DEVELOPMENT OF DESIGN EQUATIONS
FOR
SELF-DEPLOYABLE
N-STRUT TENSEGRITY SYSTEMS

By

IAN P. STERN

A THESIS PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

UNIVERSITY OF FLORIDA

1999

ACKNOWLEDGEMENTS

I would like to thank my committee chairman, Dr. Joseph Duffy, for his guidance and support throughout my pursuit of a graduate degree. Great appreciation also goes to his graduate committee members, Dr. Carl Crane and Dr. Paul Mason, for their assistance and advice. I wish to send special thanks to Jaehoon Lee, Yu Zhang and my colleagues at the Center for Intelligent Machines and Robotics (CIMAR) for sharing their knowledge and experience.

Appreciation for assistance with research goes to Roger Tobie and Robert W. Burkhardt. Finally, I would like to thank my mother and friends for all the love, support and encouragement.

TABLE OF CONTENTS

Abstract of Thesis Presented to the Graduate School
of the University of Florida in Partial Fulfillment of the
Requirements for the Degree of Master of Science

DEVELOPMENT OF DESIGN EQUATIONS
FOR SELF-DEPLOYABLE
N-STRUT TENSEGRITY SYSTEMS

By

Ian P. Stern

December 1999

Chairman: Dr. Joseph Duffy
Major Department: Mechanical Engineering

This thesis analyzes the internal force organic to n-strut tensegrity systems to obtain a better understanding of tensegrity and develop generic design equations for self-deployable tensegrity systems. A static analysis of the internal forces is conducted on the top and bottom platforms of four n-strut tensegrity systems, which determines the geometry of the systems and relationship between the internal forces.

The results of each analysis are compared with the results of the other analyses to identify patterns consistent throughout all systems by relating the results to the number of struts in the system (n). The patterns are formulated into equations based on n. The theory of elasticity is used to develop generic design equations that calculate the lengths of the struts and elastic ties needed to create a desired geometry and stiffness.

Reactions to weight and external forces on the n-strut tensegrity systems are discussed. The two methods of calculating the outcome of applying external forces are identified.

CHAPTER 1
INTRODUCTION

1.1 Background

It is well know in the engineering field that an object in pure axial stresses can withstand greater loads than an object subjected to bending or shear stresses [1]. This is the reason that many designs use truss members to carry large forces. A truss is designed to carry most of the loads in compression. However, a thin member can carry a much higher load in tension than in compression, since the member must resist buckling in compression [11]. Thus, it might be disputed that an ideal design is one with most of its members in tension and a minimum amount of its members in compression. This concept led to the development of tensegrity.

R. Buckminster Fuller is known as the innovator of the tensegrity theory. The word “tensegrity” is Fuller’s contraction of the two words “tension’ and “integrity” [6,7]. In *Synergetics* and *Synergetics 2*, Fuller describes the theories of tensegrity and demonstrates its existence in nature [7,8].

Anthony Pugh best defines a tensegrity as “a set of discontinuous compressive components interacting with a set of continuous tensile components that define a stable volume in space” [11:3]. More simply stated, a tensegrity is a stable system made of compressive members connected only by tensile members. The compressive members may be single struts or shapes designed to carry all applied loads in compression such as a truss. The tensile members may be elastic or non-elastic tendons or a cloth covering. However, the most common tensegrity is made up of single struts and ties. Pugh states that the use of the word “system’ as opposed to “structure” is deliberate because tensegrity has applications other than structures. The term “tensegrity system” is therefore used in this study.

The development of tensile structures is relatively new and the development of tensegrity systems has only existed for 25 years. Kenner conducted the earliest geometrical analysis on prismatic and spherical tensegrity systems in 1976 [9]. He demonstrated the relation in prismatic systems between the rotation of the prism and the number of struts in the system.

Most of the geometrical and structural analyses, however, have been performed in the last ten years [2]. Chassagonoux, Motro and Hanaor have done extensive studies in form finding and stiffness testing during this time. Hanaor and Motro have restricted most of their work to dome and spherical tensegrity systems, while Chassagonoux has analyzed mostly prismatic systems.

Duffy and Rooney have recently conducted studies showing how a Stewart platform (in-parallel device) can be deployed into a prismatic tensegrity position using the screw theory [5]. This system is defined as an “n-strut tensegrity” and is similar to the systems analyzed by Kenner.

The “n” is used to denote the number of struts in the tensegrity system, such that a 3-strut tensegrity contains three struts [12]. It also denotes the number of sides the top and bottom platforms have. A 3-strut tensegrity would have triangular platforms, whereas a 4-strut tensegrity would have rectangular platforms.

The n-strut tensegrity is studied by rotating the top platform of a Stewart platform to a certain predicted range [5]. However, it is more accurate to state that the n-strut tensegrity is obtained by rotating the top “platform” on one of several semi-regular Archimedean anti-prism. The anti-prisms used to form an n-strut tensegrity are “formed from two parallel congruent regular polygons joined by equilateral triangle” [12:1279]. Figure 1.1 shows a 3-3, 4-4 and 5-5 Archimedean anti-prism. Though not depicted, the top and bottom platforms of the anti-prism do not have to be equal in size to form an n-strut tensegrity.

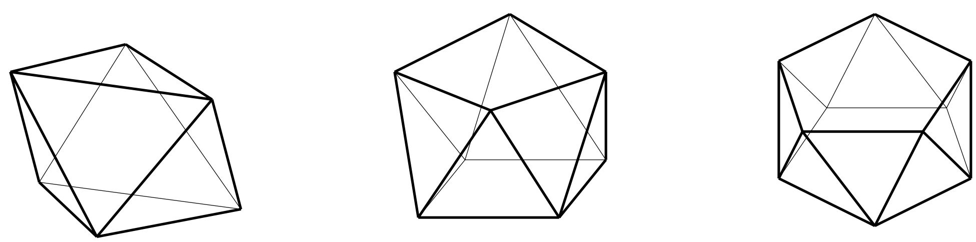

Figure 1.1
Some semi-regular Archimedean anti-prisms

An n-strut tensegrity is obtained by rotating the top platform of a wire frame anti-prism about the vertical axis while keeping the bottom platform fixed [12]. As the top platform rotates, the length of the legs (members along the side) of the anti-prism will change appropriately. The top platform will continue to rotate about the vertical axis until the system becomes a stable tensegrity. Rooney, Duffy and Lee state that the system becomes a stable tensegrity system once the top platform is rotated π/n radians and remains in stable tensegrity until the top platform is rotated $(n\text{-}1)\pi/n$ radians [12]. In this configuration, all of the members in the top and bottom platform and alternate legs will be in tension, leaving only n-struts in compression.

1.2 Definitions

A method to analyze the various n-strut tensegrity systems under all possible positions was necessary, since Rooney, Duffy and Lee stated that the system was in stable tensegrity within a range of rotations of the top platform [12]. The following definitions are used to develop the analytical methods in this study.

The initial rotation of the ant-prism's top platform defines the tensegrity as positive twist or a negative twist. If the top platform is rotated in the positive Z direction (counter clockwise looking down on the anti-prism), the system is a positive twist tensegrity. If the top platform is rotated in the negative Z direction, the system is a negative twist tensegrity. The twist of the tensegrity determines which legs are struts.

The initial position of tensegrity, when the top platform is rotated π/n radians, is defined as the minimum limit and the final position at $(n\text{-}1)\pi/n$ radians is defined as the maximum limit. The angle of rotation of the top platform beyond the minimum limit is defined as alpha (α), whereas alpha is equal to zero at the minimum limit position and is equal to $(n\text{-}2)\pi/n$ at the maximum limit. Alpha is always in the direction of the initial rotation of the anti-prism.

Figures 1.2, 1.3, 1.4 and 1.5 show the top view of the minimum and maximum limits of a positive twist and negative twist 3-strut tensegrity. The top platform is smaller than the bottom platform and is shaded to differentiate the platforms. The thick lines represent the struts and the thin lines represent the ties. An arrow depicts the direction of α. Note that the struts and leg ties are opposite on the opposing twists.

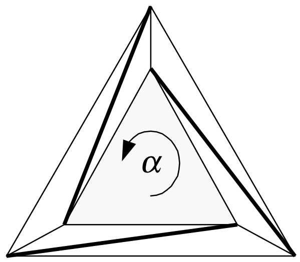

Figure 1.2

Positive twist
minimum limit

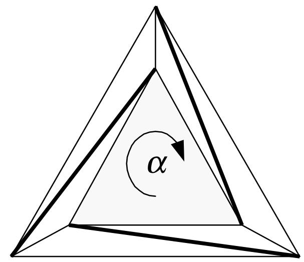

Figure 1.3

Negative twist
minimum limit

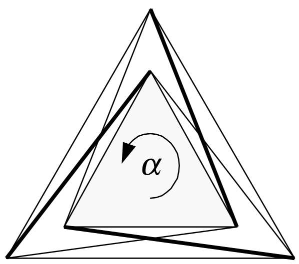

Figure 1.4

Positive twist
maximum limit

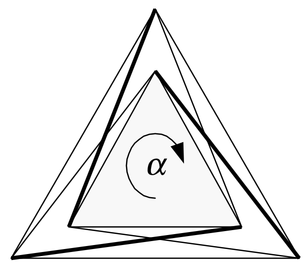

Figure 1.5

Negative twist
maximum limit

1.3 Objective and Scope of Study

The concept of using tensegrity systems as deployable systems has generated studies on tensegrity within the recent years. There have been several studies conducted on the form finding and stress analysis of dome tensegrity system [2]. However, there have been very few studies conducted on prismatic systems. There are no accessible static analyses or form finding equations for the n-strut tensegrity or similar systems as a deployable system. This leaves the need to develop design equations for deployable prismatic systems that meet specific geometric and rigidity requirements.

The purpose of this study is to conduct a static analysis on n-strut tensegrity systems, identify patterns throughout all of the systems and develop generic design equations for self-deployable systems. The static analysis is conducted on 3-strut, 4-strut, 5-strut and 6-strut tensegrity systems. The top and bottom platforms are parallel to each other and the lengths of the ties that compose a platform are equal. Each tensegrity was studied at the minimum limit position and a coordinate system is established in the center of the bottom platform. The coordinates of the top platform are defined dependent on α, so that the coordinates of each joint change as the top platform rotates. The weight of the system is assumed to be negligible and is not considered for the study. There are no external forces applied to the system.

Symmetry of forces within the tensegrity is assumed based on the symmetry of the geometry. Using the established coordinate system, the forces in each member at a bottom "joint" are related to each other based on static equilibrium. This procedure was repeated at a top joint so all forces in the member can be related to each other. The results for each n-strut tensegrity are used to develop the design calculations and present consistencies throughout the various systems.

The results of the analysis on each system are compared and the patterns throughout all systems are identified to create generic force equations based on *n*. The theory of elasticity converts the generic force equations into design equations. The force in an elastic tie is directly related to the displacement (amount stretched) of the tie based on a spring constant. This allows the equations to be solved for the original lengths of the ties. The considerations due to weight and external loads are discussed and two methods of finding the geometry and forces of the loaded n-strut tensegrity system are described.

CHAPTER 2
STATIC ANALYSIS

2.1 3-Strut Tensegrity System

The first step in statically analyzing the 3-strut tensegrity is to develop the vector systems of the internal forces. A coordinate system is placed in the center of the bottom platform. The starting position of the tensegrity system is the positive twist minimum limit position. The top platform will be free to rotate about the Z-axis while maintaining a constant distance between the top and bottom platforms (h). The length of the struts (L_s) and leg ties (L_t) must adjust as the top platform rotates to keep the distance constant. The rotation of the top platform, or twist angle (α), will be positive in the positive Z-direction. Note that the direction of α would be in the opposite direction if the starting position was the negative twist minimum limit.

Figure 2.1 shows a top view of the 3-strut tensegrity at the minimum limit and the variables used in the static analysis. The thin lines represent ties, the thick lines represent struts and the top platform is shaded. The direction of rotation (α) is depicted with an arrow and is equal to zero at this position. The lengths of the top and bottom ties are a and b respectively and the distance between the base and top platform is h (not shown). The length of the struts is L_s and the length of the leg ties is L_t. The coordinate at each "joint" (end points of the members) is given a double letter variable. The vectors symbolized by $\underline{S}$ are direction unit vectors of the members. Only the vectors along the member at AA and DD are needed since those are the two points where static equilibrium will be analyzed (see Figure 2.2 and Figure 2.5) .

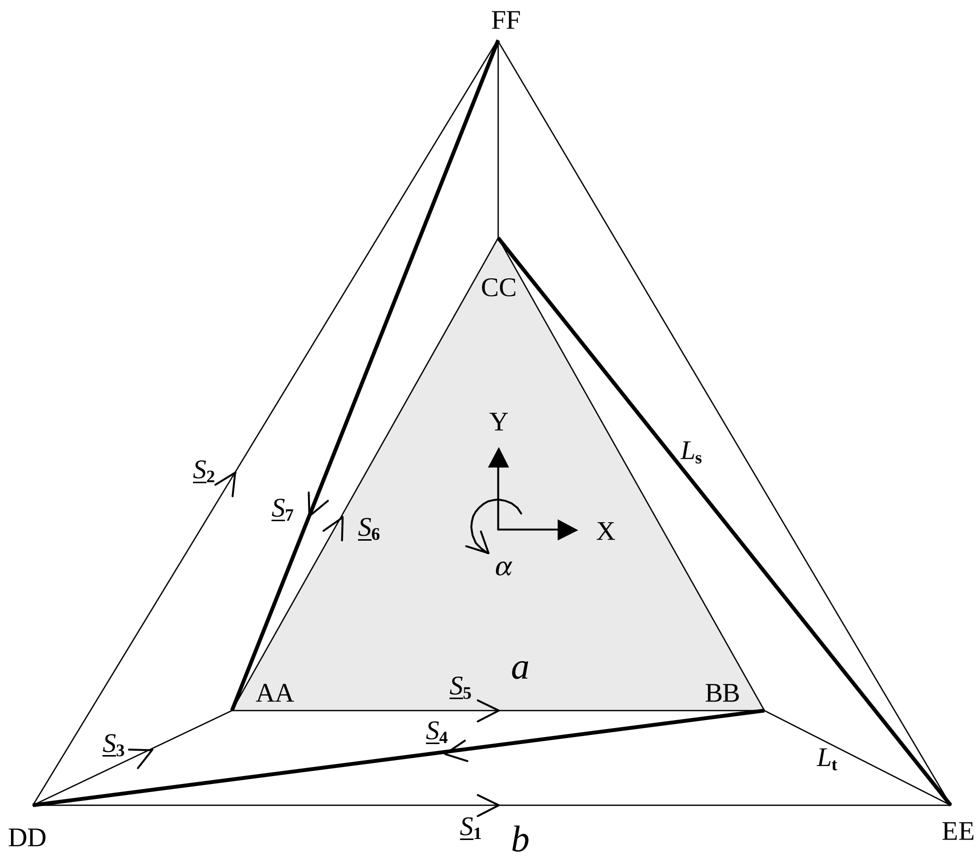

Figure 2.1
3-strut tensegrity [minimum limit]

2.1.1 Coordinate System

The coordinate system was initially developed at the minimum limit and the coordinates for the joints of the top platform were redeveloped to accommodate changes in α. The coordinates of the joints are used to determine the 3-dimensional vectors of each member. The unit vector is obtained by dividing the member's 3-dimesional vector by length of the member [14].

