moments. According to the restrictions of this study, only one external force and two external moments may be applied per strut. In addition the directions of the external moments are along the axis of the universal joint used to model the strut.

To be able to perform the static analysis the components $(F_x,\ F_y,\ F_z)$ and the point of application L_F measured along the strut for each force must be known, together with the directions of the external moments M_ε and M_β, see Figures 4.1 and 3.5.

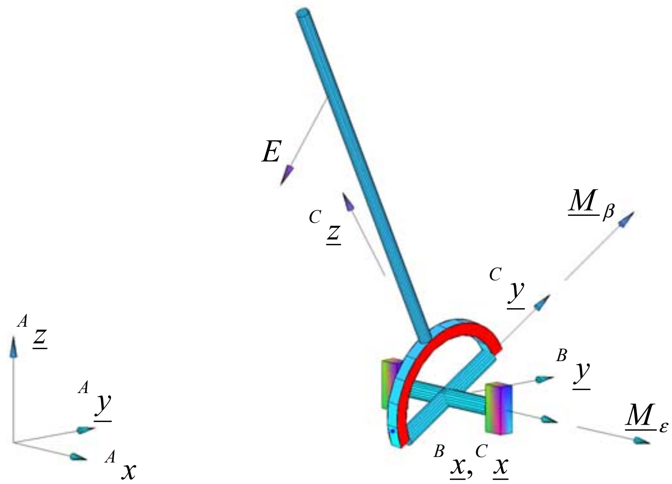


Figure 4.1. External loads applied to one of the struts of a tensegrity structure.

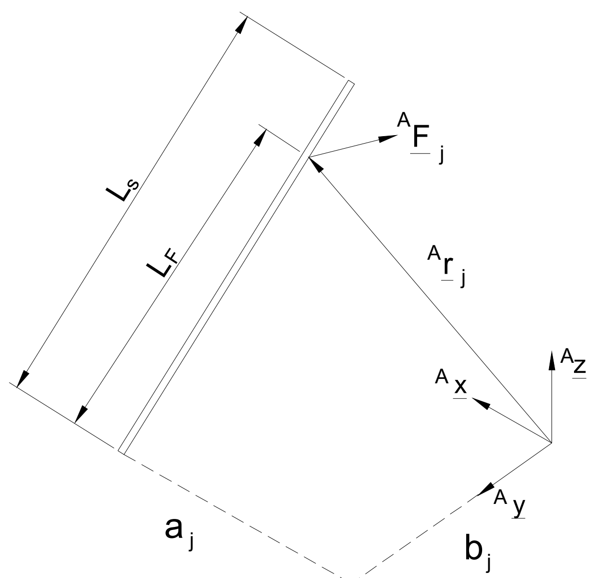


Figure 3.5. Location of the external force acting on the strut j.

Any strut of a tensegrity structure constrained to remain on the horizontal plane has four degrees of freedom, two associated with its longitudinal displacements a and b, and two associated with its rotations ε and β, see Figure 4.2. Therefore the whole structure posses $4*n$ degrees of freedom where n is the number of struts. However if some of the freedoms of the system are constrained the degrees of freedom decrease. Hence in addition to the knowledge of the external loads it is necessary to know the number of freedoms of the structure.

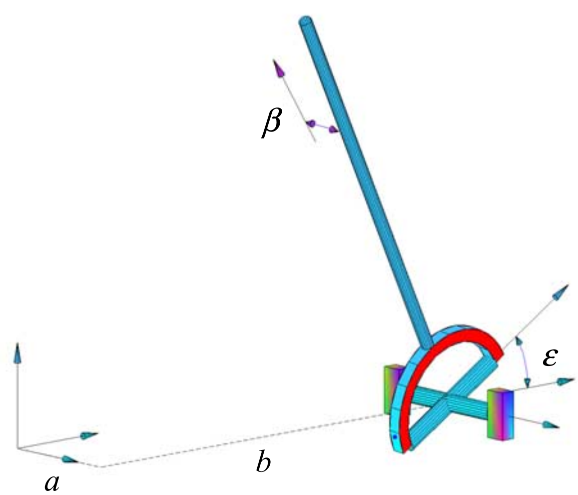


Figure 4.2. Degrees of freedom associated with one of the struts of a tensegrity structure.

The equilibrium position of the structure is determined for a system of p equations where p is the number of freedoms of the system. These equations are obtained by expanding equations (3.43) through (3.46) for each one of the generalized coordinates of the system.

$$f_i = {}^{A}F_{i\ x}$$

$$- \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial a_i}$$

$$- \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial a_i}$$

$$- \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial a_i} \qquad (3.43)$$

$$i = 1,\ 2,\ .\ .\ .\ .\ ,\ n$$

$$f_{n+i} = {}^{A}F_{i\ y}$$

$$- \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial b_i}$$

$$- \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial b_i}$$

$$- \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial b_i} \qquad (3.44)$$

$$i = 1,\ 2,\ .\ .\ .\ .\ ,\ n$$

$$f_{2n+i} = L_{Fi} \left[-{}^{A}F_{i\ y} \cos\varepsilon_i \cos\beta_i - {}^{A}F_{i\ z} \sin\varepsilon_i \cos\beta_i \right]$$

$$+ M_{\varepsilon\ i}$$

$$- \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial \varepsilon_i}$$

$$- \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial \varepsilon_i}$$

$$- \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial \varepsilon_i} \tag{3.45}$$

$i = 1,\ 2,\ .\ .\ .\ .\ ,\ n$

$$f_{3n+i} = L_{Fi} \left[{}^{A}F_{i\ x} \cos\beta_i + {}^{A}F_{i\ y} \sin\varepsilon_i \sin\beta_i - {}^{A}F_{i\ z} \cos\varepsilon_i \sin\beta_i \right]$$

$$+ M_{\beta\ i}$$

$$- \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial \beta_i}$$

$$- \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial \beta_i}$$

$$- \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial \beta_i} \tag{3.46}$$

$i = 1,\ 2,\ .\ .\ .\ .\ ,\ n$

As the resultant system must be solved numerically then the initial values of the generalized coordinates must be evaluated prior to the implementation of the numerical method. The generalized coordinates $a_{j,0}$, $b_{j,0}$, $\varepsilon_{j,0}$ and $\beta_{j,0}$

corresponding to the initial values for the strut j are obtained from equations (3.20), (3.23) and (3.24)

$$\begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos\left((j-1)\ \gamma \right) \\ R_B \sin\left((j-1)\ \gamma \right) \\ 0 \end{bmatrix} \quad , \ j \ = \ 1,\ 2,\ ....\ ,\ n \tag{3.20}$$

$$\tan \varepsilon_{j,0} \ = \ \frac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{H} \tag{3.23}$$

$$\tan \beta_{j,0} \ = \ \frac{R_T \cos(\ (j-1)\ \ \gamma + \alpha) - a_{j,0}}{\left(\dfrac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{\sin \varepsilon_{j,0}} \right)} \tag{3.24}$$

And all the terms in (3.20), (3.23) and (3.24) have been defined previously.

Now the equations can be solved and numerical values for the generalized coordinates a_j, b_j, ε_j and β_j are obtained, therefore the equilibrium position for the tensegrity structure has been found.

In order to enhance the performance of the numerical method it is advisable to increase the external loads gradually in a step by step procedure. In this way the generalized coordinates evaluated at each step are the initial values for the next step.

Equations (3.6) and (3.7) determine the coordinates of the lower and upper ends of the struts, ${}^{A}\underline{E}_j$ and ${}^{A}\underline{A}_j$ respectively, in the global reference system A.

$$ {}^{A}\underline{E}_j = \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} \quad (3.6) $$

$$ {}^{A}\underline{A}_j = \begin{bmatrix} L_s \sin\beta_j + a_j \\ -L_s \sin\varepsilon_j \cos\beta_j + b_j \\ L_s \cos\varepsilon_j \cos\beta_j \end{bmatrix} \quad (3.7) $$

4.3 Example 1: Analysis of a Tensegrity Structure with 3 Struts

4.3.1 Analysis for the Unloaded Position.

A tensegrity structure with 3 struts has the stiffness and free lengths shown in Table 4.1. Each of its struts has a length $L_s = 100mm$. It is required to evaluate its unloaded equilibrium position.

Table 4.1. Stiffness and free lengths for the structure of example 1.

	Stiffness (N/mm)	Free lengths (mm)
Top ties	$k_T = 0.5$	$T_0 = 35$
Bottom ties	$k_B = 0.3$	$B_0 = 52$
Connecting ties	$k_L = 1$	$L_0 = 80$

The solution of the system

$$ k_L \left(1 - \frac{L_o}{L}\right) R_B - 2k_T \left(R_T - R_{To}\right) \sin\frac{\gamma}{2} = 0 \quad (3.16) $$

$$ k_L \left(1 - \frac{L_o}{L}\right) R_T - 2k_B \left(R_B - R_{Bo}\right) \sin\frac{\gamma}{2} = 0 \quad (3.17) $$

$$ L - \sqrt{L_s^2 + 2R_B R_T \left[\cos(\alpha+\gamma) - \cos\alpha\right]} = 0 \quad (3.18) $$

where

$$\gamma = \frac{2\pi}{n} = \frac{360°}{3} = 120° \tag{3.15}$$

$$\alpha = \frac{\pi}{2} - \frac{\pi}{n} = \frac{90°}{2} - \frac{90°}{3} = 15° \tag{3.19}$$

$$R_{T0} = \frac{T_o}{2\sin\frac{\gamma}{2}} = \frac{35 \;\; mm}{2\sin 60°} = 20.207 \;\; mm \tag{3.11}$$

$$R_{B0} = \frac{B_o}{2\sin\frac{\gamma}{2}} = \frac{52 \;\; mm}{2\sin 60°} = 32.02 \;\; mm \tag{3.12}$$

yields

$$R_B = 33.0568 \;\; mm$$

$$R_T = 22.8422 \;\; mm$$

The coordinates of the ends of the struts for the unloaded position are obtained from

$$^{A}\underline{E}_{j,0} = \begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos(\ (j-1)\ \gamma) \\ R_B \sin(\ (j-1)\ \gamma) \\ 0 \end{bmatrix} \quad , \ j = 1,\ 2,\ ....\ ,\ n \tag{3.20}$$

$$^{A}\underline{A}_{j,0} = \begin{bmatrix} R_T \cos(\ (j-1)\ \gamma + \alpha) \\ R_T \sin(\ (j-1)\ \gamma + \alpha) \\ H \end{bmatrix} \quad , \ j = 1,\ 2,\ ....\ ,\ n \tag{3.21}$$

where if $j = 1$ then $j-1 = n$, and

$$H = \sqrt{L_s^2 - R_B^2 - R_T^2 - 2R_B R_T \sin\frac{\gamma}{2}} = 84.1287 \;\; mm$$

The results are summarized in Table 4.2. Figure 4.3 shows the structure in its unloaded position.

Table 4.2. Lower and upper coordinates for the unloaded position for the structure of example 1 (mm).

	Strut 1	Strut 2	Strut 3
E_x	33.0568	-16.5284	-16.5284
E_y	0	28.6280	-28.6280
E_z	0	0	0
A_x	-19.7819	0	19.7819
A_y	11.4211	-22.8422	11.4211
A_z	84.1287	84.1287	84.1287

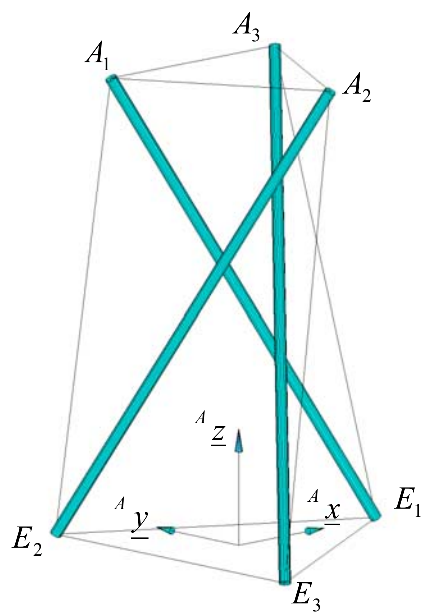


Figure 4.3. Unloaded position for the structure of example 1.

4.3.2 Analysis for the Loaded Position.

If one external force is applied at the upper end of each strut which components and point of application are presented in Table 4.3 and there is no constraints acting on the struts of the structure, it is required to evaluate the final equilibrium position of the structure.

Table 4.3. External forces acting on the structure of example 1.

	Strut 1	Strut 2	Strut 3
F_x (N)	0	0	0
F_y (N)	0	0	0
F_z (N)	-10	-10	-10
L_F (mm)	100	100	100

Since the system has 3 struts and there is no constraints then there are 12 degrees of freedom and therefore 12 equations are required, one per each generalized coordinate. The equations are generated following the procedure described in section 3.8:

Equation (3.43) yields f_1, f_2 and f_3

Equation (3.44) yields f_4, f_5 and f_6

Equation (3.45) yields f_7, f_8 and f_9

Equation (3.46) yields f_{10}, f_{11} and f_{12}

Each f_i is equated to zero and then the system is solved numerically. As an example the first equation, (3.43), is shown in the appendix A. It is clear that the complete set is extremely large and coupled. Before attempting to obtain a solution it is necessary to evaluate the initial conditions, i.e. the values of the generalized coordinates in the unloaded position. This is accomplished using (3.20), (3.23) and (3.24)

$$\begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos\left((j-1)\ \gamma \right) \\ R_B \sin\left((j-1)\ \gamma \right) \\ 0 \end{bmatrix} \quad , \ j = 1,\ 2,\ ....\ ,\ n \tag{3.20}$$

$$\tan\varepsilon_{j,0} \quad = \quad \frac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{H} \tag{3.23}$$

$$\tan\beta_{j,0} \quad = \quad \frac{R_T \cos(\ (j-1)\ \ \gamma + \alpha) - a_{j,0}}{\left(\dfrac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{\sin\varepsilon_{j,0}}\right)} \tag{3.24}$$

And all the terms in (3.20), (3.23) and (3.24) have been defined previously. The results are summarized in Table 4.4.

Table 4.4. Initial values of the generalized coordinates for the structure of example 1.

	Strut 1	Strut 2	Strut 3
a (mm)	33.0568	-16.5284	-16.5284
b (mm)	0	28.6280	-28.6280
ε (rad)	-0.1349	0.5491	-0.4443
β (rad)	-0.5567	0.1660	0.3716

It is now possible to implement the numerical method. The magnitude of the external force is increased in steps of 1 N and the equilibrium position is evaluated for each step. The final values for the generalized coordinates of the structure for an external force of 10 N are shown in Table 4.5.

Table 4.5. Generalized coordinates for the final position for the structure of example 1.

	Strut 1	Strut 2	Strut 3
a (mm)	40.8573	-20.4241	-20.4332
b (mm)	-0.0053	35.3861	-35.3808
ε (rad)	-0.0269	0.6808	-0.6643
β (rad)	-0.7434	0.3271	0.3635

Using these values, equations (3.6) and (3.7) can now be used to obtain the coordinates of the ends of the struts for the final position.

$$^{A}\underline{E}_j = \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} \tag{3.6}$$

$$^{A}\underline{A}_j = \begin{bmatrix} L_s \sin\beta_j + a_j \\ -L_s \sin\varepsilon_j \cos\beta_j + b_j \\ L_s \cos\varepsilon_j \cos\beta_j \end{bmatrix} \tag{3.7}$$

The results are summarized in Table 4.6 and Figure 4.4 shows the structure in its final equilibrium position.

Table 4.6. Lower and upper coordinates for the final position of the structure of example 1 (mm).

	Strut 1	Strut 2	Strut 3
E_x	40.8573	-20.4241	-20.4332
E_y	-0.0053	35.3861	-35.3808
E_z	0	0	0
A_x	-26.8241	11.7016	15.1224
A_y	1.9750	-24.2179	22.2429
A_z	73.5888	73.5888	73.5888

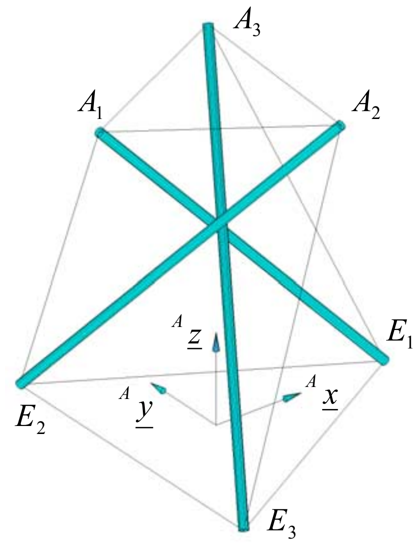
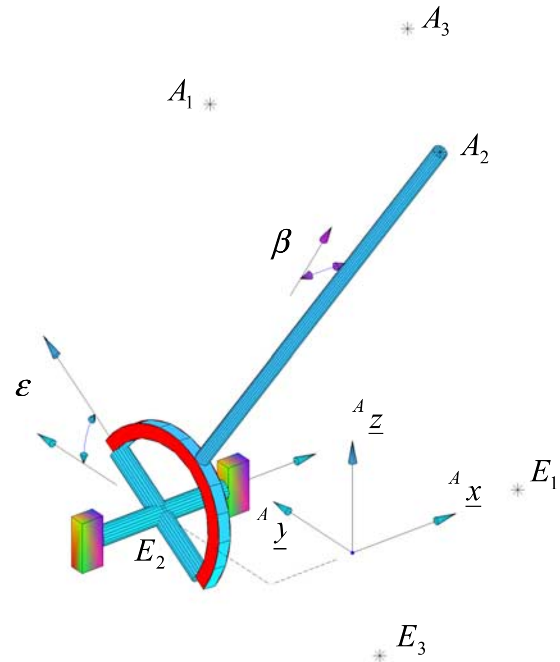


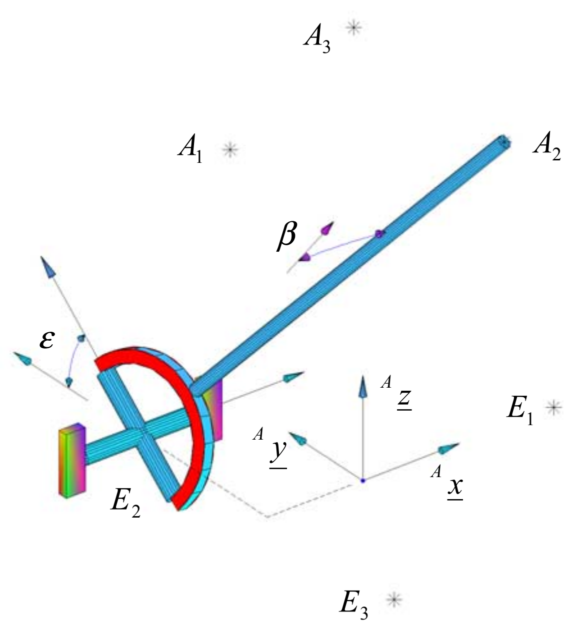
Figure 4.4. Final equilibrium position for the structure of example 1.

Figure 4.5 illustrates the second strut modeled with an universal joint in its first and final position. It can be appreciated its longitudinal and angular displacements.

Figure 4.6 shows a top view of the structure in its initial and final positions. It should be noted that the base E_1, E_2, E_3 increases in size but maintains its orientation whilst the top A_1, A_2, A_3 increases in size and also undergoes a rotation.

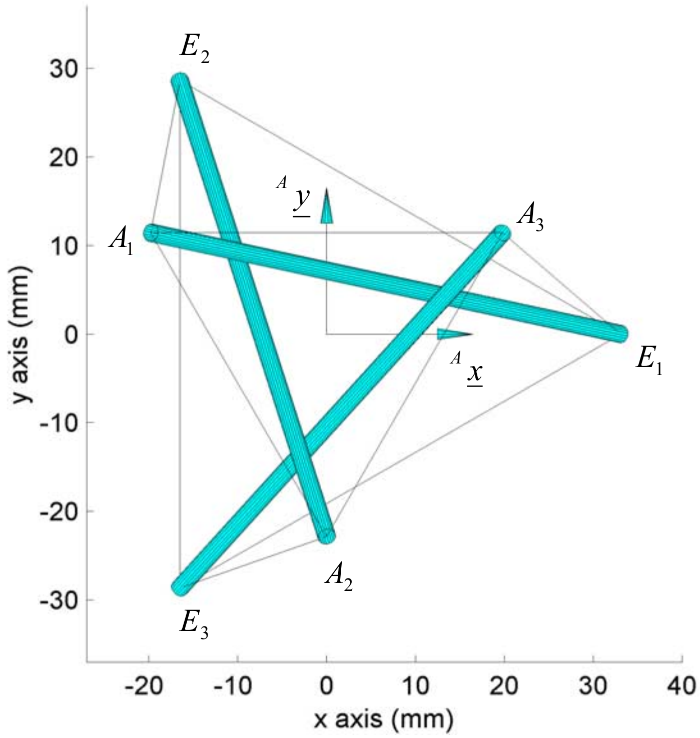


(a)

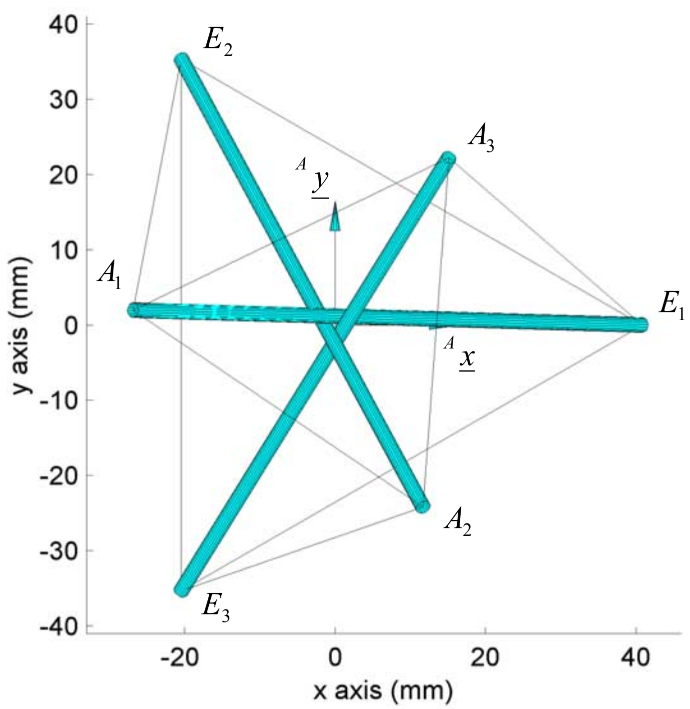


(b)

Figure 4.5. Second strut of the structure of example 1.
a)Unloaded position; b)Last position.