The coordinates of the joints before the top platform is rotated are

$$\text{AA} \quad \left(-\frac{1}{2}a\,, \quad -\frac{\sqrt{3}}{6}a\,, \quad h \right), \tag{2.1}$$

$$\text{BB} \quad \left(\frac{1}{2}a\,, \quad -\frac{\sqrt{3}}{6}a\,, \quad h \right),$$

$$\text{CC} \quad \left(0\,, \quad \frac{\sqrt{3}}{3}a\,, \quad h \right),$$

$$\text{DD} \quad \left(-\frac{1}{2}b\,, \quad -\frac{\sqrt{3}}{6}b\,, \quad 0 \right),$$

$$\text{EE} \quad \left(\frac{1}{2}b\,, \quad -\frac{\sqrt{3}}{6}b\,, \quad 0 \right),$$

$$\text{FF} \quad \left(0\,, \quad \frac{\sqrt{3}}{3}b\,, \quad 0 \right).$$

The coordinates top platform free to rotate based on α are

$$\text{AA} \quad \left(-\frac{\sqrt{3}}{3}a\cos\left(\frac{\pi}{6}+\alpha\right), \quad -\frac{\sqrt{3}}{3}a\sin\left(\frac{\pi}{6}+\alpha\right), \quad h \right), \tag{2.2}$$

$$\text{BB} \quad \left(\frac{\sqrt{3}}{3}a\sin\left(\frac{\pi}{3}+\alpha\right), \quad -\frac{\sqrt{3}}{3}a\cos\left(\frac{\pi}{3}+\alpha\right), \quad h \right),$$

$$\text{CC} \quad \left(-\frac{\sqrt{3}}{3}a\sin(\alpha), \quad \frac{\sqrt{3}}{3}a\cos(\alpha), \quad h \right).$$

The direction vector of each member is the unit vector along that member. This can be determined by subtracting the coordinates of one joint (P_2) from the coordinates of the other joint (P_1) and dividing the results by the length of the member [14]. The resultant unit vector will be in the direction from P_1 to P_2. The unit vectors in this analysis as shown on Figure 2.1 are

$\underline{S}_1$	DD $\rightarrow$ EE,	$\underline{S}_2$	DD $\rightarrow$ FF,
$\underline{S}_3$	DD $\rightarrow$ AA,	$\underline{S}_4$	BB $\rightarrow$ DD,
$\underline{S}_5$	AA $\rightarrow$ BB,	$\underline{S}_6$	AA $\rightarrow$ CC,
$\underline{S}_7$	FF $\rightarrow$ AA.		

The values of these vectors derived from Eqs. (2.1) and (2.2), are

$$\underline{S}_1 = \left\{1, \quad 0, \quad 0\right\}, \tag{2.3}$$

$$\underline{S}_2 = \left\{\frac{1}{2}, \quad \frac{\sqrt{3}}{2}, \quad 0\right\},$$

$$\underline{S}_3 = \frac{1}{L_t}\left\{\frac{1}{2}b - \frac{\sqrt{3}}{3}a\cos\left(\frac{\pi}{6}+\alpha\right), \quad \frac{\sqrt{3}}{6}b - \frac{\sqrt{3}}{3}a\sin\left(\frac{\pi}{6}+\alpha\right), \quad h\right\},$$

$$\underline{S}_4 = \frac{1}{L_s}\left\{-\frac{1}{2}b - \frac{\sqrt{3}}{3}a\sin\left(\frac{\pi}{3}+\alpha\right), \quad -\frac{\sqrt{3}}{6}b + \frac{\sqrt{3}}{3}a\cos\left(\frac{\pi}{3}+\alpha\right), \quad -h\right\},$$

$$\underline{S}_5 = \left\{\cos(\alpha), \quad \sin(\alpha), \quad 0\right\},$$

$$\underline{S}_6 = \left\{\frac{1}{2}\cos(\alpha) - \frac{\sqrt{3}}{2}\sin(\alpha), \quad \frac{\sqrt{3}}{2}\cos(\alpha) + \frac{1}{2}\sin(\alpha), \quad 0\right\},$$

$$\underline{S}_7 = \frac{1}{L_s}\left\{-\frac{\sqrt{3}}{3}a\cos\left(\frac{\pi}{6}+\alpha\right), \quad -\frac{\sqrt{3}}{3}b - \frac{\sqrt{3}}{3}a\sin\left(\frac{\pi}{6}+\alpha\right), \quad h\right\},$$

where the lengths of the struts (L_s) and the length of the leg ties (L_t) are

$$L_s = \frac{\sqrt{3}}{3}\sqrt{b^2 + 2ab\sin\left(\frac{\pi}{6}+\alpha\right) + a^2 + 3h^2}\ , \tag{2.4}$$

$$L_t = \frac{\sqrt{3}}{3}\sqrt{b^2 - 2ab\cos(\alpha) + a^2 + 3h^2}\ . \tag{2.5}$$

2.1.2 Static Analysis at Joint DD

With the unit vectors established, the forces can be analyzed at a point. Since a force is a vector, it can be expressed as a magnitude times a unit vector [13, 14]

$$\underline{F} = F\underline{S}.$$

For static equilibrium, the sum of the forces acting on a static object must equal [13]

$$\sum_{i=1}^{n} F_i \underline{S}_i = 0. \tag{2.6}$$

Figure 2.2 shows the forces acting at joint DD. Due to the symmetry of the platform tensegrity, it is assumed that all the forces along the bottom ties are the same. All the forces in the leg ties, the top ties and the struts are assumed to be equal to their other components as well. F_b is the internal force of the bottom ties, F_a is the internal force of the top ties, F_t is the internal force of the leg ties, and F_s is the force in the struts.

Note in this study the subscript *a* denotes values of the top ties, the subscript *b* denotes values of the bottom ties, the subscript *t* denotes values of the leg ties, and the subscript *s* denotes values of the struts.

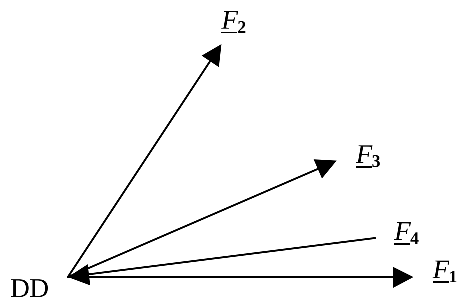

Figure 2.2

Forces at Joint DD

The forces acting on joint DD are

$$\underline{F}_1 = F_b\, \underline{S}_1, \quad (2.7)$$

$$\underline{F}_2 = F_b\, \underline{S}_2,$$

$$\underline{F}_3 = F_t\, \underline{S}_3,$$

$$\underline{F}_4 = F_s\, \underline{S}_4.$$

Because this study is analyzing simply supported members at their joints, there are no moments that act on any members. Only the axial forces need to be considered. The equation for the sum of the forces at joint DD is

$$\sum_{DD} \underline{F}_i = F_b\, \underline{S}_1 + F_b\, \underline{S}_2 + F_t\, \underline{S}_3 + F_s\, \underline{S}_4 = 0. \quad (2.8)$$

Equation (2.8) can now be resolved into three scalar equations, the sum of the forces in the X, Y and Z directions [14]

$$\sum_{DD} F_X = F_b + \frac{1}{2}F_b + \left[\frac{1}{2}b - \frac{\sqrt{3}}{3}a\cos\left(\frac{\pi}{6}+\alpha\right)\right]\frac{F_t}{L_t} - \left[\frac{1}{2}b + \frac{\sqrt{3}}{3}a\sin\left(\frac{\pi}{3}+\alpha\right)\right]\frac{F_s}{L_s} = 0, \tag{2.9}$$

$$\sum_{DD} F_Y = \frac{\sqrt{3}}{2}F_b + \left[\frac{\sqrt{3}}{6}b - \frac{\sqrt{3}}{3}a\sin\left(\frac{\pi}{6}+\alpha\right)\right]\frac{F_t}{L_t} - \left[\frac{\sqrt{3}}{6}b - \frac{\sqrt{3}}{3}a\cos\left(\frac{\pi}{3}+\alpha\right)\right]\frac{F_s}{L_s} = 0, \tag{2.10}$$

$$\sum_{DD} F_Z = \frac{h}{L_t}F_t - \frac{h}{L_s}F_s = 0. \tag{2.11}$$

Solving Eq. (2.11) for F_t yields

$$F_t = \frac{L_t}{L_s}F_s\,. \tag{2.12}$$

This equation will be found to be constant throughout all n-strut tensegrity systems because only the leg ties and the struts carry the forces in the Z direction. Thus the relation between the lengths of the leg ties and the length of the struts defines the relation between the internal force of the leg ties to the internal forces of the struts.

Using equation (2.12) to substitute for F_t into Eqs. (2.9) and (2.10) and simplifying yields

$$\sum_{DD} F_X = \frac{3}{2} F_b - \frac{a}{L_S} F_s \cos(\alpha) = 0, \tag{2.13}$$

$$\sum_{DD} F_Y = \frac{\sqrt{3}}{2} F_b - \frac{a}{L_S} F_s \sin(\alpha) = 0. \tag{2.14}$$

Both equations must be satisfied for the platform tensegrity to be in static equilibrium. Solving Eqs. (2.13) and (2.14) for F_b yields

$$F_b = \frac{2a}{3L_s} F_s \cos(\alpha), \tag{2.15}$$

$$F_b = \frac{2\sqrt{3}\,a}{3L_s} F_s \sin(\alpha). \tag{2.16}$$

Setting the right side of Eq. (2.15) equal to the right side of Eq. (2.16) and solving for α results in the equation

$$\tan(\alpha) = \frac{\sqrt{3}}{3}, \tag{2.17}$$

and yields a value for α

$$\alpha = \frac{\pi}{6}, \quad \frac{7\pi}{6}. \tag{2.18}$$

Though there are mathematically two stable solutions for the angles of rotation past the minimum limit, substituting $7\pi/6$ radians for α into the equations of static equilibrium produces a negative force on the leg ties and the struts. Switching the leg ties and struts at this configuration would result in a stable negative twist tensegrity.

The initial rotation to the proposed minimum limit of π/n radians is added to α to determine for the total rotation of the top platform from the Archimedean 3-3 anti-prism.

$$\frac{\pi}{3}+\frac{\pi}{6}=\frac{\pi}{2}.$$

Therefore, there are only two stable configurations of a 3-strut tensegrity system: positive twist and negative twist. Both configurations have the top platform rotated $\pi/2$ radians (90^{O}) from the original 3-3 anti-prism. The analysis on all n-strut tensegrity systems in this study will show that all n-strut tensegrity systems are anti-prisms with the top platform rotated 90^{O}. This concurs with the mathematical finding of Kenner on the prism tensegrity [9].

The reason there is only one angle of twist for a 3-strut tensegrity in static equilibrium is due to the geometry of the system. The distances between the joints of the leg ties are at a minimum when the top platform is twisted 90^{O} and the length of the struts is fixed. If a 3-strut tensegrity is made with any other angle of twist, the leg ties will not be stressed (taut) and the system will be free to move until the leg ties are stressed. Only when the top platform is rotated 90^{O} ($\alpha = 30^{O}$) are all the ties stressed and the system will be stable.

Lee, Duffy and Hunt prove that the Jacobian matrix of a 3-3 Stewart platform (in-parallel device) is singular when the top platform is rotated 90^{O} [10]. It is believed that the Jacobian matrix of all stable n-strut tensegrity systems is singular. This may be due to what is called "super stability." A "given tensegrity [is] super stable if any comparable configuration of vertices either violates one of the distance constraints—one of the struts is too short or one of the cables is too long" or it is "congruent to the original" [4:144].

Figure 2.3 and 2.4 show the top view of a stable positive twist and negative twist 3-strut tensegrity (respectively). The thin lines represent the ties, the thick lines represent struts and the top platform is shaded (note that the struts are visible throughout since the top platform is truly three ties and not a solid platform).

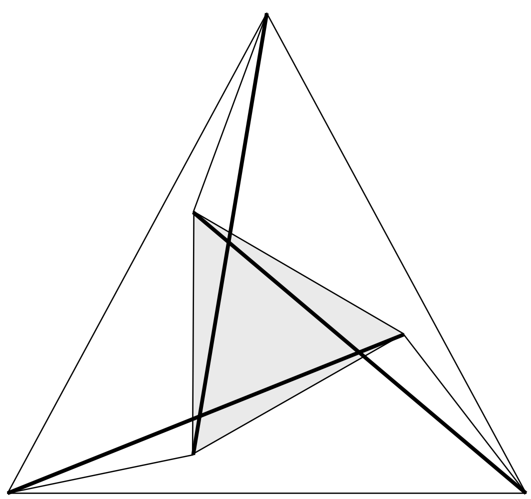

Figure 2.3

Stable positive twist
3-strut tensegrity

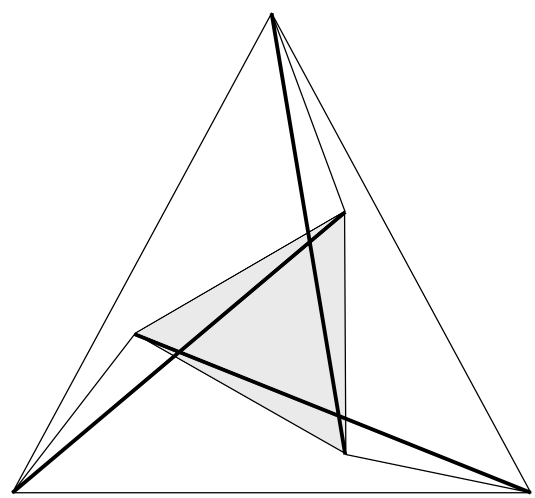

Figure 2.4

Stable negative twist
3-strut tensegrity

Knowing α can only have the value of 30^{O} for a positive twist tensegrity, it can be substituted back into our equations of static equilibrium.

The sum of the forces at DD is now

$$\sum_{DD} F_X = \frac{3}{2} F_b - \frac{\sqrt{3}\,a}{2\,L_S} F_s = 0, \tag{2.19}$$

$$\sum_{DD} F_Y = \frac{\sqrt{3}}{2} F_b - \frac{a}{2\,L_S} F_s = 0, \tag{2.20}$$

$$\sum_{DD} F_Z = \frac{h}{L_t} F_t - \frac{h}{L_s} F_s = 0, \tag{2.21}$$

where the lengths of the struts and leg ties are

$$L_s = \frac{\sqrt{3}}{3}\sqrt{b^2 + \sqrt{3}\,a\,b + a^2 + 3h^2}\,, \tag{2.22}$$

$$L_t = \frac{\sqrt{3}}{3}\sqrt{b^2 - \sqrt{3}\,a\,b + a^2 + 3h^2}\,. \tag{2.23}$$

Because the three equations of static equilibrium are linearly dependent, Eqs. (2.19) and (2.20) are the same equation. Therefore, the value of each force can only be related to another. This shows, as one member is stressed, the others members are also stressed. Solving Eq. (2.19) for the forces in the struts and substituting Eq. (2.12) in to find the forces in the leg ties in terms of the force in the bottom ties yields

$$F_s = \frac{\sqrt{3}\,L_s}{a} F_b, \tag{2.24}$$

$$F_t = \frac{\sqrt{3}\,L_t}{a} F_b\,. \tag{2.25}$$

2.1.3 Static Analysis at Joint AA

The force experienced by the top ties can be related to the force in the bottom ties by substituting Eqs. (2.24) and (2.25) into the equations of static equilibrium at joint AA. Figure 2.5 shows the forces acting at joint AA.

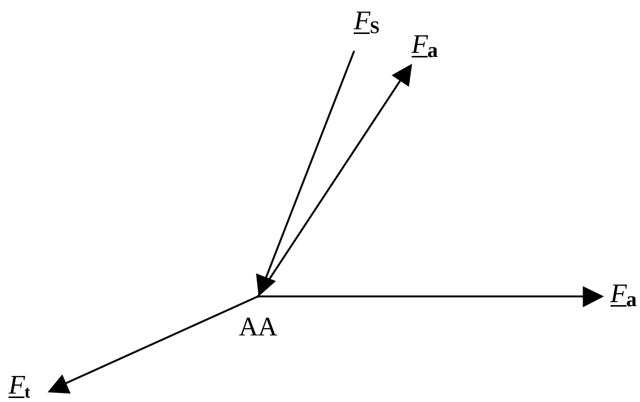

Figure 2.5
Forces at Joint AA

The equation for the sum of the forces at joint AA is

$$\sum_{AA} \underline{F}_i = F_a\, \underline{S}_5 + F_a\, \underline{S}_6 - F_t\, \underline{S}_3 + F_s\, \underline{S}_7 = 0. \qquad (2.26)$$

Note that the force produced by the leg tie is in the opposite direction of the unit vector $\underline{S}_3$. This is why the sign for that force is negative.

Expanding the Eq. (2.26) in the X, Y and Z directions produces

$$\sum_{AA} \underline{F}_X = \frac{\sqrt{3}}{2} F_a - \frac{3b - \sqrt{3}a}{6L_t} F_t - \frac{\sqrt{3}a}{6L_s} F_s = 0, \tag{2.27}$$

$$\sum_{AA} \underline{F}_Y = \frac{1}{2} F_a + F_a - \frac{\sqrt{3}b - 3a}{6L_t} F_t - \frac{2\sqrt{3}b + 3a}{6L_s} F_s = 0, \tag{2.28}$$

$$\sum_{AA} \underline{F}_Z = -\frac{h}{L_t} F_t + \frac{h}{L_s} F_s = 0. \tag{2.29}$$

Substituting Eqs. (2.24) and (2.25) into these equations and simplifying yields

$$\sum_{AA} \underline{F}_X = \frac{\sqrt{3}}{2} F_a - \frac{\sqrt{3}b}{2a} F_b = 0, \tag{2.30}$$

$$\sum_{AA} \underline{F}_Y = \frac{3}{2} F_a - \frac{3b}{2a} F_b = 0, \tag{2.31}$$

$$\sum_{AA} \underline{F}_Z = -\frac{\sqrt{3}h}{a} F_b + \frac{\sqrt{3}h}{a} F_b = 0. \tag{2.32}$$

It is apparent that Eqs. (2.30) and (2.31) are the same equations and Eq. (2.32) reduces to zero on both sides of the equations.

Solving Eq. (2.30) for F_a yields

$$F_a = \frac{b}{a} F_b. \tag{2.33}$$

This equation will also be found to be constant throughout all n-strut tensegrity systems.

Restating the forces in the members of the tensegrity system related to F_b

$$F_s = \frac{\sqrt{3}\,L_s}{a} F_b, \tag{2.24}$$

$$F_t = \frac{\sqrt{3}\,L_t}{a} F_b, \tag{2.25}$$

$$F_a = \frac{b}{a} F_b, \tag{2.33}$$

and the lengths of the struts and leg ties

$$L_s = \frac{\sqrt{3}}{3}\sqrt{b^2 + \sqrt{3}\,a\,b + a^2 + 3h^2}\,, \tag{2.22}$$

$$L_t = \frac{\sqrt{3}}{3}\sqrt{b^2 - \sqrt{3}\,a\,b + a^2 + 3h^2}\,. \tag{2.23}$$

2.2 4-Strut Tensegrity System

The 4-strut tensegrity system will be analyzed in the exact same method as the 3-strut tensegrity was analyzed, although there is uncertainty that the minimum limit position is stable for a 4-strut tensegrity. A coordinate system is again placed in the center of the bottom platform. The starting position is the positive twist at the "minimum limit" position (top platform rotated $\pi/4$ from the original anti-prism position). The top platform will be free to rotate in the positive Z direction. All variables used are the same as those in the previous analysis.

Figure 2.6 shows a top view of the 4-strut tensegrity at the minimum limit and the variables in the static analysis. The thin lines represent ties, the thick lines represent struts and the top platform is shaded. The direction of rotation (α) is depicted with an arrow and is equal to zero at this position. The lengths of the top and bottom ties are *a* and *b* respectively and the height of the tensegrity is *h* (not shown). The length of the struts is L_s and the length of the leg ties is L_t.

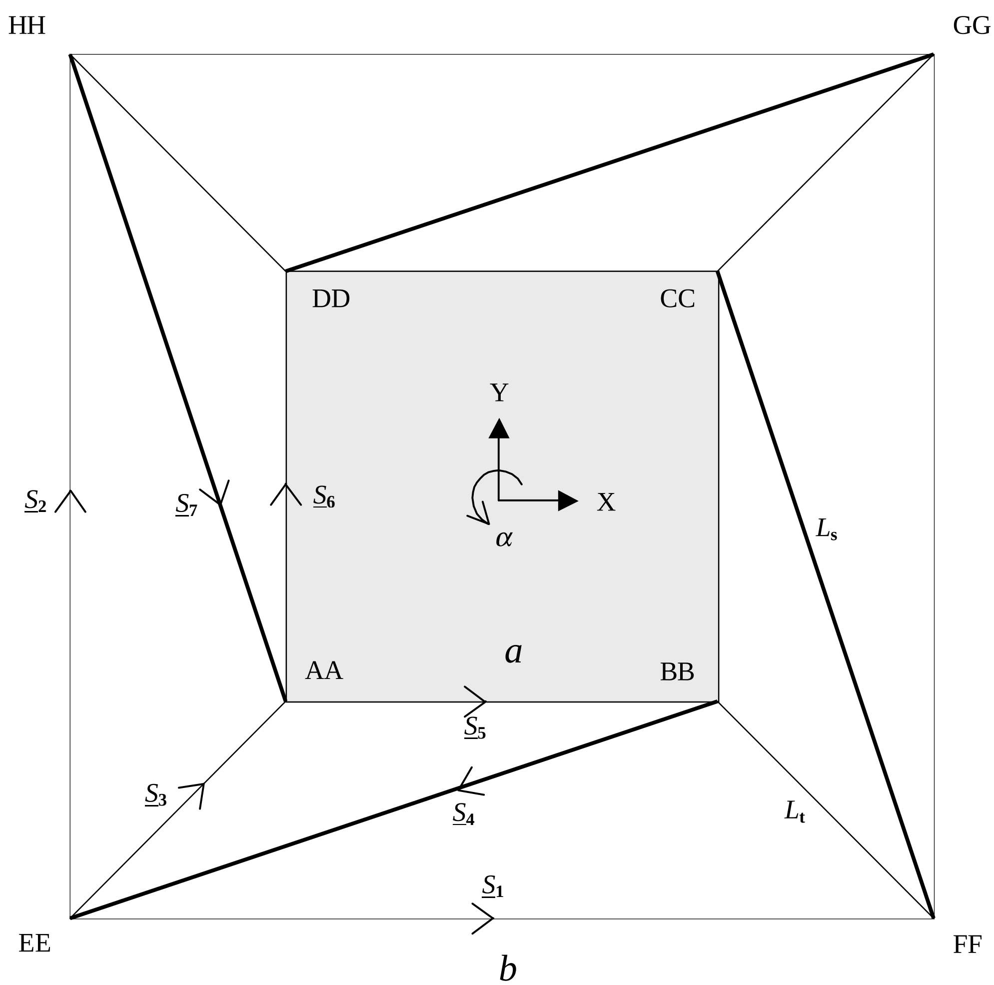

Figure 2.6

4-strut tensegrity [“minimum limit”]

2.2.1 Coordinate System

The coordinate of the joints are used to calculate the unit vectors of each member of the tensegrity. The coordinates of the joints in the top platform will be dependent on α. The coordinates of the joints are

$$\text{AA}\quad \left(-\frac{\sqrt{2}}{2} a\cos\left(\frac{\pi}{4}+\alpha\right),\quad -\frac{\sqrt{2}}{2} a\sin\left(\frac{\pi}{4}+\alpha\right),\quad h \right), \tag{2.34}$$

$$\text{BB}\quad \left(\frac{\sqrt{2}}{2} a\sin\left(\frac{\pi}{4}+\alpha\right),\quad -\frac{\sqrt{2}}{2} a\cos\left(\frac{\pi}{4}+\alpha\right),\quad h \right),$$

$$\text{CC}\quad \left(\frac{\sqrt{2}}{2} a\cos\left(\frac{\pi}{4}+\alpha\right),\quad \frac{\sqrt{2}}{2} a\sin\left(\frac{\pi}{4}+\alpha\right),\quad h \right),$$

$$\text{DD}\quad \left(-\frac{\sqrt{2}}{2} a\sin\left(\frac{\pi}{4}+\alpha\right),\quad \frac{\sqrt{2}}{2} a\cos\left(\frac{\pi}{4}+\alpha\right),\quad h \right),$$

$$\text{EE}\quad \left(-\frac{1}{2}b,\quad -\frac{1}{2}b,\quad 0 \right),$$

$$\text{FF}\quad \left(\frac{1}{2}b,\quad -\frac{1}{2}b,\quad 0 \right),$$

$$\text{GG}\quad \left(\frac{1}{2}b,\quad \frac{1}{2}b,\quad 0 \right),$$

$$\text{HH}\quad \left(-\frac{1}{2}b,\quad \frac{1}{2}b,\quad 0 \right).$$

The values of these vectors derived from the coordinates of the joints are

$$\underline{S}_1 = \{1, \quad 0, \quad 0\}, \tag{2.35}$$

$$\underline{S}_2 = \{0, \quad 1, \quad 0\},$$

$$\underline{S}_3 = \frac{1}{L_t}\left\{\frac{1}{2}b - \frac{\sqrt{2}}{2}a\cos\left(\frac{\pi}{4}+\alpha\right), \quad \frac{1}{2}b - \frac{\sqrt{2}}{2}a\sin\left(\frac{\pi}{4}+\alpha\right), \quad h\right\},$$

$$\underline{S}_4 = \frac{1}{L_s}\left\{-\frac{1}{2}b - \frac{\sqrt{2}}{2}a\sin\left(\frac{\pi}{4}+\alpha\right), \quad -\frac{1}{2}b + \frac{\sqrt{2}}{2}a\cos\left(\frac{\pi}{4}+\alpha\right), \quad -h\right\},$$

$$\underline{S}_5 = \{\cos(\alpha), \quad \sin(\alpha), \quad 0\},$$

$$\underline{S}_6 = \{-\sin(\alpha), \quad \cos(\alpha), \quad 0\},$$

$$\underline{S}_7 = \frac{1}{L_s}\left\{\frac{1}{2}b - \frac{\sqrt{2}}{2}a\cos\left(\frac{\pi}{4}+\alpha\right), \quad -\frac{1}{2}b - \frac{\sqrt{2}}{2}a\sin\left(\frac{\pi}{4}+\alpha\right), \quad h\right\},$$

where L_s and L_t are

$$L_s = \frac{\sqrt{2}}{2}\sqrt{b^2 - \sqrt{2}\,a\,b\cos\left(\frac{\pi}{4}+\alpha\right) + \sqrt{2}\,a\,b\sin\left(\frac{\pi}{4}+\alpha\right) + a^2 + 2h^2}\,, \tag{2.36}$$

$$L_t = \frac{\sqrt{2}}{2}\sqrt{b^2 - \sqrt{2}\,a\,b\cos\left(\frac{\pi}{4}+\alpha\right) - \sqrt{2}\,a\,b\sin\left(\frac{\pi}{4}+\alpha\right) + a^2 + 2h^2}\,. \tag{2.37}$$

2.2.2 Static Analysis at Joint EE

The summation of forces at joint EE is the initial step in the analysis of the 4-strut tensegrity. Figure 2.7 shows the forces acting at joint EE. Symmetry of the forces is assumed based on the symmetry of the tensegrity system. The variables used in the analysis are the same as the variables in the previous analysis.

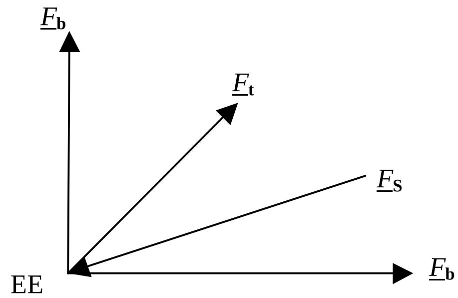

Figure 2.7

Forces at Joint EE

The equation for the summation of the forces at joint EE is

$$\sum_{EE} \underline{F}_i = F_b\, \underline{S}_1 + F_b\, \underline{S}_2 + F_t\, \underline{S}_3 + F_s\, \underline{S}_4 = 0. \tag{2.38}$$

Equation (2.35) is broken into the sum of the forces in the X, Y and Z directions.

$$\sum_{EE} F_X = F_b + \left[\frac{1}{2}b - \frac{\sqrt{2}}{2}a\cos\left(\frac{\pi}{4}+\alpha\right)\right]\frac{F_t}{L_t} - \left[\frac{1}{2}b + \frac{\sqrt{2}}{2}a\sin\left(\frac{\pi}{4}+\alpha\right)\right]\frac{F_s}{L_s} = 0, \tag{2.39}$$

$$\sum_{EE} F_Y = F_b + \left[\frac{1}{2}b - \frac{\sqrt{2}}{2}a\sin\left(\frac{\pi}{4}+\alpha\right)\right]\frac{F_t}{L_t} - \left[\frac{1}{2}b - \frac{\sqrt{2}}{2}a\cos\left(\frac{\pi}{4}+\alpha\right)\right]\frac{F_s}{L_s} = 0, \tag{2.40}$$

$$\sum_{EE} F_Z = \frac{h}{L_t}F_t - \frac{h}{L_s}F_s = 0. \tag{2.41}$$

Solving Eq. (2.41) for F_t yields

$$F_t = \frac{L_t}{L_s}F_s\,. \tag{2.42}$$

Note Eq. (2.42) is the same as Eq. (2.12) from the 3-strut tensegrity analysis.

Equation (2.42) is substituted into Eqs. (2.39) and (2.40) for F_t and simplified. The resulting equations in the X and Y direction are

$$\sum_{EE} F_X = F_b - \frac{a}{L_s} F_s \cos(\alpha) = 0, \tag{2.43}$$

$$\sum_{EE} F_Y = F_b - \frac{a}{L_s} F_s \sin(\alpha) = 0. \tag{2.44}$$

Both equations must be satisfied for the platform tensegrity to be in static equilibrium. Solving Eqs. (2.43) and (2.44) for F_b yields

$$F_b = \frac{a}{L_s} F_s \cos(\alpha), \tag{2.45}$$

$$F_b = \frac{a}{L_s} F_s \sin(\alpha). \tag{2.46}$$

Setting the right side of Eq. (2.45) equal to the right side of Eq. (2.46) and solving for α results in the equation

$$\tan(\alpha) = 1, \tag{2.47}$$

and yields a value for α

$$\alpha = \frac{\pi}{4}, \quad \frac{5\pi}{4}. \tag{2.48}$$

When α is $5\pi/4$ radians, the struts and leg ties must be exchanged to form the negative twist 4-strut tensegrity.

The initial rotation to the proposed minimum limit of π/n radians is added to α to determine for the total rotation of the top platform from the Archimedean 4-4 anti-prism.

$$\frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}.$$

This shows there are only two configurations of a 4-strut tensegrity system: positive twist and negative twist. Both configurations have the top platform rotated $\pi/2$ radians (90^{O}) from the original 4-4 anti-prism.

Figure 2.8 and 2.9 show the top view of a stable positive twist and negative twist 4-strut tensegrity (respectively). The thin lines represent the ties, the thick lines represent struts and the top platform is shaded.