(a)



(b)

Figure 4.6. Plan view structure example 1.
a)Unloaded position; b)Last position.

Because of the symmetry of the external loads the height of the structure decreases uniformly, i.e. the z coordinate for the points $\underline{E}_1$, $\underline{E}_2$ and $\underline{E}_3$ remains the same for each position. These results are illustrated in Figure 4.7.

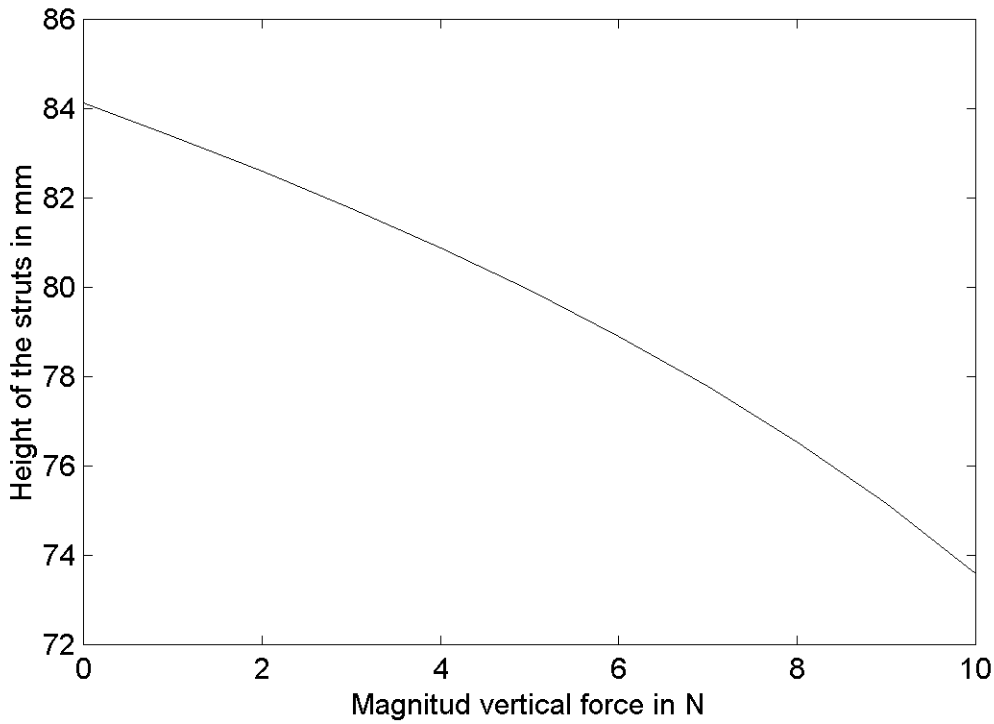


Figure 4.7. Height of the 3 struts Vs magnitude of the external force for the structure of example 1.

Figure 4.8 illustrates the variation of the ties lengths for each increment in the externally applied load. It should be noted that for the last position the length of the connecting ties is 81 mm which is approaching the free length. This means that if a larger force is applied to the structure, it cannot longer remain as a tensegrity structure. Although there could be other equilibrium positions the model developed in this research is not valid anymore.

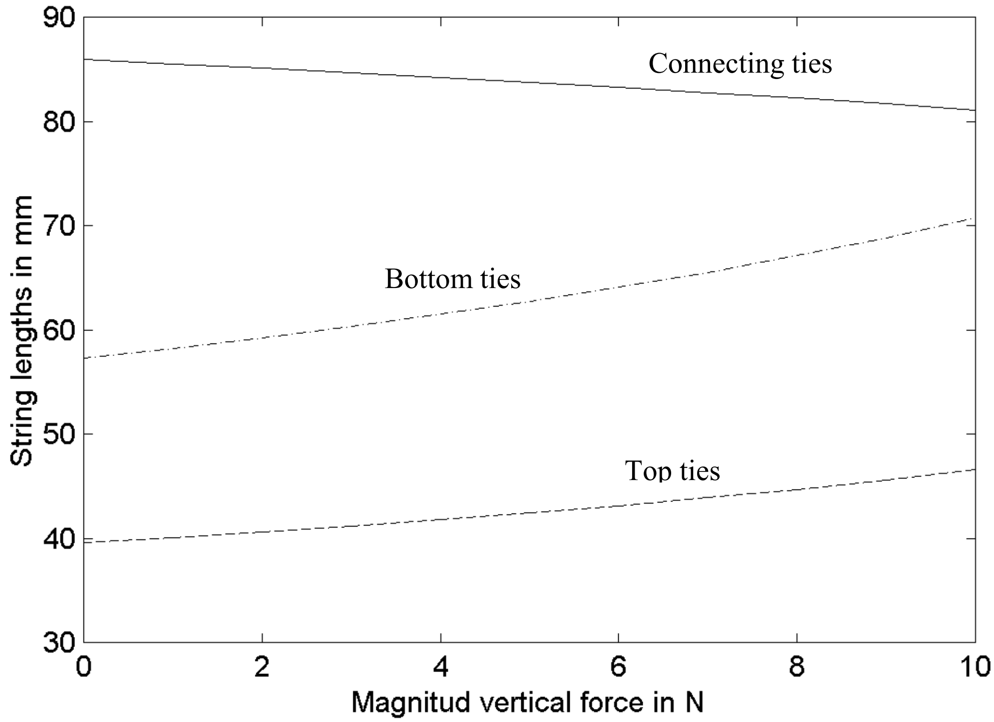


Figure 4.8. String lengths of the 3 struts structure Vs magnitude of the external force.

Because of the complexity of the equations that define equilibrium positions it is essential to verify the answers obtained independently. The software developed for this purpose performs this internally for each strut and for each position of the strut. To clarify this point, the verification of the answer is demonstrated here for the strut 2 in the last position.

Figure 4.9 shows a free body diagram for the second strut and the location of all the end points of the structure for the last position. It also includes the reaction force $\underline{R}$. Note that the direction for the force in the ties is considered as positive when the force goes from a point with subindex j to another point with subindex $j+1$. For example in Figure 4.9 the force $\underline{F}_{T2}$ goes from A_2 to A_3 therefore is positive while the force $\underline{F}_{T1}$ goes from A_2 to A_1 hence is negative.

If the system is in an equilibrium position, then the summation of moments with respect to E_2 must be zero and

$$\underline{r}\times\underline{F}+\underline{r}\times\underline{F}_{T2}-\underline{r}\times\underline{F}_{T1}+\underline{r}\times\underline{F}_{L2} = \underline{0} \tag{4.1}$$

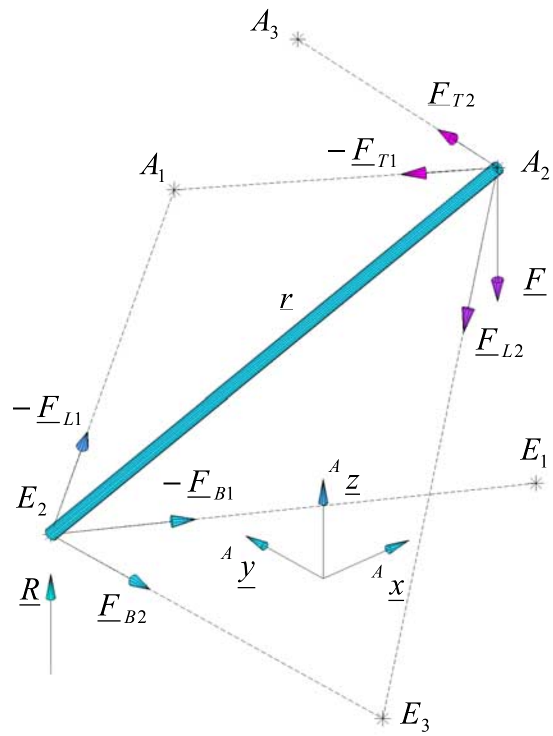


Figure 4.9. Free body diagram for the second strut of the structure of example 1 in the last position.

The vector $\underline{r}$ from $\underline{E}_2$ to $\underline{A}_2$ is given by

$$\underline{r} = \underline{A}_2-\underline{E}_2 = \begin{bmatrix} 11.7016 \\ -24.2179 \\ 73.5888 \end{bmatrix} - \begin{bmatrix} -20.4241 \\ 35.3861 \\ 0 \end{bmatrix} = \begin{bmatrix} 32.1257 \\ -59.6039 \\ 73.5888 \end{bmatrix} mm \tag{4.2}$$

And from Table 4.3 the external force at the strut 2 is

$$\underline{F}_2 = \begin{bmatrix} 0 \\ 0 \\ -10 \end{bmatrix} N \tag{4.3}$$

The force acting in the top tie T_2 is given by

$$\underline{F}_{T2} = \left|\underline{F}_{T2}\right| \ \underline{S}_{A2A3} \tag{4.4}$$

where

$$\left|\underline{F}_{T2}\right| = k_T \ \left(T_2 - T_o\right) \tag{4.5}$$

T_2 is the current length of the top tie 2

$$T_2 = \left|\underline{A}_3 - \underline{A}_2\right| = 46.5865 \ \ mm$$

k_T and T_0 are given in Table 4.1

$$k_T = 0.5 \ \frac{N}{mm}$$

$$T_o = 35 \ \ mm$$

Substituting the values of T_2, T_0 and k_T into (4.5)

$$\left|\underline{F}_{T2}\right| = 5.7932 \ \ N \tag{4.6}$$

$\underline{S}_{A2A3}$ are the unitized Plücker coordinates of the line passing through $\underline{A}_2\underline{A}_3$ and form equations (2.15), (2.16), (2.17) and (2.22)

$$\underline{S}_{A2A3} = \begin{bmatrix} 0.0734 \\ 0.9973 \\ 0 \end{bmatrix} \tag{4.7}$$

Substituting (4.6) and (4.7) into (4.4) yields

$$\underline{F}_{T2} = \begin{bmatrix} 0.4254 \\ 5.7776 \\ 0 \end{bmatrix} N \tag{4.8}$$

Similarly, the force acting in the top tie T_1 is given by

$$\underline{F}_{T1} = \left|\underline{F}_{T1}\right| \underline{S}_{A1A2} \tag{4.9}$$

where

$$\left|\underline{F}_{T1}\right| = k_T \left(T_1 - T_o\right) \tag{4.10}$$

The current length of the top tie 1, T_1, is

$$T_1 = \left|\underline{A}_2 - \underline{A}_1\right| = 46.5865 \ mm$$

Therefore

$$\left|\underline{F}_{T1}\right| = 5.7932 \ N \tag{4.11}$$

The unitized Plücker coordinates of the line passing through $\underline{A}_1 \underline{A}_2$ are

$$\underline{S}_{A1A2} = \begin{bmatrix} 0.8270 \\ -0.5622 \\ 0 \end{bmatrix} \tag{4.12}$$