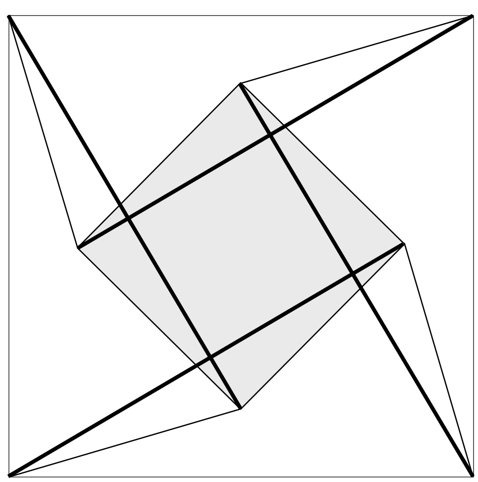

Figure 2.8

Stable positive twist
4-strut tensegrity

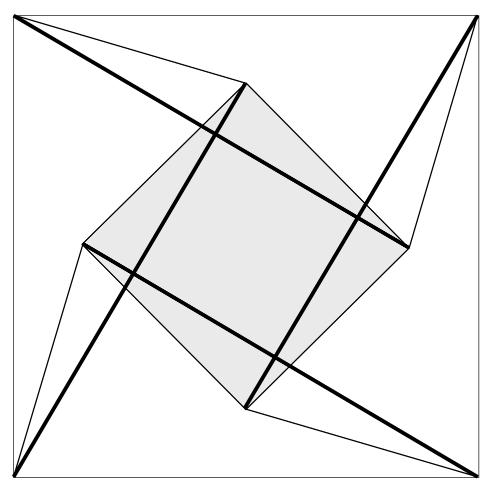

Figure 2.9

Stable negative twist
4-strut tensegrity

Knowing α can only be 45^{O} for a positive twist tensegrity, it can be substituted back into our equations of static equilibrium. The sum of the forces at EE is now

$$\sum_{EE} F_X = F_b - \frac{\sqrt{2}\,a}{2\,L_S} F_s = 0, \tag{2.49}$$

$$\sum_{EE} F_Y = F_b - \frac{\sqrt{2}\,a}{2\,L_S} F_s = 0, \tag{2.50}$$

$$\sum_{EE} F_Z = \frac{h}{L_t} F_c - \frac{h}{L_s} F_s = 0, \tag{2.51}$$

where the lengths of the struts and leg ties are

$$L_s = \frac{\sqrt{2}}{2}\sqrt{b^2 + \sqrt{2}\,a\,b + a^2 + 2\,h^2}\,, \tag{2.52}$$

$$L_t = \frac{\sqrt{2}}{2}\sqrt{b^2 - \sqrt{2}\,a\,b + a^2 + 2\,h^2}\,. \tag{2.53}$$

Because the three equations of static equilibrium are linearly dependent, Eqs. (2.49) and (2.50) are the same equation. The value of each force can only be related to another. Solving Eq. (2.49) for the forces in the struts and substituting Eq. (2.42) in to find the forces in the leg ties in terms of the force in the bottom ties yields

$$F_s = \frac{\sqrt{2}\,L_s}{a} F_b, \tag{2.54}$$

$$F_t = \frac{\sqrt{2}\,L_t}{a} F_b. \tag{2.55}$$

2.2.3 Static Analysis at Joint AA

The force experienced by the top ties can be related to the force in the bottom tie by substituting equations (2.54) and (2.55) in to the equations of static equilibrium at joint AA. Figure 2.10 shows the forces acting at joint AA.

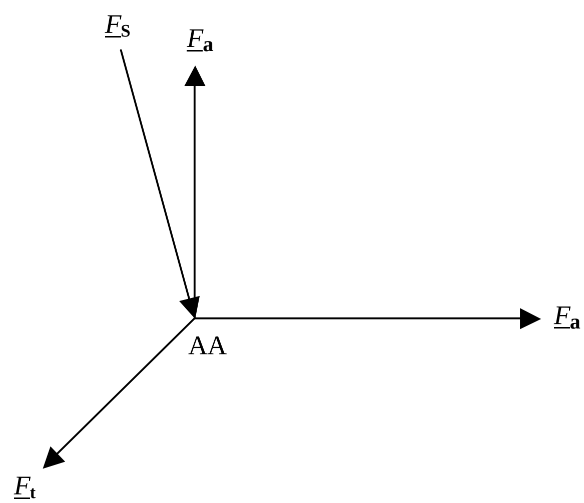

Figure 2.10
Forces at Joint AA

The equation for the sum of the forces at joint AA is

$$\sum_{AA} \underline{F}_i = F_a\, \underline{S}_5 + F_a\, \underline{S}_6 - F_t\, \underline{S}_3 + F_s\, \underline{S}_7 = 0. \tag{2.56}$$

Note that the force produced by the leg tie is in the opposite direction of the unit vector $\underline{S}_3$. This is why the sign for that force is negative.

Expanding the equation in the X, Y and Z directions produces

$$\sum_{AA} \underline{F}_X = \frac{\sqrt{2}}{2} F_a - \frac{\sqrt{2}}{2} F_a - \frac{b}{2\,L_t} F_t + \frac{b}{2\,L_s} F_s = 0, \tag{2.57}$$

$$\sum_{AA} \underline{F}_Y = \frac{\sqrt{2}}{2} F_a + \frac{\sqrt{2}}{2} F_a - \frac{b - \sqrt{2}\,a}{2\,L_t} F_t - \frac{b + \sqrt{2}\,a}{2\,L_s} F_s = 0, \tag{2.58}$$

$$\sum_{AA} \underline{F}_Z = - \frac{h}{L_t} F_t + \frac{h}{L_s} F_s = 0. \tag{2.59}$$

Substituting Eqs. (2.54) and (2.55) into these equations and simplifying yields

$$\sum_{AA} \underline{F}_X = \frac{\sqrt{2}}{2} F_a - \frac{\sqrt{2}}{2} F_a - \frac{\sqrt{2}\,b}{2\,a} F_b + \frac{\sqrt{2}\,b}{2\,a} F_b = 0, \tag{2.60}$$

$$\sum_{AA} \underline{F}_Y = \sqrt{2}\,F_a - \frac{\sqrt{2}\,b}{2\,a} F_b - \frac{\sqrt{2}\,b}{2\,a} F_b = 0, \tag{2.61}$$

$$\sum_{AA} \underline{F}_Z = -\frac{\sqrt{2}\,h}{a} F_b + \frac{\sqrt{2}\,h}{a} F_b = 0. \tag{2.62}$$

It is apparent that Eqs. (2.60) and (2.62) reduce to zero on both sides of the equations. Solving Eq. (2.61) for F_a yields

$$F_a = \frac{b}{a} F_b. \tag{2.63}$$

Note this equation is the same as Eq. (2.33).

Restating the forces in the members of the tensegrity system related to F_b

$$F_s = \frac{\sqrt{2}\, L_s}{a} F_b\,, \tag{2.54}$$

$$F_t = \frac{\sqrt{2}\, L_t}{a} F_b\,, \tag{2.55}$$

$$F_a = \frac{b}{a} F_b, \tag{2.63}$$

and the lengths of the struts and leg ties

$$L_s = \frac{\sqrt{2}}{2}\sqrt{b^2 + \sqrt{2}\, a\, b + a^2 + 2h^2}\,, \tag{2.52}$$

$$L_t = \frac{\sqrt{2}}{2}\sqrt{b^2 - \sqrt{2}\, a\, b + a^2 + 2h^2}\,. \tag{2.53}$$

2.3 5-Strut Tensegrity System

The 5-strut tensegrity system will be analyzed in the exact same method as the 3-strut and 4-strut tensegrity systems were analyzed. A coordinate system is placed in the center of the bottom platform. The starting position is the positive twist at the "minimum limit" position (top platform rotated $\pi/5$ from the original anti-prism position). The top platform will be free to rotate in the positive Z direction. All variables used are the same as those in the previous analyses.

Figure 2.11 shows a top view of the 5-strut tensegrity at the minimum limit and the variables in the static analysis. The thin lines represent ties, the thick lines represent struts and the top platform is shaded.

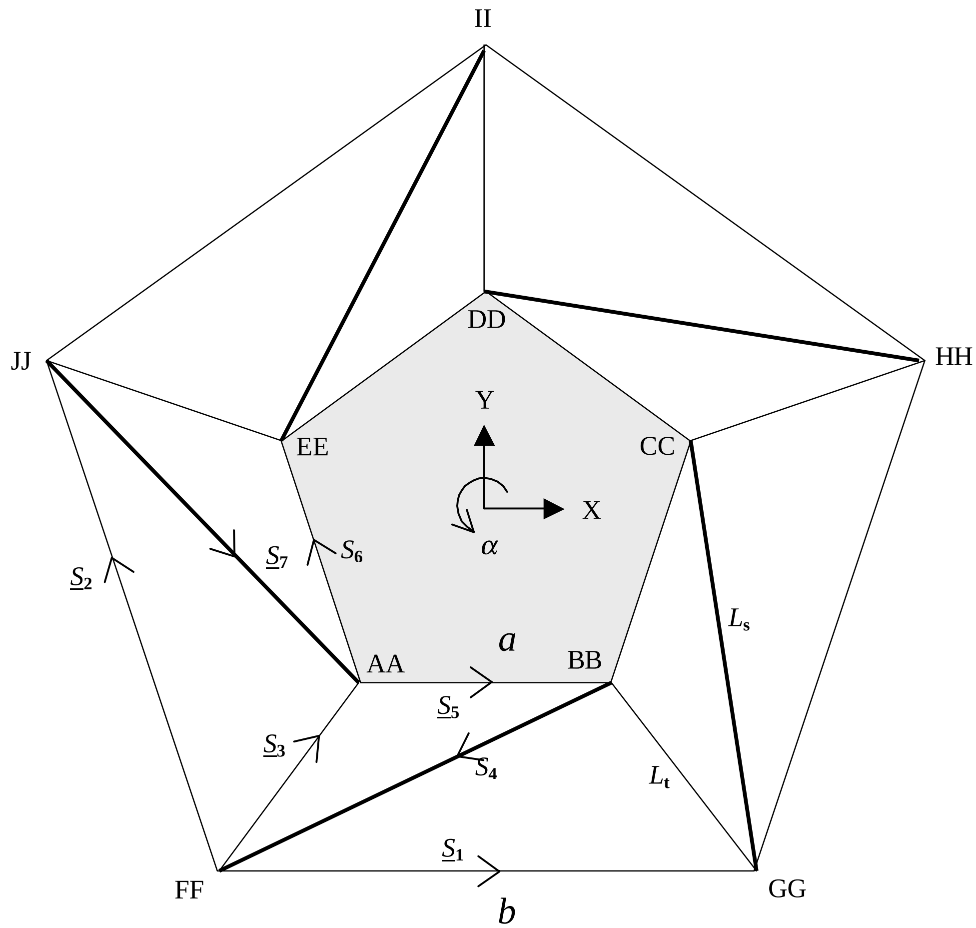

Figure 2.11

5-strut tensegrity [“minimum limit”]

2.3.1 Coordinate System

The coordinate of the joints are used to calculate the unit vectors of each member of the tensegrity. The coordinates of the joints in the top platform are dependent on α.

The coordinates of the joints used to calculate the direction vectors in this analysis are

$$\text{AA} \quad \left(-\frac{\sin\left(\frac{\pi}{5}-\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\,,\quad -\frac{\sin\left(\frac{3\pi}{10}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\,,\quad h \right), \tag{2.64}$$

$$\text{BB} \quad \left(\frac{\sin\left(\frac{\pi}{5}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\,,\quad -\frac{\sin\left(\frac{3\pi}{10}-\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\,,\quad h \right),$$

$$\text{EE} \quad \left(-\frac{\sin\left(\frac{2\pi}{5}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\,,\quad \frac{\sin\left(\frac{\pi}{10}-\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\,,\quad h \right),$$

$$\text{FF} \quad \left(-\frac{1}{2}b\,,\quad -\frac{\sin\left(\frac{3\pi}{10}\right)}{2\sin\left(\frac{\pi}{5}\right)}b\,,\quad 0 \right),$$

$$\text{GG} \quad \left(\frac{1}{2}b\,,\quad -\frac{\sin\left(\frac{3\pi}{10}\right)}{2\sin\left(\frac{\pi}{5}\right)}b\,,\quad 0 \right),$$

$$\text{JJ} \quad \left(-\frac{\sin\left(\frac{2\pi}{5}\right)}{2\sin\left(\frac{\pi}{5}\right)}b\,,\quad \frac{\sin\left(\frac{\pi}{10}\right)}{2\sin\left(\frac{\pi}{5}\right)}b\,,\quad 0 \right).$$

The values for the sine of the various angles will be substituted latter in the analysis.

The values of these vectors derived from the coordinates of the joints are

$$\underline{S}_1 = \{1,\quad 0,\quad 0\}, \tag{2.65}$$

$$\underline{S}_2 = \left\{ \frac{1}{2} - \sin\left(\frac{3\pi}{10}\right),\quad \frac{1+4\sin^2\left(\frac{3\pi}{10}\right)}{4\sin\left(\frac{2\pi}{5}\right)},\quad 0 \right\},$$

$$\underline{S}_3 = \frac{1}{L_t}\left\{ \frac{1}{2}b - \frac{\sin\left(\frac{\pi}{5}-\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a,\quad \frac{\sin\left(\frac{3\pi}{10}\right)}{2\sin\left(\frac{\pi}{5}\right)}b - \frac{\sin\left(\frac{3\pi}{10}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a,\quad h \right\},$$

$$\underline{S}_4 = \frac{1}{L_s}\left\{ -\frac{1}{2}b - \frac{\sin\left(\frac{\pi}{5}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a,\quad -\frac{\sin\left(\frac{3\pi}{10}\right)}{2\sin\left(\frac{\pi}{5}\right)}b + \frac{\sin\left(\frac{3\pi}{10}-\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a,\quad -h \right\},$$

$$\underline{S}_5 = \{\cos(\alpha),\quad \sin(\alpha),\quad 0\},$$

$$\underline{S}_6 = \left\{ \frac{\sin\left(\frac{\pi}{5}-\alpha\right) - \sin\left(\frac{2\pi}{5}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)},\quad \frac{\sin\left(\frac{\pi}{10}-\alpha\right) + \sin\left(\frac{3\pi}{10}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)},\quad 0 \right\},$$

$$\underline{S}_7 = \frac{1}{L_s}\left\{ \sin\left(\frac{3\pi}{10}\right)b - \frac{\sin\left(\frac{\pi}{5}-\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a,\quad -\frac{1}{4\sin\left(\frac{2\pi}{5}\right)}b - \frac{\sin\left(\frac{3\pi}{10}+\alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a,\quad h \right\}.$$

Where L_s and L_t are

$$L_s = \frac{1}{2\sin\left(\frac{\pi}{5}\right)}\sqrt{b^2 - 2ab\cos\left(\frac{2\pi}{5}+\alpha\right)+a^2+4h^2\sin^2\left(\frac{\pi}{5}\right)}, \qquad (2.66)$$

$$L_t = \frac{1}{2\sin\left(\frac{\pi}{5}\right)}\sqrt{b^2 - 2ab\cos(\alpha)+a^2+4h^2\sin^2\left(\frac{\pi}{5}\right)}. \qquad (2.67)$$

2.3.2 Static Analysis at FF

The summation of forces at joint FF is the initial step in the analysis of the 5-strut tensegrity. Figure 2.12 shows the forces acting at joint FF. Symmetry of the forces is assumed based on the symmetry of the tensegrity system. The variables used in the analysis are the same as the variables in the previous analyses.

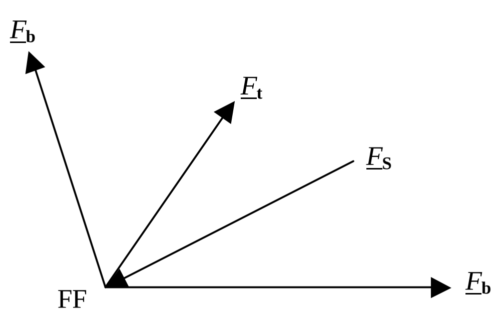

Figure 2.12
Forces at Joint FF

The equation for the summation of the forces at joint FF is

$$\sum_{FF} \underline{F}_i = F_b\,\underline{S}_1 + F_b\,\underline{S}_2 + F_t\,\underline{S}_3 + F_s\,\underline{S}_4 = 0. \qquad (2.68)$$

Equation (2.68) is broken into the sum of the forces in the X, Y and Z directions.

$$\sum_{FF} F_X = \left[\frac{3}{2} - \sin\left(\frac{3\pi}{10}\right)\right] F_b + \left[\frac{1}{2}b - \frac{\sin\left(\frac{\pi}{5} - \alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\right]\frac{F_t}{L_t} - \left[\frac{1}{2}b + \frac{\sin\left(\frac{\pi}{5} + \alpha\right)}{2\sin\left(\frac{\pi}{5}\right)}a\right]\frac{F_s}{L_s} = 0, \tag{2.69}$$

$$\sum_{FF} F_Y = \left[\frac{1 + 4\sin^2\left(\frac{3\pi}{10}\right)}{4\sin\left(\frac{2\pi}{5}\right)}\right] F_b + \left[\frac{\sin\left(\frac{3\pi}{10}\right)b - \sin\left(\frac{3\pi}{10} + \alpha\right)a}{2\sin\left(\frac{\pi}{5}\right)}\right]\frac{F_t}{L_t} - \left[\frac{\sin\left(\frac{3\pi}{10}\right)b - \sin\left(\frac{3\pi}{10} - \alpha\right)a}{2\sin\left(\frac{\pi}{5}\right)}\right]\frac{F_s}{L_s} = 0, \tag{2.70}$$

$$\sum_{FF} F_Z = \frac{h}{L_t}F_t - \frac{h}{L_s}F_s = 0. \tag{2.71}$$

Solving the Eq. (2.71) for F_t yields

$$F_t = \frac{L_t}{L_s}F_s\,. \tag{2.72}$$

Note Eq. (2.72) is the same as Eq. (2.12) from the 3-strut tensegrity analysis.

Equation (2.72) is substituted into Eqs. (2.69) and (2.70) for F_t and simplified. The resulting equations in the X and Y direction are

$$\sum_{FF} F_X = \left[\frac{3}{2} - \sin\left(\frac{3\pi}{10}\right)\right] F_b - \frac{a}{L_S} F_s \cos(\alpha) = 0, \tag{2.73}$$

$$\sum_{FF} F_Y = \left[\frac{1 + 4\sin^2\left(\frac{3\pi}{10}\right)}{4\sin\left(\frac{2\pi}{5}\right)}\right] F_b - \frac{a}{L_S} F_s \sin(\alpha) = 0. \tag{2.74}$$

Both equations must be satisfied for the platform tensegrity to be in static equilibrium. Solving Eqs. (2.73) and (2.74) for F_b yields

$$F_b = \frac{2a}{\left[1 - 2\sin\left(\frac{3\pi}{10}\right)\right] L_s} F_s \cos(\alpha), \tag{2.75}$$

$$F_b = \frac{4a\sin\left(\frac{2\pi}{5}\right)}{\left[1 + 4\sin^2\left(\frac{3\pi}{10}\right)\right] L_s} F_s \sin(\alpha). \tag{2.76}$$

Setting the right side of Eq. (2.75) equal to the right side of Eq. (2.76) and solving for α results in the equation

$$\tan(\alpha) = \frac{1 + 4\sin^2\left(\frac{3\pi}{10}\right)}{2\sin\left(\frac{2\pi}{5}\right)\left[3 - 2\sin\left(\frac{3\pi}{10}\right)\right]}. \tag{2.77}$$

Substituting the values for the sine of the angles,

$$\sin\left(\frac{3\pi}{10}\right)=\frac{1+\sqrt{5}}{4},$$

$$\sin\left(\frac{2\pi}{5}\right)=\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{4},$$

into Eq. (2.77) and simplifying produces

$$\tan(\alpha)=\frac{\sqrt{2\left(5+\sqrt{5}\right)}}{5-\sqrt{5}}. \tag{2.77}$$

Eq. (2.77) yields a value for α

$$\alpha=\frac{3\pi}{10},\quad \frac{13\pi}{10}. \tag{2.78}$$

When α is $3\pi/10$ radians, the struts and leg ties must be exchanged to form the negative twist 5-strut tensegrity.

The initial rotation to the proposed minimum limit of π/n radians is added to α to determine for the total rotation of the top platform from the Archimedean 5-5 anti-prism.

$$\frac{3\pi}{10}+\frac{\pi}{5}=\frac{\pi}{2}.$$

This shows there are only two configurations of a 5-strut tensegrity system: positive twist and negative twist. Both configurations have the top platform rotated $\pi/2$ radians (90^{O}) from the original 5-5 anti-prism.

Figure 2.13 and 2.14 show the top view of a stable positive twist and negative twist 5-strut tensegrity (respectively). The thin lines represent the ties, the thick lines represent struts and the top platform is shaded.

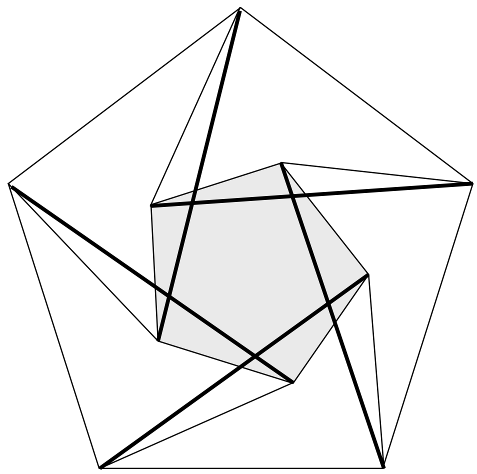

Figure 2.13

Stable positive twist
5-strut tensegrity

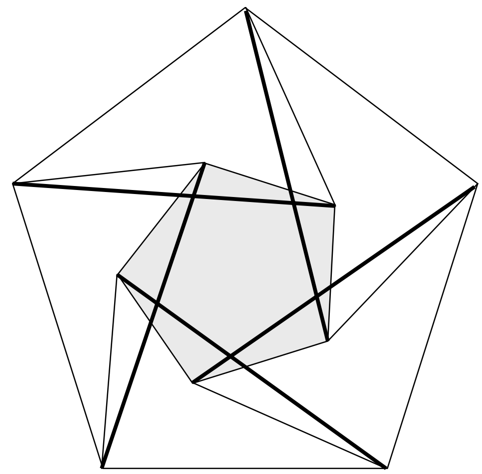

Figure 2.14

Stable negative twist
5-strut tensegrity

Knowing α can only be 54^{O} for a positive twist tensegrity, it can be substituted back into our equations of static equilibrium. The sum of the forces at FF is now

$$\sum_{FF} F_X = \left[\frac{3}{2} - \sin\left(\frac{3\pi}{10}\right)\right] F_b - \frac{a}{L_s} F_s \sin\left(\frac{\pi}{5}\right) = 0, \tag{2.79}$$

$$\sum_{FF} F_Y = \left[\frac{1 + 4\sin^2\left(\frac{3\pi}{10}\right)}{4\sin\left(\frac{2\pi}{5}\right)}\right] F_b - \frac{a}{L_s} F_s \sin\left(\frac{3\pi}{10}\right) = 0. \tag{2.80}$$

$$\sum_{FF} F_Z = \frac{h}{L_t} F_t - \frac{h}{L_s} F_s = 0, \tag{2.81}$$

where the lengths of the struts and leg ties are

$$L_s = \frac{1}{2\sin\left(\frac{\pi}{5}\right)}\sqrt{b^2 + 2ab\sin\left(\frac{\pi}{5}\right) + a^2 + 4h^2\sin^2\left(\frac{\pi}{5}\right)}, \tag{2.82}$$

$$L_t = \frac{1}{2\sin\left(\frac{\pi}{5}\right)}\sqrt{b^2 - 2ab\sin\left(\frac{\pi}{5}\right) + a^2 + 4h^2\sin^2\left(\frac{\pi}{5}\right)}. \tag{2.83}$$

Because the three equations of static equilibrium are linearly dependent, Eqs. (2.79) and (2.80) are the same equation. The value of each force can only be related to another. Substituting the values for the sine of the angle,

$$\sin\left(\frac{\pi}{5}\right) = \frac{\sqrt{2\left(5-\sqrt{5}\right)}}{4},$$

and the previously used values into Eqs. (2.79), (2.82) and (2.83) and simplifying produces

$$\sum_{FF} F_X = \sqrt{5-\sqrt{5}}\,F_b - \frac{\sqrt{2}\,a}{L_S}F_s = 0, \tag{2.79}$$

$$L_s = \sqrt{\frac{2b^2 + \sqrt{2\left(5-\sqrt{5}\right)}\,ab + 2a^2}{5-\sqrt{5}} + h^2}, \tag{2.82}$$

$$L_t = \sqrt{\frac{2b^2 - \sqrt{2\left(5-\sqrt{5}\right)}\,ab + 2a^2}{5-\sqrt{5}} + h^2}. \tag{2.83}$$

Solving Eq. (2.79) for the forces in the struts and substituting Eq. (2.72) in to find the forces in the leg ties in terms of the force in the bottom ties yields

$$F_s = \frac{\sqrt{2\left(5-\sqrt{5}\right)}L_s}{2a}F_b, \tag{2.84}$$

$$F_t = \frac{\sqrt{2\left(5-\sqrt{5}\right)}L_t}{2a}F_b. \tag{2.85}$$

2.3.3 Static Analysis at Joint AA

The force experienced by the top ties can be related to the force in the bottom ties by substituting equations (2.84) and (2.85) in to the equations of static equilibrium at joint AA. Figure 2.15 shows the forces acting on at joint AA.

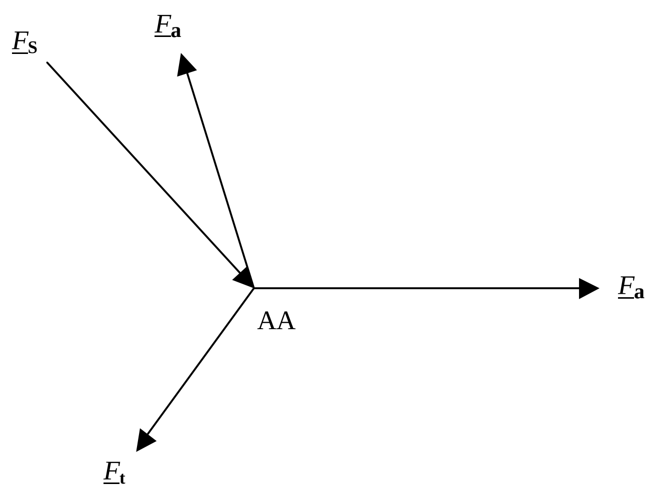

Figure 2.15

Forces at Joint AA

The equation for the sum of the forces at joint AA is

$$\sum_{AA} \underline{F}_i = F_a\, \underline{S}_5 + F_a\, \underline{S}_6 - F_t\, \underline{S}_3 + F_s\, \underline{S}_7 = 0. \qquad (2.86)$$

Note that the force produced by the leg tie is in the opposite direction of the unit vector $\underline{S}_3$. This is why the sign for that force is negative.

Expanding the equation in the X, Y and Z directions produces

$$\sum_{AA} \underline{F}_X = \sin\left(\frac{\pi}{5}\right) F_a - \frac{1 + 4\sin^2\left(\frac{3\pi}{10}\right)}{4\sin\left(\frac{2\pi}{5}\right)} F_a - \left[\frac{1}{2} b + \frac{1}{4\sin\left(\frac{2\pi}{5}\right)} a\right] \frac{F_t}{L_t} + \left[b \sin\left(\frac{3\pi}{10}\right) + \frac{1}{4\sin\left(\frac{2\pi}{5}\right)} a\right] \frac{F_s}{L_s} = 0, \qquad (2.87)$$

$$\sum_{AA} \underline{F}_Y = F_a \sin\left(\frac{3\pi}{10}\right) + \left[\sin\left(\frac{3\pi}{10}\right) - \frac{1}{2}\right] F_a - \left[\frac{\sin\left(\frac{3\pi}{10}\right)}{2\sin\left(\frac{\pi}{5}\right)} b - a \sin\left(\frac{3\pi}{10}\right)\right] \frac{F_t}{L_t} - \left[\frac{1}{4\sin\left(\frac{2\pi}{5}\right)} b + a \sin\left(\frac{3\pi}{10}\right)\right] \frac{F_s}{L_s} = 0, \qquad (2.88)$$

$$\sum_{AA} \underline{F}_Z = -\frac{h}{L_t} F_t + \frac{h}{L_s} F_s = 0. \qquad (2.89)$$

Substituting Eqs. (2.84) and (2.85) into these equations and simplifying yields

$$\sum_{AA} \underline{F}_X = \left[\sin\left(\frac{\pi}{5}\right) - \frac{1 + 4\sin^2\left(\frac{3\pi}{10}\right)}{4\sin\left(\frac{2\pi}{5}\right)} \right] F_a + \left[\sin\left(\frac{2\pi}{5}\right) - \sin\left(\frac{\pi}{5}\right) \right] \frac{b}{a} F_b = 0, \quad (2.90)$$

$$\sum_{AA} \underline{F}_Y = \left[2\sin\left(\frac{3\pi}{10}\right) - \frac{1}{2} \right] F_a - \left[\sin\left(\frac{3\pi}{10}\right) + \frac{1}{4\sin\left(\frac{3\pi}{10}\right)} \right] \frac{b}{a} F_b = 0, \quad (2.91)$$

$$\sum_{AA} \underline{F}_Z = -\frac{2h}{a} F_b \sin\left(\frac{\pi}{5}\right) + \frac{2h}{a} F_b \sin\left(\frac{\pi}{5}\right) = 0. \quad (2.92)$$

It is apparent that Eq. (2.92) reduces to zero on both sides of the equations. Solving Eqs. (2.90) and (2.91) for F_a and substituting the values of the sine of the angles yields the same results

$$F_a = \frac{b}{a} F_b. \quad (2.93)$$

Note this equation is the same as Eq. (2.33).

Restating the forces in the members of the tensegrity system related to F_b

$$F_s = \frac{\sqrt{2\left(5 - \sqrt{5}\right)} L_s}{2a} F_b, \quad (2.84)$$

$$F_t = \frac{\sqrt{2\left(5 - \sqrt{5}\right)} L_t}{2a} F_b. \quad (2.85)$$

$$F_a = \frac{b}{a} F_b, \quad (2.93)$$

and the lengths of the struts and leg ties

$$L_s = \sqrt{\frac{2b^2 + \sqrt{2\left(5-\sqrt{5}\right)}ab + 2a^2}{5-\sqrt{5}} + h^2}\ , \tag{2.82}$$

$$L_t = \sqrt{\frac{2b^2 - \sqrt{2\left(5-\sqrt{5}\right)}ab + 2a^2}{5-\sqrt{5}} + h^2}\ . \tag{2.83}$$

2.4 6-Strut Tensegrity System

The 6-strut tensegrity system will be analyzed in the exact same method as the previous tensegrity systems were analyzed. A coordinate system is placed in the center of the bottom platform. The starting position is the positive twist at the “minimum limit” position (top platform rotated $\pi/6$ from the original anti-prism position). The top platform will be free to rotate in the positive Z direction. All variables used are the same as those in the previous analyses.