Substituting (4.11) and (4.12) into (4.9)

$$\underline{F}_{T1} = \begin{bmatrix} 4.7908 \\ -3.2572 \\ 0 \end{bmatrix} \tag{4.13}$$

Finally the force acting on the lateral tie L_2 is given by

$$\underline{F}_{L2} = \left|\underline{F}_{L2}\right| \underline{S}_{A2E3} \tag{4.14}$$

where

$$\left|\underline{F}_{L2}\right| = k_L \left(L_2 - L_o\right) \tag{4.15}$$

And

$$L_2 = |\underline{E}_3 - \underline{A}_2| = 81.0714 \ mm$$

k_L and L_0 are given in Table 4.1

$$k_L = 1 \ \frac{N}{mm}$$

$$L_o = 80 \ mm$$

Substituting the values of L_2, L_0 and k_L into (4.15)

$$|\underline{F}_{L2}| = 1.0714 \ N \tag{4.16}$$

The unitized Plücker coordinates of the line passing through $\underline{A}_2\underline{E}_3$ are

$$\underline{S}_{A2E3} = \begin{bmatrix} -0.3964 \\ -0.1377 \\ -0.9077 \end{bmatrix} \tag{4.17}$$

Substituting (4.16) and (4.17) into (4.14)

$$\underline{F}_{L2} = \begin{bmatrix} -0.4247 \\ -0.1475 \\ -0.9725 \end{bmatrix} N \tag{4.18}$$

Now the summation of moments can be evaluated by substituting (4.2), (4.3), (4.6), (4.13) and (4.18) into (4.1)

$$\begin{bmatrix} 596.0394 \\ 321.2574 \\ 0 \end{bmatrix} N \cdot mm + \begin{bmatrix} -425.1663 \\ 31.3041 \\ 210.9646 \end{bmatrix} N \cdot mm - \begin{bmatrix} 239.6931 \\ 352.5526 \\ 180.9136 \end{bmatrix} N \cdot mm$$

$$+ \begin{bmatrix} 68.82 \\ -0.0089 \\ -30.0510 \end{bmatrix} N \cdot mm = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} N \cdot mm$$

Since the summation of moments with respect to $\underline{E}_2$ is $\underline{0}$, the numerical result confirms that the current position for the strut 2 of the structure corresponds to an equilibrium position.

4.4 Example 2: Analysis of a Tensegrity Structure with 4 Struts

4.4.1 Analysis for the Unloaded Position.

It is required to evaluate the unloaded equilibrium position of a tensegrity structure with 4 struts with the stiffness and free lengths shown in Table 4.7. Each of its struts has a length $L_s = 100mm$.

Table 4.7. Stiffness and free lengths for the structure of example 2.

	Stiffness (N/mm)	Free lengths (mm)
Top ties	$k_T = 0.5$	$T_0 = 40$
Bottom ties	$k_B = 0.5$	$B_0 = 40$
Connecting ties	$k_L = 0.5$	$L_0 = 40$

The solution of the system

$$k_L \left(1 - \frac{L_o}{L}\right) R_B - 2k_T \left(R_T - R_{To}\right) \sin\frac{\gamma}{2} = 0 \qquad (3.16)$$

$$k_L \left(1 - \frac{L_o}{L}\right) R_T - 2k_B \left(R_B - R_{Bo}\right) \sin\frac{\gamma}{2} = 0 \qquad (3.17)$$

$$L - \sqrt{L_s^2 + 2R_B R_T \left[\cos(\alpha + \gamma) - \cos\alpha\right]} = 0 \qquad (3.18)$$

where

$$\gamma = \frac{2\pi}{n} = \frac{360°}{4} = 90° \qquad (3.15)$$

$$\alpha \;=\; \frac{\pi}{2} \;-\; \frac{\pi}{n} \;=\; \frac{90^\circ}{2} \;-\; \frac{90^\circ}{4} \;=\; 22.5^\circ \tag{3.19}$$

$$R_{T0} \;=\; \frac{T_o}{2\sin\frac{\gamma}{2}} \;=\; \frac{40\;\;mm}{2\sin 45^\circ} \;=\; 28.2843\;\;mm \tag{3.11}$$

$$R_{B0} \;=\; \frac{B_o}{2\sin\frac{\gamma}{2}} \;=\; \frac{40\;\;mm}{2\sin 45^\circ} \;=\; 28.2843\;\;mm \tag{3.12}$$

yields

$$R_B \;=\; 41.2528\;\;mm$$

$$R_T \;=\; 41.2528\;\;mm$$

The coordinates of the ends of the struts for the unloaded position are obtained from

$$^{A}\underline{E}_{j,0} \;=\; \begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} \;=\; \begin{bmatrix} R_B\cos(\;(j-1)\;\;\gamma) \\ R_B\sin(\;(j-1)\;\;\gamma) \\ 0 \end{bmatrix} \quad,\;\; j \;=\; 1,\;\; 2,\;\; ....\;\;,\;\; n \tag{3.20}$$

$$^{A}\underline{A}_{j,0} \;=\; \begin{bmatrix} R_T\cos(\;(j-1)\;\;\gamma+\alpha) \\ R_T\sin(\;(j-1)\;\;\gamma+\alpha) \\ H \end{bmatrix} \quad,\;\; j \;=\; 1,\;\; 2,\;\; ....\;\;,\;\; n \tag{3.21}$$

where if $j=1$ then $j-1=n$, and

$$H \;=\; \sqrt{L_s^2 - R_B^2 - R_T^2 - 2R_B R_T \sin\frac{\gamma}{2}} \;=\; 64.7280\;\;mm$$

The results are summarized in Table 4.8 and Figure 4.10 shows the structure in its unloaded position.

Table 4.8. Lower and upper coordinates for the unloaded position for the structure of example 2 (mm).

	Strut 1	Strut 2	Strut 3	Strut 4
E_x	41.2528	0	-41.2528	0
E_y	0	41.2528	0	-41.2528
E_z	0	0	0	0
A_x	-29.1701	-29.1701	29.1701	29.1701
A_y	29.1701	-29.1701	-29.1701	29.1701
A_z	64.7280	64.7280	64.7280	64.7280

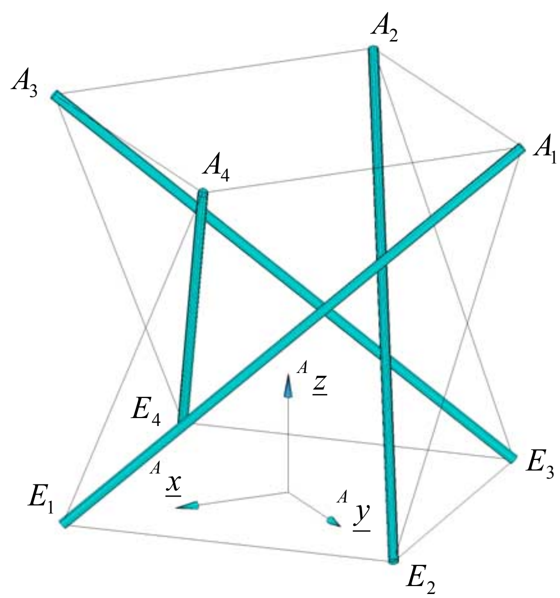


Figure 4.10. Unloaded position for the structure of example 2.

4.4.2 Analysis for the Loaded Position.

It is required to evaluate the final equilibrium position of the structure when the external moments listed in Table 4.9 are applied along the axis of the universal joints that model the structure, see Figure 4.11, and the lower ends of the struts are constrained in such a way that they cannot move in the horizontal plane.

Table 4.9. External moments acting on the structure of example 2.

	Strut 1	Strut 2	Strut 3	Strut 4
M_{ε} (N.mm)	450	-900	450	-900
M_{β} (N.mm)	450	450	450	450

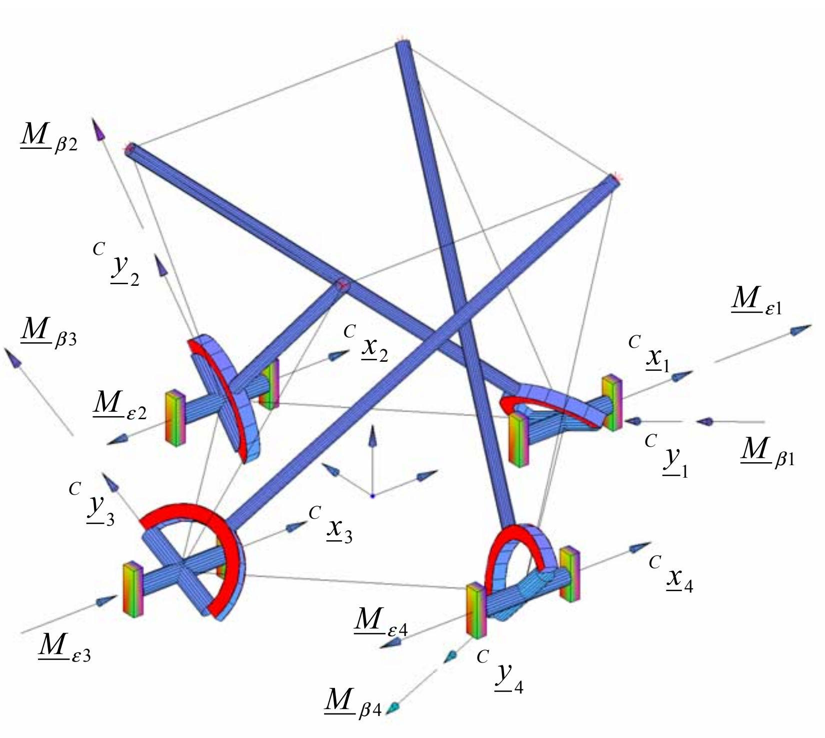


Figure 4.11. Directions of the external moments for the structure of example 2.

Because there are 2 constraints per strut there are 8 degrees of freedom for this system, and they are associated with the rotations of the struts. The generalized coordinates are ε_1, β_1, ε_2, β_2, ε_3, β_3 and ε_4, β_4, where the subscript indicates the number of the strut.

The equilibrium equations are obtained as follows

Equation (3.45) yields f_9, f_{10}, f_{11} and f_{12}

Equation (3.46) yields f_{13}, f_{14}, f_{15} and f_{16}

The initial conditions, this is the values of the generalized coordinates in the unloaded position, are obtained using (3.20), (3.23) and (3.24)

$$\begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos(\ (j-1)\ \ \gamma) \\ R_B \sin(\ (j-1)\ \ \gamma) \\ 0 \end{bmatrix} \quad , \ j \ = \ 1,\ 2,\ \ldots.\ ,\ n \tag{3.20}$$

$$\tan \varepsilon_{j,0} = \frac{b_{j,0} - R_T \sin(\ \ (j-1)\ \ \gamma + \alpha)}{H} \tag{3.23}$$

$$\tan \beta_{j,0} = \frac{R_T \cos(\ \ (j-1)\ \ \gamma + \alpha) - a_{j,0}}{\left(\dfrac{b_{j,0} - R_T \sin(\ \ (j-1)\ \ \gamma + \alpha)}{\sin \varepsilon_{j,0}} \right)} \tag{3.24}$$

And all the terms in (3.20), (3.23) and (3.24) have been defined previously. The results are presented in Table 4.10.