Figure 2.16 shows a top view of the 6-strut tensegrity at the minimum limit and the variables in the static analysis. The thin lines represent ties, the thick lines represent struts and the top platform is shaded. The direction of rotation (α) is depicted with an arrow and is equal to zero at this position. The lengths of the top and bottom ties are a and b respectively and the height of the tensegrity is h (not shown). The length of the struts is L_s and the length of the leg ties is L_t.

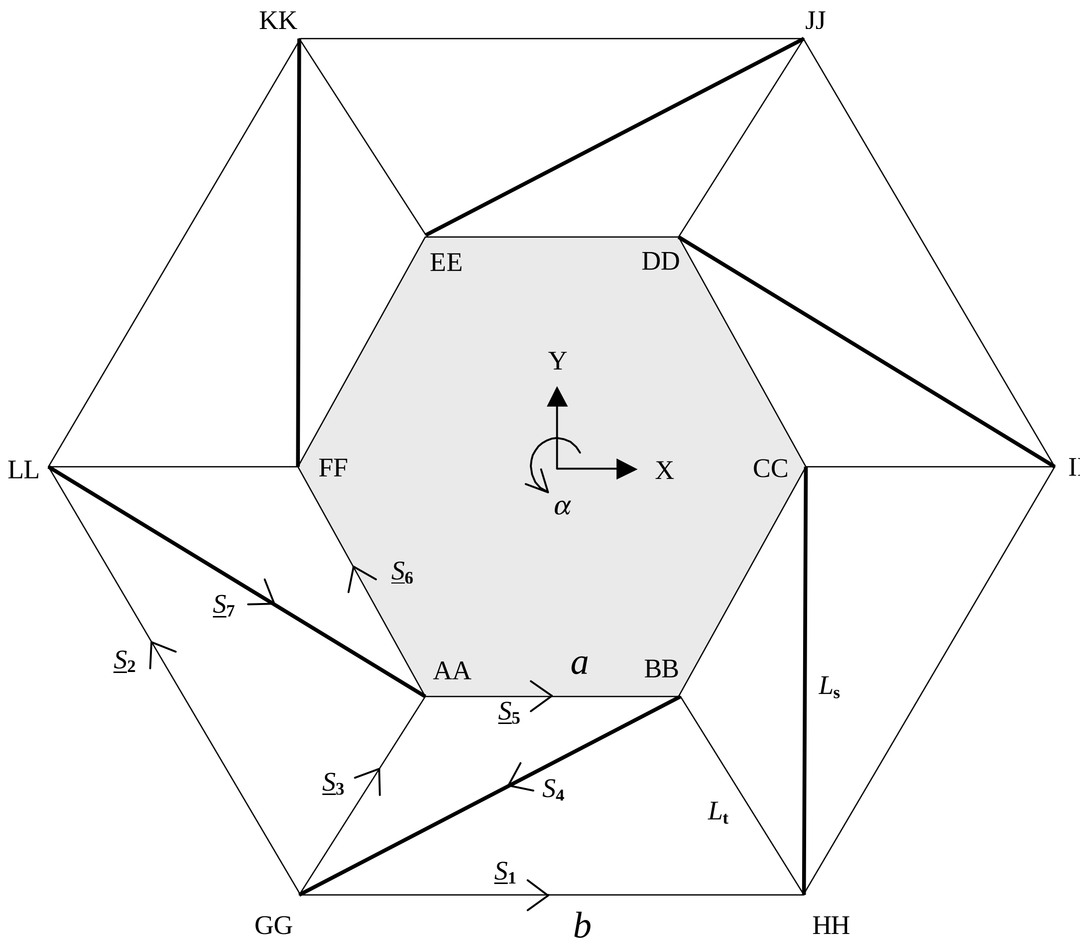

Figure 2.16

6-strut tensegrity [“minimum limit”]

2.4.1 Coordinate System

The coordinate of the joints are used to calculate the unit vectors of each member of the tensegrity. The coordinates of the joints in the top platform are dependent on α. The coordinates of the joints used to calculate the direction vectors in this analysis are

$$\text{AA} \quad \left(-a\cos\left(\frac{\pi}{3}+\alpha\right), \quad -a\sin\left(\frac{\pi}{3}+\alpha\right), \quad h \right), \tag{2.94}$$

$$\text{BB} \quad \left(a\sin\left(\frac{\pi}{6}+\alpha\right), \quad -a\cos\left(\frac{\pi}{6}+\alpha\right), \quad h \right),$$

$$\text{FF} \quad \left(-a\cos(\alpha), \quad -a\sin(\alpha), \quad h \right),$$

$$\text{GG} \quad \left(-\frac{1}{2}b, \quad -\frac{\sqrt{3}}{2}b, \quad 0 \right),$$

$$\text{HH} \quad \left(\frac{1}{2}b, \quad -\frac{\sqrt{3}}{2}b, \quad 0 \right),$$

$$\text{LL} \quad \left(-b, \quad 0, \quad 0 \right).$$

The values of these vectors derived from the coordinates of the joints are

$$\underline{S}_1 = \{1, \quad 0, \quad 0\}, \tag{2.95}$$

$$\underline{S}_2 = \left\{ -\frac{1}{2}, \quad \frac{\sqrt{3}}{2}, \quad 0 \right\},$$

$$\underline{S}_3 = \frac{1}{L_t}\left\{ \frac{1}{2}b - a\cos\left(\frac{\pi}{3}+\alpha\right), \quad \frac{\sqrt{3}}{2}b - a\sin\left(\frac{\pi}{3}+\alpha\right), \quad h \right\},$$

$$\underline{S}_4 = \frac{1}{L_s}\left\{ -\frac{1}{2}b - a\sin\left(\frac{\pi}{6}+\alpha\right), \quad -\frac{\sqrt{3}}{2}b + a\cos\left(\frac{\pi}{6}+\alpha\right), \quad -h \right\},$$

$$\underline{S}_5 = \{\cos(\alpha), \quad \sin(\alpha), \quad 0\},$$

$$\underline{S}_6 = \left\{ -\frac{\sqrt{3}}{2}\sin(\alpha) - \frac{1}{2}\cos(\alpha), \quad \frac{\sqrt{3}}{2}\cos(\alpha) - \frac{1}{2}\sin(\alpha), \quad 0 \right\},$$

$$\underline{S}_7 = \frac{1}{L_s}\left\{ b - a\cos\left(\frac{\pi}{3}+\alpha\right), \quad -a\sin\left(\frac{\pi}{3}+\alpha\right), \quad h \right\}.$$

Where L_s and L_t are

$$L_s = \sqrt{b^2 - 2\,a\,b\cos\left(\frac{\pi}{3}+\alpha\right)+a^2+h^2}\,, \tag{2.96}$$

$$L_t = \sqrt{b^2 - 2\,a\,b\cos(\alpha)+a^2+h^2}\,. \tag{2.97}$$

2.4.2 Static Analysis at Joint GG

The summation of forces at joint GG is the initial step in the analysis of the 6-strut tensegrity. Figure 2.17 shows the forces acting at joint GG. Symmetry of the forces is assumed based on the symmetry of the tensegrity system. F_b is the force in the bottom ties, F_a is the force in the top ties, F_t is the force in the leg ties, and F_s is the force in the struts.

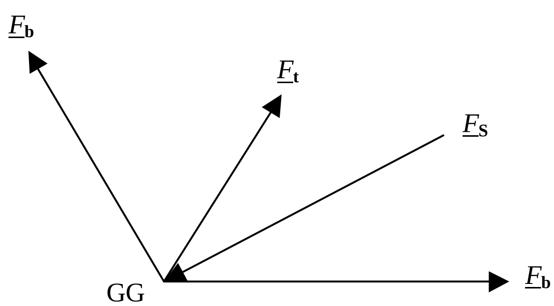

Figure 2.17
Forces at Joint GG

The equation for the summation of the forces at joint GG is

$$\sum_{GG} \underline{F}_i = F_b\,\underline{S}_1 + F_b\,\underline{S}_2 + F_t\,\underline{S}_3 + F_s\,\underline{S}_4 = 0. \tag{2.98}$$

Equation (2.35) is broken into the sum of the forces in the X, Y and Z directions.

$$\sum_{GG} F_X = F_b - \frac{1}{2}F_b + \left[\frac{1}{2}b - a\cos\left(\frac{\pi}{3}+\alpha\right)\right]\frac{F_t}{L_t} - \left[\frac{1}{2}b + a\sin\left(\frac{\pi}{6}+\alpha\right)\right]\frac{F_s}{L_s} = 0, \quad (2.99)$$

$$\sum_{GG} F_Y = \frac{\sqrt{3}}{2}F_b + \left[\frac{\sqrt{3}}{2}b - a\sin\left(\frac{\pi}{3}+\alpha\right)\right]\frac{F_t}{L_t} - \left[\frac{\sqrt{3}}{2}b - a\cos\left(\frac{\pi}{6}+\alpha\right)\right]\frac{F_s}{L_s} = 0, \quad (2.100)$$

$$\sum_{GG} F_Z = \frac{h}{L_c}F_c - \frac{h}{L_s}F_s = 0. \quad (2.101)$$

Solving the third equation (summation of the forces in the Z direction) for F_t yields

$$F_t = \frac{L_c}{L_s}F_s. \quad (2.102)$$

Note Eq. (2.102) is the same as Eq. (2.12) from the 3-strut tensegrity analysis.

Equation (2.102) is substituted into Eqs. (2.99) and (2.100) for F_t and simplified. The resulting equations in the X and Y direction are

$$\sum_{GG} F_X = \frac{1}{2}F_b - \frac{a}{L_s}F_s\cos(\alpha) = 0, \quad (2.103)$$

$$\sum_{GG} F_Y = \frac{\sqrt{3}}{2}F_b - \frac{a}{L_s}F_s\sin(\alpha) = 0. \quad (2.104)$$

Both equations must be satisfied for the platform tensegrity to be in static equilibrium. Solving Eqs. (2.103) and (2.104) for F_b yields

$$F_b = \frac{2a}{L_s} F_s \cos(\alpha), \tag{2.105}$$

$$F_b = \frac{2\sqrt{3}a}{3L_s} F_s \sin(\alpha). \tag{2.106}$$

Setting the right side of Eq. (2.105) equal to the right side of Eq. (2.106) and solving for α results in the equation

$$\tan(\alpha) = \sqrt{3}, \tag{2.107}$$

and yields a value for α

$$\alpha = \frac{\pi}{3}, \quad \frac{4\pi}{3}. \tag{2.108}$$

When α is $4\pi/3$ radians, the struts and leg ties must be exchanged to form the negative twist 6-strut tensegrity.

The initial rotation to the proposed minimum limit of π/n radians is added to α to determine for the total rotation of the top platform from the Archimedean 6-6 anti-prism.

$$\frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}.$$

This shows there are only two configurations of a 6-strut tensegrity system: positive twist and negative twist. Both configurations have the top platform rotated $\pi/2$ radians (90^{O})

from the original 6-6 anti-prism.

Figure 2.18 and 2.19 show the top view of a stable positive twist and negative twist 6-strut tensegrity (respectively). The thin lines represent the ties, the thick lines represent struts and the top platform is shaded.

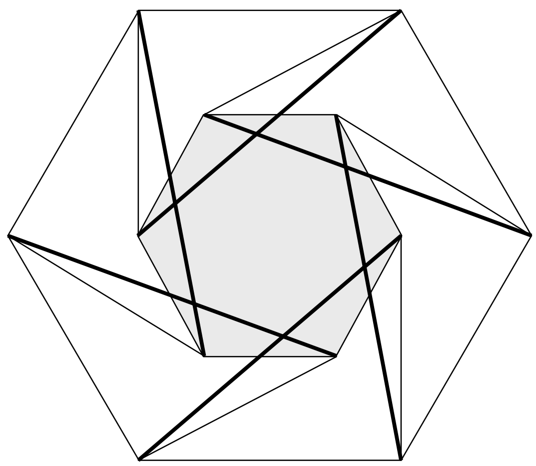

Figure 2.18

Stable positive twist
6-strut tensegrity

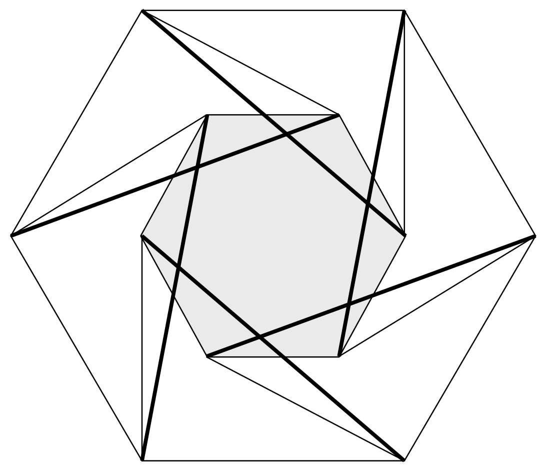

Figure 2.19

Stable negative twist
6-strut tensegrity

Knowing α can only be 60^{O} for a positive twist tensegrity, it can be substituted back into our equations of static equilibrium. The sum of the forces at GG is now

$$\sum_{GG} F_X = \frac{1}{2} F_b - \frac{a}{2 L_S} F_s = 0, \tag{2.109}$$

$$\sum_{EE} F_Y = \frac{\sqrt{3}}{2} F_b - \frac{\sqrt{3}\, a}{2 L_S} F_s = 0, \tag{2.110}$$

$$\sum_{EE} F_Z = \frac{h}{L_t} F_t - \frac{h}{L_s} F_s = 0, \tag{2.111}$$

where the lengths of the struts and leg ties are

$$L_s = \sqrt{b^2 + a\,b + a^2 + h^2}\ , \tag{2.112}$$

$$L_t = \sqrt{b^2 - a\,b + a^2 + h^2}\ . \tag{2.113}$$

Because the three equations of static equilibrium are linearly dependent, Eqs. (2.109) and (2.110) are the same equation. The value of each force can only be related to another. Solving Eq. (2.109) for the forces in the struts and substituting Eq. (2.102) in to find the forces in the leg ties in terms of the force in the bottom ties yields

$$F_s = \frac{L_s}{a} F_b\,, \tag{2.114}$$

$$F_t = \frac{L_t}{a} F_b\,. \tag{2.115}$$

2.4.3 Static Analysis at Joint AA

The force experienced by the top ties can be related to the force in the bottom ties by substituting Eqs. (2.104) and (2.105) in to the equations of static equilibrium at joint AA. Figure 2.20 shows the forces acting on at joint AA.

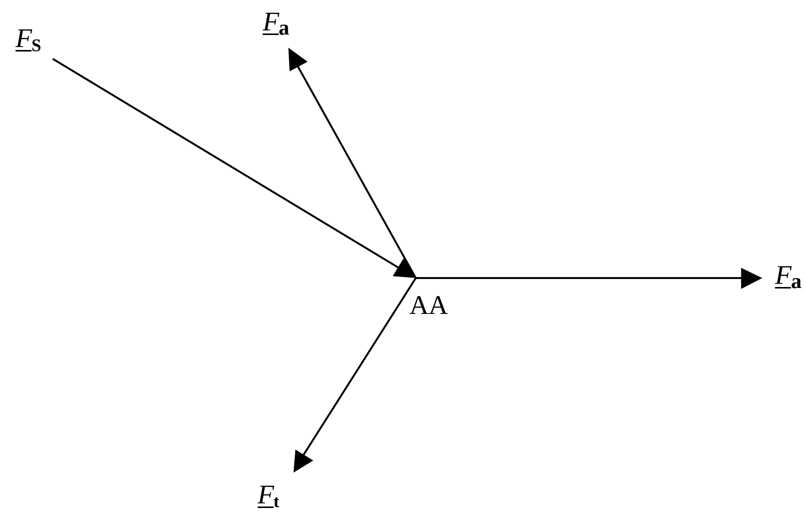

Figure 2.20

Forces at Joint AA

The equation for the sum of the forces at joint AA is

$$\sum_{AA} \underline{F}_i = F_a \, \underline{S}_5 + F_a \, \underline{S}_6 - F_t \, \underline{S}_3 + F_s \, \underline{S}_7 = 0. \tag{2.116}$$

Note that the force produced by the leg tie is in the opposite direction of the unit vector $\underline{S}_3$. This is why the sign for that force is negative. Expanding the equation in the X, Y and Z directions produces

$$\sum_{AA} \underline{F}_X = \frac{1}{2} F_a - F_a - \frac{b+a}{2\,L_t} F_t + \frac{2b+a}{2\,L_s} F_s = 0, \tag{2.117}$$

$$\sum_{AA} \underline{F}_Y = \frac{\sqrt{3}}{2} F_a - \frac{\sqrt{3}\,b - \sqrt{3}\,a}{2\,L_t} F_t - \frac{\sqrt{3}\,a}{2\,L_s} F_s = 0, \tag{2.118}$$

$$\sum_{AA} \underline{F}_Z = - \frac{h}{L_t} F_t + \frac{h}{L_s} F_s = 0. \tag{2.119}$$

Substituting Eqs. (2.114) and (2.115) into these equations and simplifying yields

$$\sum_{AA} \underline{F}_X = -\frac{1}{2} F_a + \frac{b}{2a} F_b = 0, \quad (2.120)$$

$$\sum_{AA} \underline{F}_Y = \frac{\sqrt{3}}{2} F_a - \frac{\sqrt{3}\, b}{2a} F_b = 0, \quad (2.121)$$

$$\sum_{AA} \underline{F}_Z = -\frac{h}{a} F_b + \frac{h}{a} F_b = 0. \quad (2.122)$$

It is apparent that Eqs. (2.120) and (2.121) are the same equations and Eq. (2.122) reduces to zero on both sides of the equations. Solving Eq. (2.120) for F_a yields

$$F_a = \frac{b}{a} F_b. \quad (2.123)$$

Note this equation is the same as Eq. (2.33).

Restating the forces in the members of the tensegrity system related to F_b

$$F_s = \frac{L_s}{a} F_b, \quad (2.114)$$

$$F_t = \frac{L_t}{a} F_b, \quad (2.115)$$

$$F_a = \frac{b}{a} F_b, \quad (2.123)$$

and the lengths of the struts and leg ties

$$L_s = \sqrt{b^2 + a\,b + a^2 + h^2}, \quad (2.112)$$

$$L_t = \sqrt{b^2 - a\,b + a^2 + h^2}. \quad (2.113)$$

CHAPTER 3
PATTERN ANALYSIS

This chapter compares the results of the static analyses of the n-strut tensegrity systems to identify patterns consistent though out all the systems and relate the results to the number of struts (n). It will be apparent that the results of each system are very closely related. The variables in this chapter are the same as those in Chapter 2: the length of the top ties is a; the length of the bottom ties b; distance between the platforms is h; the length of the struts is L_s; the length of the leg ties is L_t. Subscripts a, b, c, and s denotes the variables of the top ties, bottom ties, leg ties, and struts respectively.

Chapter 2 showed that all systems became stable tensegrity systems when the top platform of any n-n Archemedean anti-prism was rotated 90^{O}. This is consistent throughout all n-strut tensegrity systems and it identifies the location of the top platform with respect to the bottom platform [9].

Chapter 2 identified two equations that remain constant throughout all n-strut tensegrity systems

$$\frac{F_t}{L_t} = \frac{F_s}{L_s}, \tag{3.1}$$

$$a\, F_a = \mathrm{b}\, F_b. \tag{3.2}$$

The relation between forces in the leg ties and the struts is equivalent to the relation between the lengths of the leg ties and the struts. Such that, as the length of one is increased, so does its internal force. However, the forces in the top and bottom platforms are inversely related to their lengths. If the length of either the top or bottom ties is increased, their internal force decrease.

The relation between the platform ties and the struts is not constant throughout all n-strut tensegrity systems. The equations for the strut's force in relation to the bottom tie's force for each n-strut are

(3-strut tensegrity) $$F_s = \frac{\sqrt{3}\,L_s}{a} F_b, \quad (2.24)$$

(4-strut tensegrity) $$F_s = \frac{\sqrt{2}\,L_s}{a} F_b, \quad (2.54)$$

(5-strut tensegrity) $$F_s = \frac{\sqrt{2\left(5-\sqrt{5}\right)}L_s}{2a} F_b, \quad (2.84)$$

(6-strut tensegrity) $$F_s = \frac{L_s}{a} F_b. \quad (2.114)$$

After analyzing the equations it is discovered all equations can all be related by the sine of π/n

$$F_s = \frac{2L_s}{a} F_b \sin\left(\frac{\pi}{n}\right). \quad (3.3)$$

Equation (3.1) can be substituted into Eq. (3.3) to relate the force in the leg ties to the force in the bottom ties

$$F_t = \frac{2L_t}{a} F_b \sin\left(\frac{\pi}{n}\right). \quad (3.4)$$

Substituting Eq. (3.2) into Eqs. (3.3) and (3.4) relates the forces to the force in the top ties

$$F_s = \frac{2L_s}{b} F_a \sin\left(\frac{\pi}{n}\right), \tag{3.5}$$

$$F_t = \frac{2L_t}{b} F_a \sin\left(\frac{\pi}{n}\right). \tag{3.6}$$

The lengths of the struts and leg ties are not constant throughout the various n-strut tensegrity systems. The equations for lengths of the struts are

(3-strut tensegrity) $$L_s = \frac{\sqrt{3}}{3}\sqrt{b^2 + \sqrt{3}\,ab + a^2 + 3h^2}\ , \tag{2.22}$$

(4-strut tensegrity) $$L_s = \frac{\sqrt{2}}{2}\sqrt{b^2 + \sqrt{2}\,ab + a^2 + 2h^2}\ , \tag{2.52}$$

(5-strut tensegrity) $$L_s = \sqrt{\frac{2b^2 + \sqrt{2\left(5-\sqrt{5}\right)}\,ab + 2a^2}{5-\sqrt{5}} + h^2}\ , \tag{2.82}$$

(6-strut tensegrity) $$L_s = \sqrt{b^2 + ab + a^2 + h^2}\ . \tag{2.112}$$

After analyzing the equations it is discovered all equations can all be related by the sine of π/n

$$L_s = \sqrt{\frac{b^2 + 2ab\sin\left(\frac{\pi}{n}\right) + a^2}{4\sin^2\left(\frac{\pi}{n}\right)} + h^2}\ . \tag{3.7}$$

Doing the same analysis for length of the leg ties yields

$$L_t = \sqrt{\frac{b^2 - 2ab\sin\left(\frac{\pi}{n}\right) + a^2}{4\sin^2\left(\frac{\pi}{n}\right)} + h^2} \ . \tag{3.8}$$

These equations confirm Kenner's calculations on prismatic tensegrity systems [9].

The equations developed in this chapter are the design equations for an n-strut tensegrity with non-elastic (plastic) ties. The lengths of the ties used would be a for the top platform ties, b for the bottom platform ties, L_t for the leg ties and L_s for the struts. A method of stiffening the system to stabilize it, such as screw connectors at the joints, may be used. If so, the amount tightened on one connection must be matched by tightening the same connection on all the joints of that platform the same amount. This will insure the force is distributed equally throughout all symmetric members.

The force equations can only tell you the relation between the forces of the members in a non-elastic tensegrity. To determine the forces in the members, the force in one member must be obtained from testing.

CHAPTER 4
DEPLOYABILITY AND ELASTICITY

4.1 Deployable Tensegrity Systems

There has been a large increase in the study of deployable systems in the last twenty years greatly due to the need to reduce the cost of transporting and storing systems. A deployable system is a system that has at least two distinct configurations. One configuration(s) is the operational configuration and is known as the "deployed" configuration or state. The other configuration(s) is non-operational and is known as the "redeployed" configuration or state. The redeployed state is usually used to transport or store the system.

N-strut tensegrity systems can be designed to be a deployable systems by using elastic ties for the leg ties. By doing so, the struts of the n-strut tensegrity can be forced together along their lengths and held in place by external forces. In this new configuration, the leg ties are stretched to the length of the struts and the tension in the top and bottom platform ties is removed.

The tensegrity will remain in this configuration until the external forces are removed. When the external forces are removed, the internal force in the leg ties will pull the struts out until the system returns to the deployed tensegrity state. Because the system returns to its deployed state without external forces, it is classified as a self-deployable system. This capability makes an n-strut tensegrity an excellent platform for deployable systems in space as well as systems that need rapid deployment.

Figure 4.1 depicts a 3-strut tensegrity in the deployed and redeployed state. It can be easily observed that the leg ties are stretched to the length of the struts and are under higher stress, while the top and bottom ties are under no stress.

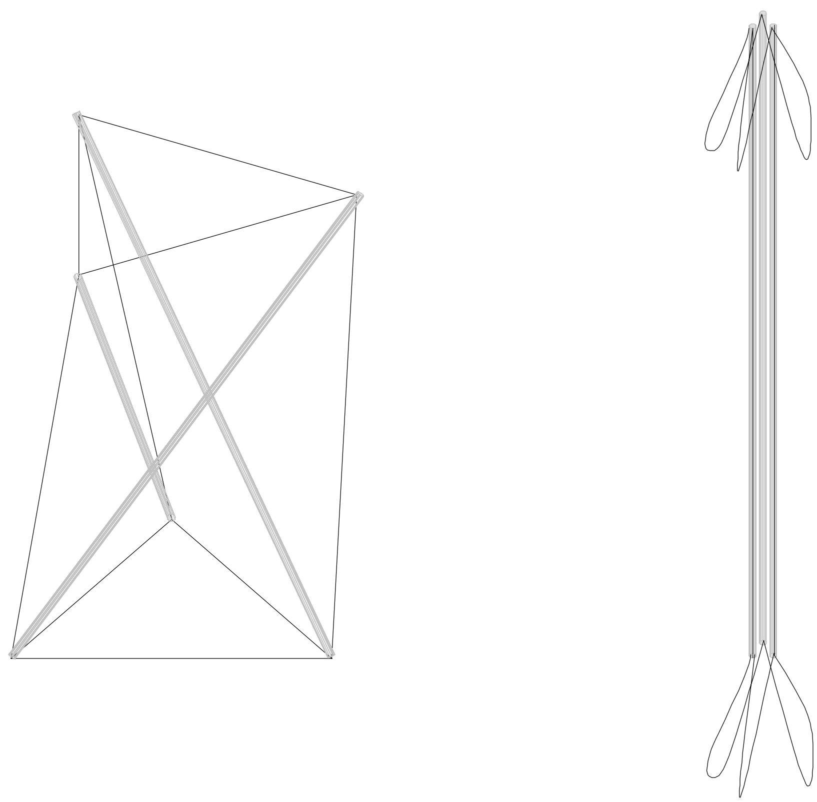

Figure 4.1

3-strut tensegrity
deployed state and redeployed state

The top and bottom ties of the tensegrity system can be replaced with elastic ties or elastic material, though this is no requirement for the system to be self-deployable. However, the system would still only require an external force to be transformed into the redeployed state and held in that configuration.

4.2 Theory of Elasticity

When the ties of the n-strut tensegrity are assumed to be linear elastic, the relation between the stress and strain of the ties is defined to be linear

$$\sigma = E\varepsilon, \tag{4.1}$$

where σ is the stress in the ties, ε is the strain of the ties and E is modulus of elasticity of the material [1]. This means that every time a tie is stressed a specific amount, the strain (amount stretched per unit length) of the tie is the same amount, no matter how much the tie is currently stretched.

Because the length of the ties is long in relation to the diameter of the ties, it can be assumed that the change in cross-sectional area, *A*, is negligible. The force is then linearly related to the stress [1]

$$F = \sigma A. \tag{4.2}$$

Equation (4.1) is substituted into Eq. (4.2)

$$F = \mathrm{E}\varepsilon A, \tag{4.3}$$

where strain is defined as the change in length divided by the initial length

$$\varepsilon = \frac{(L - l)}{l}. \tag{4.4}$$

The left side of the equation can be replaced by a constant times a displacement [1, 13]

$$F = k\left(L - l\right), \tag{4.5}$$

where

$$k = \frac{AE}{l}. \tag{4.6}$$

The constant, k, is known as the spring coefficient. The displacement, L - l, is the amount the elastic tie is stretched, where L is the final length of the tie (stretched) and l is the initial length of the tie (unstretched). Equations (4.5) and (4.6) are substituted into the equations found in Chapter 3 to produce design equations for self-deployable tensegrity systems.

4.3 Self-deployable Tensegrity Equations

The equations found in Chapter 3 relate the forces in the ties and struts to each other. By replacing the forces by the right side of Eq. (4.7), the displacement of the ties can be related to each other.

$$F_i = k_i\left(L_i - l_i\right). \tag{4.7}$$

The subscript i is represents the subscripts a, b or t. Subscripts a, b, and t label the variables associated with the top ties, bottom ties and leg ties respectively while subscript s labels the variables associated with the struts. Therefore, L_a denotes the final length of the top ties, L_b denotes the final length of the bottom ties, L_t denotes the final length of the leg ties and L_s denotes the length of the struts. The same notation applies to k_i, l_i, etc. Note, L_a equals a and L_b equals b; the variables L_a and L_b are used in the succeeding equations. Substituting Eq. (4.6) into Eq. (4.7) and simplifying yields

$$F_i = A_i E_i \left(\frac{L_i}{l_i} - 1 \right). \tag{4.8}$$

Figure 4.2 shows a 3-strut tensegrity with the final length variables depicted.

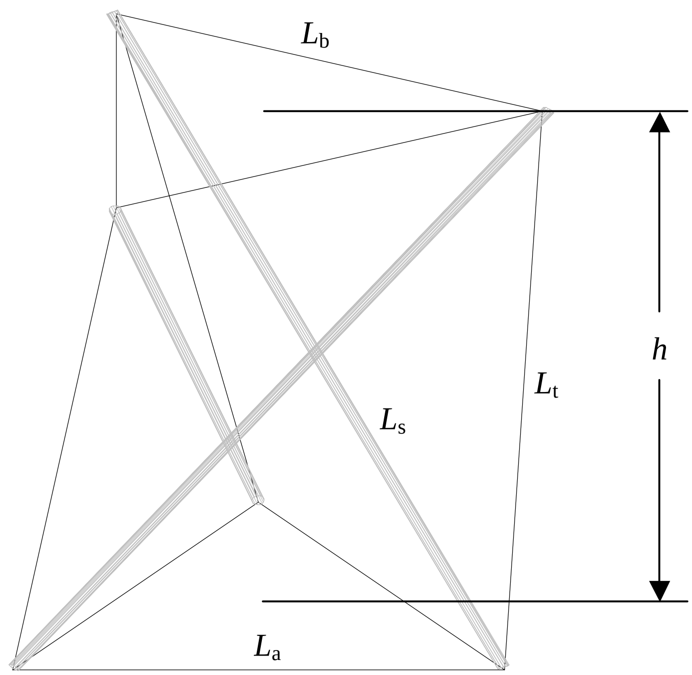

Figure 4.2
3-strut tensegrity

The equations in Chapter 3 are rewritten in terms of the force in the leg ties, because the leg ties are the ties that will be stretched in the redeployed state. The leg ties must therefore be designed to undergo additional stress with minimal plastic (non-elastic) deformation. Therefore, the stress of the leg tie in the redeployed state must be less than the materials yield stress.