Table 4.10. Initial values of the generalized coordinates for the structure of example 2.

	Strut 1	Strut 2	Strut 3	Strut 4
ε (rad)	-0.4234	0.8275	0.4234	-0.8275
β (rad)	-0.7813	-0.2960	0.7813	0.2960

In order to avoid evaluating positions that do not correspond to the real problem is essential to increase the load smoothly in small increments rather than trying to obtain the final values of the moments in a single step. The number of steps was chosen arbitrarily as 10. Table 4.11 shows the values of the external moments for each step. The results for the final position are listed in Table 4.12

Table 4.11. External moments at each step for the structure of example 2 in N.mm.

Step		1	2	3	4	5	6	7	8	9	10
Strut 1	M_ε	45	90	135	180	225	270	315	360	405	450
	M_β	45	90	135	180	225	270	315	360	405	450
Strut 2	M_ε	-90	-180	-270	-360	-450	-540	-630	-720	-810	-900
	M_β	45	90	135	180	225	270	315	360	405	450
Strut 3	M_ε	45	90	135	180	225	270	315	360	405	450
	M_β	45	90	135	180	225	270	315	360	405	450
Strut 4	M_ε	-90	-180	-270	-360	-450	-540	-630	-720	-810	-900
	M_β	45	90	135	180	225	270	315	360	405	450

Table 4.12. Generalized coordinates for the final position for the structure of example 2.

	Strut 1	Strut 2	Strut 3	Strut 4
ε (rad)	-0.4689	0.6311	0.8140	-1.1716
β (rad)	-0.2916	0.1598	1.2136	0.5948

With values listed in Table 4.12, equations (3.6) and (3.7) permit to obtain the coordinates of the ends of the struts for the final position.

$$ {}^{A}\underline{E}_j = \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} \tag{3.6} $$

$$ {}^{A}\underline{A}_j = \begin{bmatrix} L_s \sin\beta_j + a_j \\ -L_s \sin\varepsilon_j \cos\beta_j + b_j \\ L_s \cos\varepsilon_j \cos\beta_j \end{bmatrix} \tag{3.7} $$

The results are summarized in Table 4.13 and Figure 4.12 shows the structure in its final equilibrium position.

Table 4.13. Lower and upper coordinates for the final position of the structure of example 2 (mm).

	Strut 1	Strut 2	Strut 3	Strut 4
E_x	41.2528	0	-41.2528	0
E_y	0	41.2528	0	-41.2528
E_z	0	0	0	0
A_x	12.5079	15.9151	52.4370	56.0322
A_y	43.2859	-16.9992	-25.4176	35.0631
A_z	85.4404	79.7083	24.0035	32.1911

It is apparent from Figure 4.12 that the z coordinate for each strut does not change uniformly. The variations in z for each strut are shown in Figure 4.13.

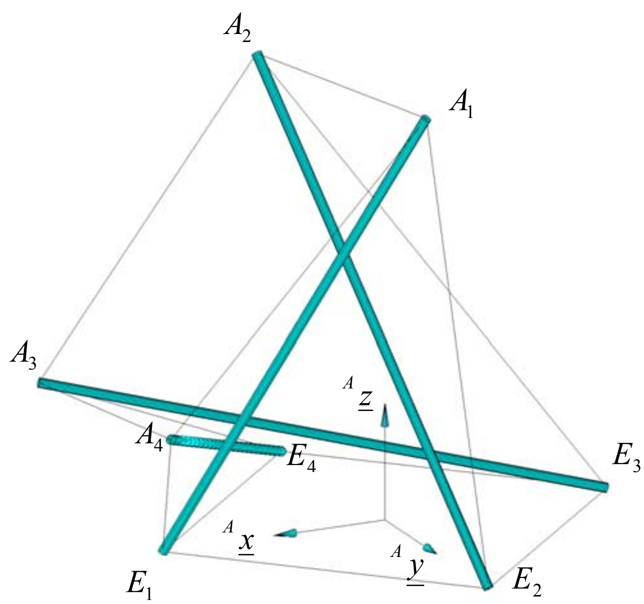


Figure 4.12. Final equilibrium position for the structure of example 2.

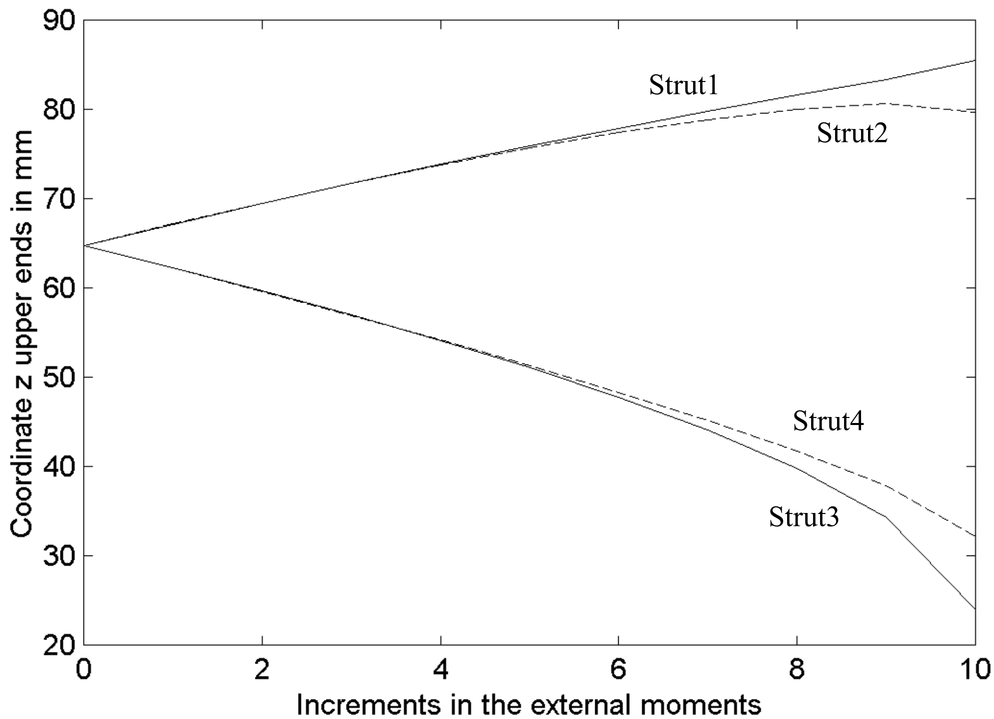


Figure 4.13. Coordinate z for each strut for each increment in the external moments applied to the structure of example 2.

In order to verify the numerical results when external moments are present, the final position of strut 4 is analyzed. Figure 4.14 shows the free body diagram for this strut in its final position modeled with an universal joint. In addition to the forces in the ties and the external moments, there is a reaction force $\underline{R}$ and a reaction moment $\underline{RM}$, both of which are unknown. Table 4.14 lists the unitized Plücker coordinates for the forces in the ties attached to the ends of the strut 4 expressed in the global reference system A.

From Newton’s Law it is known that the summation of moments about any point must be zero if the analyzed position is an equilibrium position. The forces and moments involved in the summations might be expressed in any system. If the C system is chosen, this is the system defined by the axes of the universal joint, the components of the reaction moment along ${}^{C}\underline{x}$ and ${}^{C}\underline{y}$ must be zero because a universal joint cannot exert any reaction moment along its axes.

Table 4.14. Unitized Plücker coordinates for the forces in the ties attached to the ends of strut 4 expressed in the A system.

${}^{A}\hat{\$}_{A4A1}$	${}^{A}\hat{\$}_{A4A3}$	${}^{A}\hat{\$}_{A4E1}$	${}^{A}\hat{\$}_{E4E1}$	${}^{A}\hat{\$}_{E4E3}$	${}^{A}\hat{\$}_{E4A3}$
-0.6284	-0.0588	-0.297	0.7071	-0.7071	0.8768
0.1187	-0.9892	-0.704	0.7071	0.7071	0.2648
0.7688	-0.1339	-0.646	0	0	0.4014
23.135	27.1493	0	0	0	-16.5576
-63.305	5.6108	26.6442	0	0	0
28.685	-53.3677	-29.0214	29.1701	-29.1701	36.1708

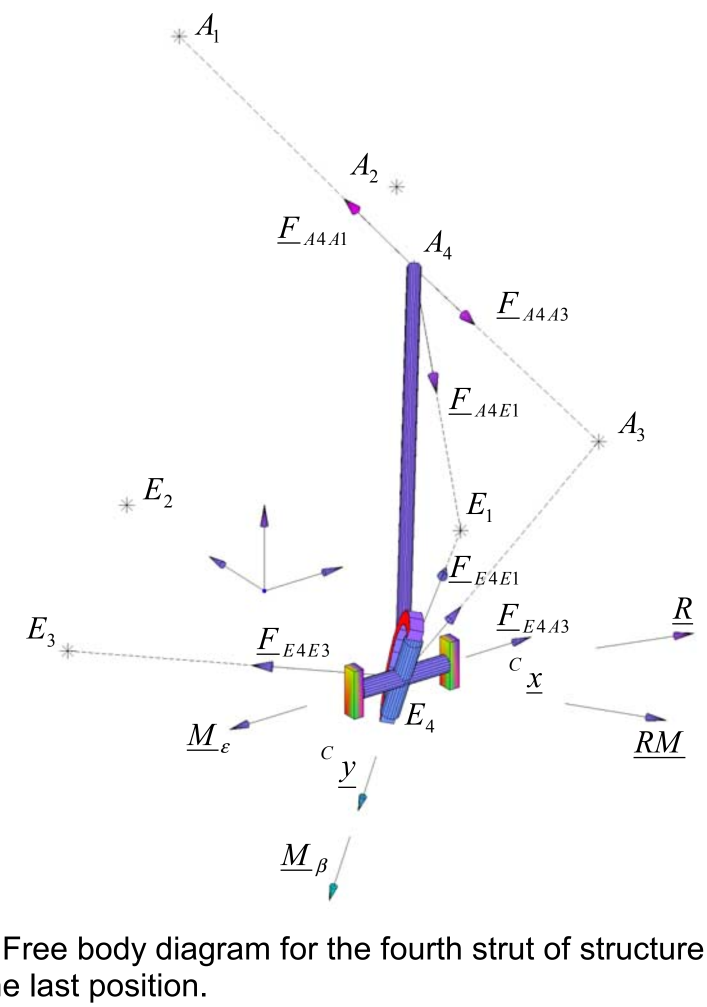


Figure 4.14. Free body diagram for the fourth strut of structure of example 2 in the last position.

In what follows all the forces and external moments are expressed in the C system, then the summation of moments is analyzed.

The same coordinates can be expressed in the C system defined by the axes of the universal joint, see figure 4.14, using the equations (2.29) , (2.30), (2.28) and (2.67) adapted to the current notation

$$ {}^{C}\underline{\$} = e^{-1}\ {}^{A}\underline{\$} \tag{1.29} $$

where

$$ e^{-1} = \begin{bmatrix} {}^{A}_{C}R^{T} & O_3 \\ {}^{A}_{C}R^{T} A_3^{T} & {}^{A}_{C}R^{T} \end{bmatrix} \tag{1.30} $$

$$ A_3 = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \tag{1.28} $$

$$ {}^{A}_{C}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & \sin\varepsilon & \cos\varepsilon \end{bmatrix} \tag{1.67} $$

where x, y and z are given by the values of E_x, E_y and E_z for strut 4 in Table 4.13. Angles ε and β are given for the values for strut 4 in Table 4.12.

Table 4.15 shows the unitized Plücker coordinates of the forces in the ties in the C system obtained after substituting numerical values in the previous expressions. This table also includes the unitized Plücker coordinates for the external moments $\underline{M}_{\varepsilon}$, $\underline{M}_{\beta}$ expressed in the C system. Table 4.16 lists the Plücker coordinates for the reaction force $\underline{R}$ and the reaction moment $\underline{RM}$ expressed in the C system.

Table 4.15. Unitized Plücker coordinates for forces in the ties and the external moments acting on strut 4 expressed in the C system.

${}^{C}\hat{\$}_{A4A1}$	${}^{C}\hat{\$}_{A4A3}$	${}^{C}\hat{\$}_{A4E1}$	${}^{C}\hat{\$}_{E4E1}$	${}^{C}\hat{\$}_{E4E3}$	${}^{C}\hat{\$}_{E4A3}$	${}^{C}\$_{M\varepsilon}$	${}^{C}\hat{\$}_{M\beta}$
-0.628	-0.0588	-0.2965	0.7071	-0.7071	0.8768	0	0
-0.662	-0.2611	0.3217	0.2748	0.2748	-0.2669	0	0
0.408	-0.9635	-0.8992	0.6515	0.6515	0.4	0	0
54.8493	21.6248	-26.644	0	0	0	-1	0
-74.918	49.118	25.824	0	0	0	0	1
-37.105	-14.629	18.025	0	0	0	0	0

Table 4.16. Plücker coordinates for reaction force and reaction moment acting on strut 4 expressed in the C system.

${}^{C}\underline{\$}_{R}$	${}^{C}\underline{\$}_{RM}$
${}^{C}R_x$	0
${}^{C}R_y$	0
${}^{C}R_z$	0
0	${}^{C}RM_x$
0	${}^{C}RM_y$
0	${}^{C}RM_z$

The equilibrium equation for the strut in the C system can be stated as follows

$$F_{A4A1}\,{}^{C}\hat{\$}_{A4A1} + F_{A4A3}\,{}^{C}\hat{\$}_{A4A3} + F_{A4E1}\,{}^{C}\hat{\$}_{A4E1} + F_{E4E1}\,{}^{C}\hat{\$}_{E4E1}$$

$$+ F_{E4E3}\,{}^{C}\hat{\$}_{E4E3} + F_{E4A3}\,{}^{C}\hat{\$}_{E4A3} + M_{\varepsilon}\,{}^{C}\$_{M\varepsilon} + M_{\beta}\,{}^{C}\hat{\$}_{M\beta}$$

$$+ {}^{C}\underline{\$}_{R} + {}^{C}\underline{\$}_{RM} = \underline{0} \tag{4.19}$$

where

$$F_{A4A1} = k_T\left(\left|\underline{A}_4 - \underline{A}_1\right| - T_O\right)$$

$$F_{A4A3} = k_T\left(\left|\underline{A}_4 - \underline{A}_3\right| - T_O\right)$$

$$
\begin{aligned}
F_{A4E1} &= k_L\left(\left|\underline{A}_4-\underline{E}_1\right|-L_O\right)\\
F_{E4E1} &= k_B\left(\left|\underline{E}_4-\underline{E}_1\right|-B_O\right)\\
F_{E4E3} &= k_B\left(\left|\underline{E}_4-\underline{E}_3\right|-B_O\right)\\
F_{E4A3} &= k_L\left(\left|\underline{E}_4-\underline{A}_3\right|-L_O\right) \qquad (4.20)
\end{aligned}
$$

After substituting the values given in Tables 4.7 and 4.13 into equation (4.20) yields

$$
\begin{aligned}
F_{A4A1} &= 14.6319N\\
F_{A4A3} &= 10.5692N\\
F_{A4E1} &= 4.9205N\\
F_{E4E1} &= 9.1702N\\
F_{E4E3} &= 9.1702N\\
F_{E4A3} &= 9.9022N \qquad (4.21)
\end{aligned}
$$

Substituting (4.21) and the values listed in Tables (4.15) and (4.16) into (4.19) yields

$$
14.6319\begin{bmatrix}-0.628\\-0.662\\0.408\\54.8493\\-74.9186\\-37.1052\end{bmatrix}+10.5692\begin{bmatrix}-0.0588\\-0.2611\\-0.9635\\21.6248\\49.118\\-14.629\end{bmatrix}+4.9205\begin{bmatrix}-0.2965\\0.3217\\-0.8992\\-26.6442\\25.824\\18.025\end{bmatrix}+9.1702\begin{bmatrix}0.7071\\0.2748\\0.6515\\0\\0\\0\end{bmatrix}
$$

$$+9.1702\begin{bmatrix}-0.7071\\0.2748\\0.6515\\0\\0\\0\end{bmatrix}+9.9022\begin{bmatrix}0.8768\\-0.2669\\0.4\\0\\0\\0\end{bmatrix}+900\begin{bmatrix}0\\0\\0\\-1\\0\\0\end{bmatrix}+450\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}+\begin{bmatrix}{}^{C}R_x\\{}^{C}R_y\\{}^{C}R_Z\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\{}^{C}RM_x\\{}^{C}RM_y\\{}^{C}RM_Z\end{bmatrix}=\underline{0} \tag{4.22}$$

Rows four and five in (4.22) yield

$$802.549 + 228.556 - 131.1027 - 900 + {}^{C}RM_x = 0 \tag{4.23}$$

and

$$-1096.201 + 519.138 + 127.067 + 450 + {}^{C}RM_y = 0 \tag{4.24}$$

From (4.23) and (4.24)

$RM_x = 0$

$RM_y = 0$

Since the universal joint cannot exert any reaction moment along its axes, the foregoing results confirm that the current position is an equilibrium position.

4.5 Example 3: Analysis of a Tensegrity Structure with 6 Struts

4.5.1 Analysis for the Unloaded Position.

A tensegrity structure with 6 struts has the stiffness and free lengths shown in Table 4.17. Each of its struts has a length $L_s = 80mm$. It is required to evaluate its equilibrium unloaded position.

Table 4.17. Stiffness and free lengths for the structure of example 3.

	Stiffness (N/mm)	Free lengths (mm)
Top ties	$k_T = 0.5$	$T_0 = 15$
Bottom ties	$k_B = 0.3$	$B_0 = 25$
Connecting ties	$k_L = 0.3$	$L_0 = 30$

The solution of the system

$$k_L \left(1 - \frac{L_o}{L}\right) R_B - 2k_T \left(R_T - R_{To}\right) \sin\frac{\gamma}{2} = 0 \tag{3.16}$$

$$k_L \left(1 - \frac{L_o}{L}\right) R_T - 2k_B \left(R_B - R_{Bo}\right) \sin\frac{\gamma}{2} = 0 \tag{3.17}$$

$$L - \sqrt{L_s^2 + 2R_B R_T \left[\cos(\alpha + \gamma) - \cos\alpha\right]} = 0 \tag{3.18}$$

where

$$\gamma = \frac{2\pi}{n} = \frac{360°}{6} = 60° \tag{3.15}$$

$$\alpha = \frac{\pi}{2} - \frac{\pi}{n} = \frac{90°}{2} - \frac{90°}{6} = 30° \tag{3.19}$$

$$R_{T0} = \frac{T_o}{2\sin\frac{\gamma}{2}} = \frac{15 \ mm}{2\sin 30°} = 15 \ mm \tag{3.11}$$

$$R_{B0} = \frac{B_o}{2\sin\frac{\gamma}{2}} = \frac{25 \ mm}{2\sin 30°} = 25 \ mm \tag{3.12}$$

yields

$$R_B = 39.9154 \ mm$$

$$R_T = 27.8338 \ mm$$

The coordinates of the ends of the struts for the unloaded position are obtained from

$$^{A}\underline{E}_{j_1 o} = \begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos(\ (j-1)\ \gamma) \\ R_B \sin(\ (j-1)\ \gamma) \\ 0 \end{bmatrix} \quad , \ j = 1,\ 2,\ ....\ ,\ n \qquad (3.20)$$

$$^{A}\underline{A}_{j_1 o} = \begin{bmatrix} R_T \cos(\ (j-1)\ \gamma+\alpha) \\ R_T \sin(\ (j-1)\ \gamma+\alpha) \\ H \end{bmatrix} \quad , \ j = 1,\ 2,\ ....\ ,\ n \qquad (3.21)$$

where if $j=1$ then $j-1=n$, and

$$H = \sqrt{L_s^2 - R_B^2 - R_T^2 - 2R_B R_T \sin\frac{\gamma}{2}} = 84.1287 \ \ mm$$

The results are summarized in Table 4.18. Figure 4.15 shows the structure in its unloaded position.

Table 4.18. Lower and upper coordinates for the unloaded position for the structure of example 3 (mm).

	Strut 1	Strut 2	Strut 3	Strut 4	Strut 5	Strut 6
E_x	39.9154	19.9577	-19.9577	-39.9154	-19.9577	19.9577
E_y	0	34.5677	34.5677	0	-34.5677	-34.5677
E_z	0	0	0	0	0	0
A_x	-13.9169	-27.83	-13.916	13.9169	27.8338	13.9169
A_y	24.1047	0	-24.1047	-24.1047	0	24.1047
A_z	54.0468	54.0468	54.0468	54.0468	54.0468	54.0468

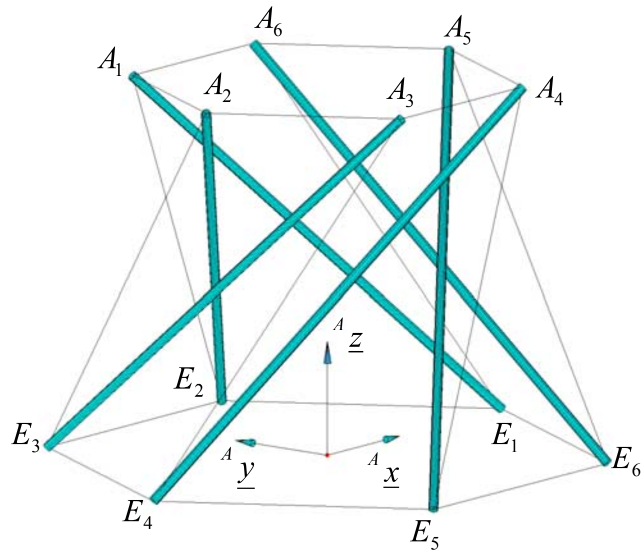


Figure 4.15. Unloaded position for the structure of example 3.

4.5.2 Analysis for the Loaded Position.

It is required to evaluate the final equilibrium position of the structure when external forces each of one of magnitude $16N$ are applied at the upper ends of each strut and the lower ends of all the struts are constrained in the horizontal plane. Each force $\underline{F}_i$ has no vertical component and its direction forms at every moment and angle $\gamma_i = \theta_i + \frac{3\pi}{2}$ with the ${}^A\underline{x}$ axis, see Figure 4.16, where θ_i is the angle between ${}^A\underline{x}$ and the projection of $\underline{OE}_i$ on the horizontal plane.

Since the system has 6 struts and the lower ends are constrained, there are two degrees of freedoms per strut and therefore 12 generalized coordinates which require 12 equations. The equations are obtained as follows

Equation (3.45) yields f_{13}, f_{14}, f_{15}, f_{16}, f_{17} and f_{18}.

Equation (3.46) yields f_{19}, f_{20}, f_{21}, f_{22}, f_{23} and f_{24}.

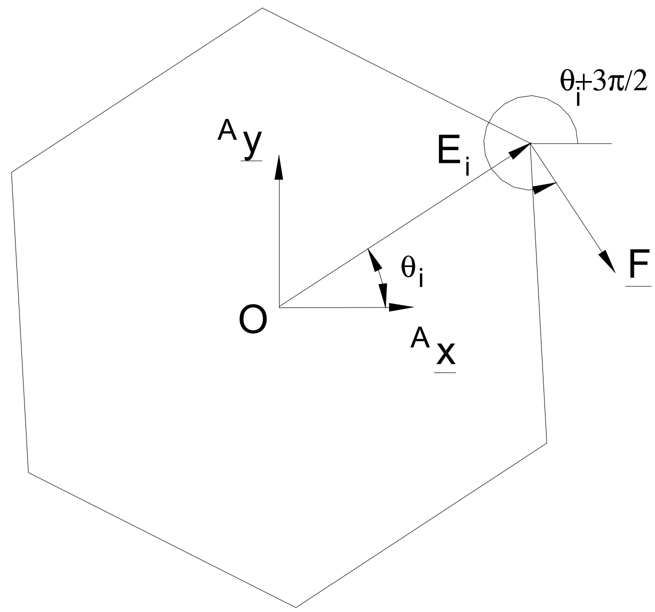


Figure 4.16. Direction external force in the struts for the structure of example 3.

The initial conditions to solve the last set of equations, i.e. the values for $\varepsilon_1,\ \varepsilon_2,\ \cdots\ ,\varepsilon_6$ and $\beta_1,\ \beta_2,\ \cdots\ ,\beta_6$ in the unloaded position are obtained using (3.20), (3.23) and (3.24).

$$\begin{bmatrix} a_{j,0} \\ b_{j,0} \\ 0 \end{bmatrix} = \begin{bmatrix} R_B \cos\left((j-1)\ \gamma\right) \\ R_B \sin\left((j-1)\ \gamma\right) \\ 0 \end{bmatrix} \quad , \ j \ = \ 1,\ 2,\ ....\ ,\ n \tag{3.20}$$

$$\tan \varepsilon_{j,0} \ = \ \frac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{H} \tag{3.23}$$

$$\tan \beta_{j,0} \ = \ \frac{R_T \cos(\ (j-1)\ \ \gamma + \alpha) - a_{j,0}}{\left(\dfrac{b_{j,0} - R_T \sin(\ (j-1)\ \ \gamma + \alpha)}{\sin \varepsilon_{j,0}} \right)} \tag{3.24}$$

And all the terms in (3.20), (3.23) and (3.24) have been defined previously. The results are summarized in Table 4.19.

Table 4.19. Initial values of the generalized coordinates for the structure of example 3.

	Strut 1	Strut 2	Strut 3	Strut 4	Strut 5	Strut 6
ε (rad)	-0.4195	0.5690	0.8264	0.4195	-0.5690	-0.8264
β (rad)	-0.7381	-0.6402	0.0756	0.7381	0.6402	-0.0756

As before is advisable to divide the solution in several steps. To increase the forces gradually helps to guide the solution obtained by numerical methods, but here in contrast with the previous examples, not only the magnitude but also the direction of the forces is changing. This fact should be consider before trying to solve the equilibrium equations.

The direction of the forces depend on angle θ_i, see Figure 4.16, which can be evaluated as

$$\theta_i = \tan^{-1}\left(\frac{{}^{A}E_{i,y}}{{}^{A}E_{i,x}}\right) \tag{4.25}$$

Therefore the components of each one of the external forces are given by

$${}^{A}\underline{F}_i = \left|{}^{A}\underline{F}_i\right| \begin{bmatrix} \cos(\theta_i + \frac{3\pi}{2}) \\ \sin(\theta_i + \frac{3\pi}{2}) \\ 0 \end{bmatrix} \tag{4.26}$$

where $\left|{}^{A}\underline{F}_i\right|$ is the magnitude of the external force for the current step. If the number of steps for this analysis is chosen arbitrarily as 4, then the magnitude of the force for the first step is $16N/4 = 4N$, for each one of the external forces. The components of the forces for the first step are computed using (4.25) and (4.26) together with the coordinates for E_i listed in Table 4.18, the results are presented in Table 4.20.

Table 4.20. Direction and components of the external forces for the initial position of example 3.

	Strut 1	Strut 2	Strut 3	Strut 4	Strut 5	Strut 6
θ_i (rad)	2.0944	3.1416	-2.0944	-1.0472	0	1.0472
F_x	3.4641	0	-3.4641	-3.4641	0	3.4641
F_y	2	4	2	-2	-4	-2
F_z	0	0	0	0	0	0

With the initial values for the generalized coordinates listed in Table 4.19 and the components of the forces presented in Table 4.20, the set of 12 equations is solved. The results obtained correspond to the generalized coordinates for the equilibrium position after the first increment in the external forces. To continue the process the magnitude of the forces is increased in $4N$ and their new components are evaluated by using (4.25) and (4.26). The process ends when the magnitude of the external forces reaches its final value of $16N$ and Table 4.21 shows the results.

Table 4.21. Generalized coordinates for the final position for the structure of example 3.

	Strut 1	Strut 2	Strut 3	Strut 4	Strut 5	Strut 6
ε (rad)	0.5952	0.9542	0.6329	-0.5952	-0.9542	-0.6329
β (rad)	-0.7978	-0.0189	0.7712	0.7978	0.0189	-0.7712

Using these values, equations (3.6) and (3.7) can now be used to obtain the coordinates of the ends of the struts for the final position.

$$^{A}\underline{E}_{j} = \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} \tag{3.6}$$

$$^{A}\underline{A}_{j} = \begin{bmatrix} L_s \sin\beta_j + a_j \\ -L_s \sin\varepsilon_j \cos\beta_j + b_j \\ L_s \cos\varepsilon_j \cos\beta_j \end{bmatrix} \tag{3.7}$$

Table 4.22 lists the coordinates of the ends of the struts for the final position and Figures 4.17 through 4.20 illustrate the positions assumed by the structure after each increment in the magnitude of the external forces. It is apparent that the struts of the structure changed their orientation upon the application of this particular load.

Table 4.22. Lower and upper coordinates for the final position for the structure of example 3 (mm).

	Strut 1	Strut 2	Strut 3	Strut 4	Strut 5	Strut 6
E_x	39.9154	19.9577	-19.9577	-39.9154	-19.9577	19.9577
E_y	0	34.5677	34.5677	0	-34.5677	-34.5677
E_z	0	0	0	0	0	0
A_x	-17.3515	18.4484	35.7999	17.3515	-18.4484	-35.7999
A_y	-31.3202	-30.6870	0.6333	31.3202	30.6870	-0.6333
A_z	46.2553	46.2553	46.2553	46.2553	46.2553	46.2553

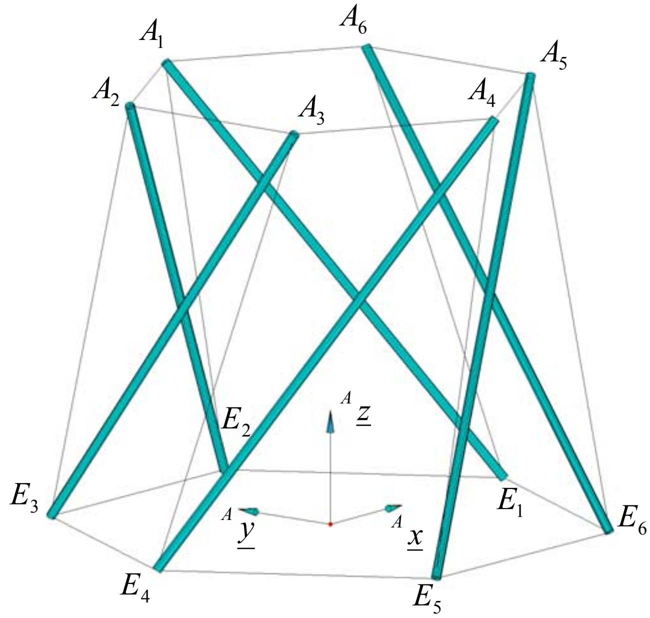


Figure 4.17. Position of the structure of example 3 when the magnitudes of the external forces are $4N$.

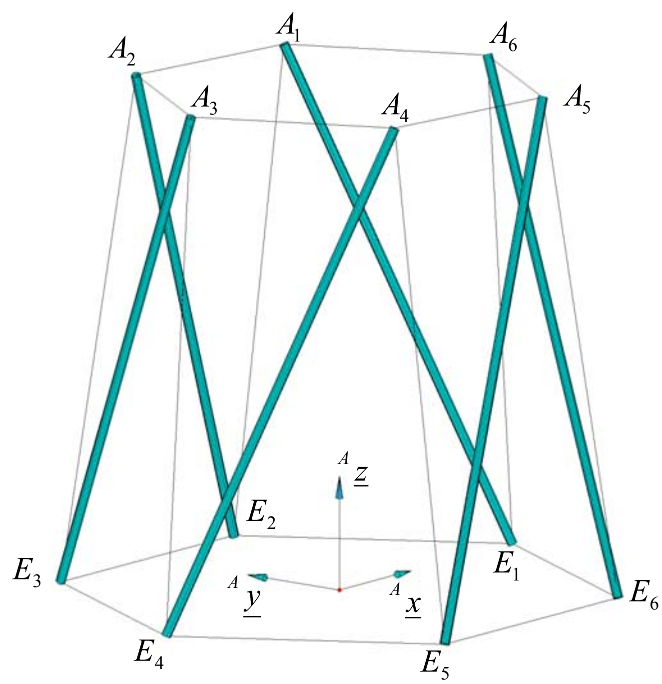


Figure 4.18. Position of the structure of example 3 when the magnitudes of the external forces are $8N$.

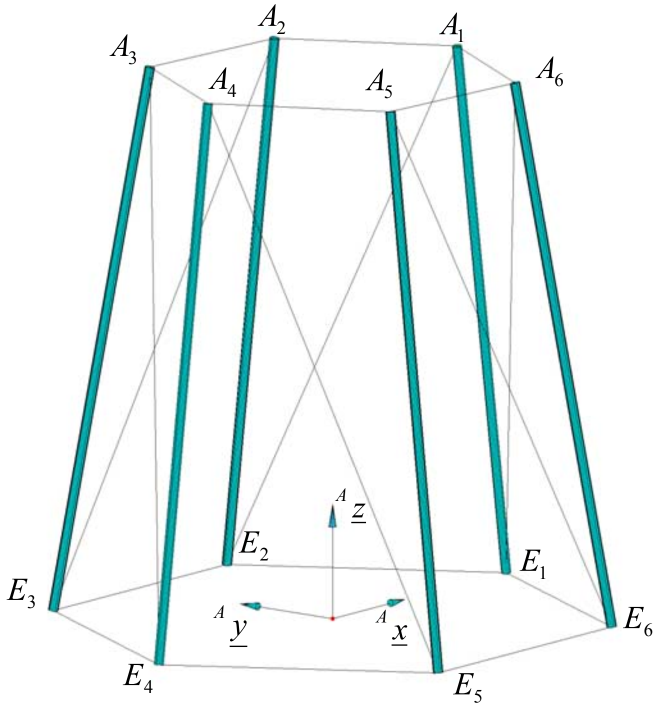


Figure 4.19. Position of the structure of example 3 when the magnitudes of the external forces are $12N$.

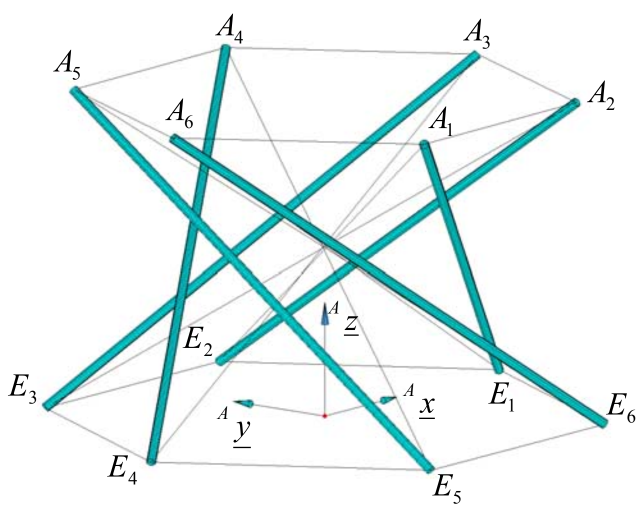


Figure 4.20. Position of the structure of example 3 when the magnitudes of the external forces are $16N$.

CHAPTER 5
CONCLUSIONS

This research has addressed and solved the problem of finding the equilibrium position of a tensegrity structure subjected to external loads. No previous references addressing this problem have appeared as far as the author is aware.

The model allows one to analyze a general anti-prism tensegrity structure subjected to a wide variety of external loads and the software developed is able to solve the system of equations generated for the model. The results are presented both numerically and in a three dimension graphical representation, which permits one to visualize the behavior of the structure.

The model is developed using the virtual work approach and all the results are checked using the Newton's Third Law. This verification assures one that the answers produced by the numerical method accurately correspond to equilibrium positions.

Mathematical models for variations of the basic configuration of tensegrity structures such as the reinforced tensegrity prisms might be developed following the same procedure presented in this research.

None tensegrity structure can be loaded indefinitely without collapsing. This fact cannot be predicted by the model developed here, but the position of the structure after that the collapse occurs can be identified by the results of the simulations. In general there were detected three situations that lead to the crash of the software and they are associated with particular physical situations.

The first occurs when the distance between two strut ends which are connected by a tie is less than the free length for that tie. The mathematical model always assumes that the ties are in tension, if this is not the case the model is no longer a valid representation of the structure and as result no convergence is found and the software crashes. The second occurs when two struts or a tie and a strut intersect. This fact is associated with the vanishing of the Jacobian for the structure [8], and corresponds to a singular configuration that the software cannot solve and in consequence it crashes.

The occurrence of the third physical situation is the less evident than the previous two. In certain situations even though there are no singularities and all the ties are in tension, one small increase in the external load makes impossible for the software the convergence to a solution, i.e. it is not possible to find a new equilibrium position. The system suffers a sudden change and it jumps from one equilibrium position to another for a smooth transition force. This is known as a catastrophe [11] and [12]. Catastrophe Theory is a well developed classical method. It describes sudden changes caused by a gradually changing input. It offers a better understanding of the phenomena reported here which is beyond the scope of this work.

The configuration of tensegrity structures is similar to the in-parallel devices which consist of a pair of rigid platforms connected by legs. The stability in rigid platforms is closed related to the coordinates of the leg lines [8]. In the future the use the quality indexes defined for platforms could be implemented for tensegrity structures to predict the stability of the elastic system subjected to external loads.

When the first position is not its unloaded position it is possible to determine the generalized coordinates of this new first position by gradually applying the external load to its unloaded position using the model described here.

Numerical methods are a good approach to solve complex systems of equations as those that arise in this research provided that a proper set of initial conditions is available. However, they do present disadvantages related to the speed of computations and the impossibility to make predictions about the behavior of the system. A challenging future work may address the problem of finding a closed solution for the analysis of a tensegrity structure. It will require the establishment of a better nomenclature that simplifies and reduces the size of the equations and at the same time permits one to follow the physical behavior of the structure.

Another interesting problem is the related to the inverse analysis, this is given a desired position for the structure to find the required external loads that may locate the structure in the desired position.

Finally it may be possible to find alternative numerical approaches. Finite element techniques have been well proved in fields like fluids, thermal sciences, vibrations and strength of materials. Such techniques could provide interesting possibilities to solving the problem.

APPENDIX A
FIRST EQUILIBRIUM EQUATION FOR THE STATICS OF A TENSEGRITY STRUCTURE WITH 3 STRUTS

F1x-1/2*kT*(((Ls*sin(□1)+a1-Ls*sin(□2)-a2)^2+(-Ls*sin(□1)*cos(□1)+

b1+Ls*sin(□2)*cos(□2)-b2)^2+(Ls*cos(□1)*cos(□1)-

Ls*cos(□2)*cos(□2))^2)^(1/2)-T0)/((Ls*sin(□1)+a1-Ls*sin(□2)-a2)^2+(-

Ls*sin(□1)*cos(□1)+b1+

Ls*sin(□2)*cos(□2)-b2)^2+(Ls*cos(□1)*cos(□1)-Ls*cos(□2)*cos(□2))^2)^(1/2)*

(2*Ls*sin(□1)+2*a1-2*Ls*sin(□2)-2*a2)-1/2*kB*((a1^2-2*a1*a2+a2^2+b1^2-

2*b1*b2+b2^2)^(1/2)-B0)/(a1^2-2*a1*a2+a2^2+b1^2-2*b1*b2+b2^2)^(1/2)*(2*a1-

2*a2)-1/2*kL*(((Ls*sin(□1)+a1-a2)^2+(-Ls*sin(□1)*cos(□1)+b1-b2)^2+Ls^2*

cos(□1)^2*cos(□1)^2)^(1/2)-L0)/((Ls*sin(□1)+a1-a2)^2+(-

Ls*sin(□1)*cos(□1)+b1-

b2)^2+Ls^2*cos(□1)^2*cos(□1)^2)^(1/2)*(2*Ls*sin(□1)+2*a1-2*a2)-1/2*kT*

(((Ls*sin(□3)+a3-Ls*sin(□1)-a1)^2+(-

Ls*sin(□3)*cos(□3)+b3+Ls*sin(□1)*cos(□1)-b1)^2+(Ls*cos(□3)*cos(□3)-

Ls*cos(□1)*cos(□1))^2)^(1/2)-T0)/((Ls*sin(□3)+a3-Ls*sin(□1)-a1)^2+(-

Ls*sin(□3)*cos(□3)+b3+Ls*sin(□1)*cos(□1)-b1)^2+(Ls*cos(□3)

*cos(□3)-Ls*cos(□1)*cos(□1))^2)^(1/2)*(-2*Ls*sin(□3)-2*a3+2*Ls*sin(□1)+2*a1)-

1/2*kB*((a3^2-2*a3*a1+a1^2+b3^2-2*b3*b1+b1^2)^(1/2)-B0)/(a3^2-2*a3*a1+

a1^2+b3^2-2*b3*b1+b1^2)^(1/2)*(-2*a3+2*a1)-1/2*kL*(((Ls*sin(□3)+a3-a1)^2+(-

Ls*sin(□3)*cos(□3)+b3-b1)^2+Ls^2*cos(□3)^2*cos(□3)^2)^(1/2)-

L0)/((Ls*sin(□3)+

a3-a1)^2+(-Ls*sin(□3)*cos(□3)+b3-b1)^2+Ls^2*cos(□3)^2*cos(□3)^2)^(1/2)*(-

2*Ls*sin(□3)-2*a3+2*a1)

APPENDIX B
SOFTWARE FOR THE STATIC ANALYSIS OF A TENSEGRITY STRUCTURE

A software able to generate and solve the equations necessary for the static analysis of a tensegrity structure is developed in Matlab. The software is open and structured which permits one to modify any of its parts easily.

Initially the general equilibrium equations are obtained in symbolic form for several tensegrity structures. This procedure is performed only once and the results are stored in the hard drive.

In order to be able to analyze a particular system is necessary to provide the following information: number of struts, length of the struts, stiffness and free length for each tie, external forces and their points of application, external moments, the number of steps desired for the analysis and to specify the generalized coordinates for each strut if some of the struts are subjected to additional constraints.

The information related to the ties and struts is sufficient to evaluate the coordinates of the ends of the strut for its unloaded position and therefore the initial conditions .

Originally the set of equations required to analyze a structure conformed by n struts is $4*n$, but if there are constraints not all the $4*n$ degrees of freedom are present. In this case the software automatically disregards any redundant equations.

The external loads are divided according to the number of steps required for the analysis. At this time all the information necessary to solve the equations is available. The program calls the function fsolve of Matlab which implements the Newton-Raphson method and yields the results for the current generalized coordinates. The process is repeated for each step and all the data is stored.

For each position the program evaluates the coordinates for the ends of the struts, the length of the strings, the forces in the ties, the Plücker coordinates for the lines joining points connected by ties and the reaction forces and reaction moments if they exist.

The program continues with the verification of the results applying the Newton's Third Law. The maximum deviation from the value of zero is reported. Generally this value is less than 0.0001.

The last step is to present an animation of the platform. All the results are stored and they are available in planar format to be used for other software if this is required.

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BIOGRAPHICAL SKETCH

Mr. Correa pursued his undergraduate studies at Universidad Nacional de Colombia in Medellín, Colombia, where he received the title of Mechanical Engineer. He worked as a teacher at University Pontificia Bolivariana in Medellín, between 1989 and 1999. In that institution he conformed a research group in mechanical design in 1997 which was recognized by the Colombian Government as one of the best in the country in 1999. He enrolled as a graduate student at the University of Florida in the fall of 1999 to pursue the Master of Science in mechanical engineering.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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Graduate Research Professor of
Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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