Rewriting Eqs. (3.4), (3.6) and (3.1) yields

$$F_{\mathrm{b}} = \frac{L_a}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} F_t\,, \tag{4.9}$$

$$F_{\mathrm{a}} = \frac{L_b}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} F_t\,, \tag{4.10}$$

$$F_s = \frac{L_s}{L_t} F_t\,. \tag{4.11}$$

Substituting Eq. (4.8) into Eqs. (4.9) and (4.10) and Eq. (4.7) into Eq. (4.11) yields

$$A_b\,E_b\left(\frac{L_b}{l_b}-1\right) = \frac{L_a}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} A_t\,E_t\left(\frac{L_t}{l_t}-1\right), \tag{4.12}$$

$$A_a\,E_a\left(\frac{L_a}{l_a}-1\right) = \frac{L_b}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} A_t\,E_t\left(\frac{L_t}{l_t}-1\right), \tag{4.13}$$

$$F_s = \frac{L_s}{L_t} k_t\left(L_t - l_t\right). \tag{4.14}$$

Note that the force in the strut is not related to its displacement because the displacement of the struts is assumed to be negligible.

Solving the above Eqs. (4.12) and (4.13) for the initial lengths of the ties, yields

$$l_b = \frac{2\,L_t\,L_b\,A_b\,E_b \sin\left(\frac{\pi}{n}\right)}{2\,L_t\,A_b\,E_b \sin\left(\frac{\pi}{n}\right) + L_a\,A_t\,E_t\left(\frac{L_t}{l_t} - 1\right)}, \tag{4.15}$$

$$l_a = \frac{2\,L_t\,L_a\,A_a\,E_a \sin\left(\frac{\pi}{n}\right)}{2\,L_t\,A_a\,E_a \sin\left(\frac{\pi}{n}\right) + L_b\,A_t\,E_t\left(\frac{L_t}{l_t} - 1\right)}, \tag{4.16}$$

The relation between the initial and final lengths of the ties provides us with a dimensionless stiffness ratio. The greater the difference between the two lengths is, the stiffer the tensegrity shall be. Let the ratio of the initial length compared to the final length be S (geometric stiffness ratio), such that

$$S_i = \frac{L_i}{l_i}. \tag{4.17}$$

When S_i = 1, the tie is unstressed and the tensegrity system is at its minimum stiffness. Note, if S_a = 1, then S_b = 1 and S_t = 1. As S_i increases, the system becomes stiffer. This continues until the material is stretched to the limit of its yield strength. When the stress in the tie is equal to the yield stress, S is at its maximum value. Introducing the appropriate subscript to Eq. (4.2) and setting it equal to Eq. (4.8)

$$\sigma_i\,A_i = A_i\,E_i\left(\frac{L_i}{l_i} - 1\right). \tag{4.18}$$

Substituting (4.17) into Eq. (4.18) and simplifying

$$\sigma_i = E_i \ (S_i - 1). \tag{4.19}$$

Solving for S and setting stress (σ) equal to material's yield stress (σ) shows each material has its own maximum value for S

$$S_{i\,\max} = \frac{\sigma_{Y\,i}}{E_i} + 1. \tag{4.20}$$

The greater the ratio of the yield stress of a material to its modulus of elasticity, the greater the maximum value of S shall be.

Substituting Eq. (4.17) into Eqs. (4.15) and (4.16) yields

$$l_b = \frac{2\, L_t\, L_b\, A_b\, E_b \sin\left(\frac{\pi}{n}\right)}{2\, L_t\, A_b\, E_b \sin\left(\frac{\pi}{n}\right) + L_a\, A_t\, E_t (S_t - 1)}, \tag{4.21}$$

$$l_a = \frac{2\, L_t\, L_a\, A_a\, E_a \sin\left(\frac{\pi}{n}\right)}{2\, L_t\, A_a\, E_a \sin\left(\frac{\pi}{n}\right) + L_b\, A_t\, E_t (S_t - 1)}. \tag{4.22}$$

All variables in these equations must be known to solve for the initial lengths of the bottom and top ties. If the bottom or top ties are assumed to be inelastic, the modulus of elasticity is assumed to be infinite ($E_i = \infty$) and the length of the ties will be L_i.

Dividing Eq. (4.21) by L_b and Eq.(2.22) by L_a yields the equations for S_b and S_a

$$S_b = \frac{2\,L_t\,A_b\,E_b\sin\left(\frac{\pi}{n}\right)}{2\,L_t\,A_b\,E_b\sin\left(\frac{\pi}{n}\right)+L_a\,A_t\,E_t\left(S_t-1\right)}, \tag{4.23}$$

$$S_a = \frac{2\,L_t\,A_a\,E_a\sin\left(\frac{\pi}{n}\right)}{2\,L_t\,A_a\,E_a\sin\left(\frac{\pi}{n}\right)+L_b\,A_t\,E_t\left(S_t-1\right)}. \tag{4.24}$$

The values of S_b and S_a cannot exceed the maximum geometry stiffness from Eq. (4.20).

The value of S_t must be determined from the desired stiffness, but can not exceed the maximum geometric stiffness of the material. If the tensegrity is to be used as a deployable system, the leg ties must be able to stretch the length of the struts. Substituting the length of the strut (L_s) into the Eq. (4.18) yields

$$\sigma_t\,A_t = A_t\,E_t\left(\frac{L_s}{l_t}-1\right), \tag{4.25}$$

This identified the maximum internal force of the leg ties. Substituting Eq. (4.17) into Eq. (4.25)

$$\sigma_t\,A_t = A_t\,E_t\left(S_t\,\frac{L_s}{L_t}-1\right), \tag{4.26}$$

and solving for S_t gives the maximum value of S_t for a self-deployable tensegrity

$$S_{t\max} = \left(\frac{\sigma_{Y\,t}}{E_t}+1\right)\frac{L_t}{L_s}. \tag{4.27}$$

A factor of safety (G) should be used to prevent the stress in the leg ties from reaching the yield stress (σ_Y) of the material

$$S_t = \left(\frac{\sigma_{Y\,t}}{G\,E_t} + 1 \right) \frac{L_t}{L_s} . \tag{4.28}$$

The initial length of the leg ties can be found by substituting Eq. (3.8) into Eq. (4.17) and solving for l_t

$$l_t = \frac{1}{S_t} \sqrt{\frac{L_b{}^2 - 2\,L_a\,L_b \sin\left(\frac{\pi}{n}\right) + L_a{}^2}{4 \sin^2\left(\frac{\pi}{n}\right)} + h^2} , \tag{4.29}$$

In Chapter 3, the length of the struts was found to be

$$L_s = \sqrt{\frac{L_b{}^2 + 2\,L_a\,L_b \sin\left(\frac{\pi}{n}\right) + L_a{}^2}{4 \sin^2\left(\frac{\pi}{n}\right)} + h^2} . \tag{3.7}$$

Substituting Eqs. (4.2) and (4.17) into Eq. (4.14) and solving for stress yields

$$\sigma_s = \frac{L_s}{A_s} k_t \left(1 - \frac{1}{S_t} \right). \tag{4.30}$$

The strut is under the most stress when the system is in the redeployed state. The force exerted on the strut is equal to the force required to fully stretch the leg ties to the redeployed state

$$F_{s\max} = k_t \left(L_s - l_t\right). \tag{4.31}$$

Substituting Eqs. (4.2) and (4.17) into Eq. (4.31) and solving for stress yields

$$\sigma_{s\max} = \frac{k_t}{A_s}\left(L_s - \frac{L_t}{S_t}\right). \tag{4.32}$$

4.4 Design Process

Assuming the desired geometry of the system is know, the first step to designing the self-deployable tensegrity is to determine the materials to be used. Depending on the function of the system, certain materials may be required. The yield stresses and modulus of elasticity can be identified from material property handbooks or results of testing. If the materials to be use can vary, then the desired stiffness can assist in conducting the material selection.

Equation (4.22) shows the material with the highest yield stress to modulus of elasticity ratio will obtain the maximum stiffness. A material with a lower yield stress to modulus of elasticity ratio may be selected if the maximum stiffness is not desired.

Let an example tensegrity system be a 4-strut tensegrity with the dimensions: a (L_a) = 4 ft., b (L_b) = 6 ft. and h = 8 ft. The material selection for the ties is a polyurethane/ABS blend. It has a yield stress of σ_Y = 5.1 ksi and a modulus of elasticity of E = 12.0 ksi.

Once the materials of the ties are identified, the geometric stiffness ratio of the tensegrity is calculated using Eq. (4.28)

$$S_t = \left(\frac{\sigma_{Y\,t}}{G\,E_t} + 1 \right) \frac{L_t}{L_s} . \tag{4.28}$$

where

$$L_s = \sqrt{\frac{{L_b}^2 + 2\,L_a\,L_b \sin\left(\frac{\pi}{n}\right) + {L_a}^2}{4 \sin^2\left(\frac{\pi}{n}\right)} + h^2} , \tag{3.7}$$

$$L_t = \sqrt{\frac{{L_b}^2 - 2\,L_a\,L_b \sin\left(\frac{\pi}{n}\right) + {L_a}^2}{4 \sin^2\left(\frac{\pi}{n}\right)} + h^2} . \tag{3.8}$$

The lengths of the struts and leg ties in the example are

$$L_s = \sqrt{\frac{6^2 + 2 \cdot 4 \cdot 6 \sin\left(\frac{\pi}{4}\right) + 4^2}{4 \sin^2\left(\frac{\pi}{4}\right)} + 8^2} = 10.35 \text{ ft.}, \tag{3.7}$$

$$L_t = \sqrt{\frac{6^2 - 2 \cdot 4 \cdot 6 \sin\left(\frac{\pi}{4}\right) + 4^2}{4 \sin^2\left(\frac{\pi}{4}\right)} + 8^2} = 8.55 \text{ ft.} \tag{3.8}$$

Using a factor of safety of G = 1.3, the geometric stiffness ratio is found to be

$$S_t = \left(\frac{5.1}{1.3 \cdot 12.0} + 1\right)\frac{8.55}{10.35} = 1.10,$$

Note, the largest factor of safety the tensegrity can be is G = 2, where S_t = 1.

The initial length of the leg ties can now be calculated using

$$l_t = \frac{1}{S_t}\sqrt{\frac{L_b{}^2 - 2\,L_a\,L_b \sin\left(\frac{\pi}{n}\right) + L_a{}^2}{4\sin^2\left(\frac{\pi}{n}\right)} + h^2}\ . \qquad (4.29)$$

The initial length of the leg ties for the example system is

$$l_t = \frac{1}{1.10}\sqrt{\frac{6^2 - 2 \cdot 4 \cdot 6 \sin\left(\frac{\pi}{4}\right) + 4^2}{4\sin^2\left(\frac{\pi}{4}\right)} + 8^2} = 7.80 \text{ ft.}$$

Substituting Eq. (4.6) into Eq. (4.7) and solving for A_i yields

$$A_i = \frac{F_i\, l_i}{E_i\,(L_i - l_i)}. \qquad (4.33)$$

Because the maximum length the leg ties will be stretched to the length of the struts, L_i is replaced by L_s to solve for A_t

$$A_{t\max} = \frac{F\, l_t}{E_t\,(L_s - l_t)}. \qquad (4.34)$$

The force in Eq. (4.34) is the force that can be generated against the internal force of the leg ties to redeploy the system (not the force applied perpendicular to the struts). This force must be calculated or estimated based on the method of redeployment.

Let the internal force generated against the leg ties of the example system be $F = 50$ lb. The cross-sectional area of the leg ties is

$$A_{t\,\max} = \frac{50 \cdot 7.80}{12000\left(10.35 - 7.80\right)} = 0.013 \text{ in}^2.$$

This is a cord with a 1/8$^{\text{th}}$-in. diameter.

The cross-sectional area of the strut is derived by introducing a safety of factor to Eq. (4.32) and solving for A_s

$$A_s = \frac{k_t}{\sigma_{Y\,s}} G\left(L_s - \frac{L_t}{S_t}\right), \tag{4.35}$$

where

$$k_t = \frac{A_t\,E_t}{l_t}. \tag{4.6}$$

The critical buckling load of a column is defined as a function of moment of inertia of the cross-section of the column [1]. To prevent the struts from buckling, the critical stress equation [1] is substituted into Eq. (4.35) and solved for the moment of inertia (I)

$$I_s = \frac{L_s^2}{\pi^2 E_s} k_t\, G\left(L_s - \frac{L_t}{S_t}\right). \tag{4.36}$$

This is the minimum moment of inertia of the cross-section of the struts about the axis of bending. If the cross-section of the struts is circular, the moment of inertia is about any axis on the cross-section.

This identifies the geometrical requirements of the struts. The cross-sectional area of the struts can be maximized to reduce the material requirements or minimized to reduce the weight of a specific material. Both Eqs. (4.35) and (4.36) must be satisfied to prevent the strut from failing in the redeployed state.

The leg ties of the example tensegrity system have a spring coefficient

$$k_t = \frac{0.013 \cdot 12000}{7.80 \cdot 12} = 1.67 \text{ lb./in.}$$

Let the struts of the example system be made of aluminum alloy (1100-H12) which has a yield stress of σ_Y = 15 ksi and a modulus of elasticity E = 10200 ksi. Keeping the factor of safety of 1.5 and solving for the cross-sectional area of the strut

$$A_s = \frac{1.67}{15000} 1.5 \left(10.35 - \frac{8.55}{1.10} \right) \cdot 12 = 0.0052 \text{ in}^2,$$

and moment of inertia of the strut

$$I_s = \frac{10.35^2 \cdot 12^2}{\pi^2 10{,}200{,}000} 1.67 \cdot 1.5 \left(10.35 - \frac{8.55}{1.10} \right) \cdot 12 = 0.012 \text{ in}^4.$$

The radiuses of an extruded aluminum pole that meets these minimum requirements can be identified.

Substituting Eqs. (4.8) and (4.17) into Eqs. (4.9) and (4.10) for F_t produces

$$F_b = \frac{L_a\left(S_t - 1\right)}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} A_t\,E_t\,, \tag{4.37}$$

$$F_a = \frac{L_b\left(S_t - 1\right)}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} A_t\,E_t\,. \tag{4.38}$$

The cross-sectional areas for the bottom and top ties are found by substituting Eq. (4.2) into Eqs. (4.37) and (4.38) and solving for A_i

$$A_b = \frac{L_a\left(S_t - 1\right)}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} \frac{A_t\,E_t\,G}{\sigma_{Y\,b}}\,, \tag{4.39}$$

$$A_a = \frac{L_b\left(S_t - 1\right)}{2\,L_t\,\sin\left(\frac{\pi}{n}\right)} \frac{A_t\,E_t\,G}{\sigma_{Y\,a}}\,. \tag{4.40}$$

The cross-sectional area of the ties can be maximized to reduce the material requirements or minimized to reduce the weight of a specific material. The greater the cross-sectional area of the ties, the greater the stiffness of the tensegrity.

Using the same material for all of the ties, the cross-sectional areas are

$$A_b = \frac{4\left(1.10 - 1\right)}{2 \cdot 8.55 \sin\left(\frac{\pi}{4}\right)} \frac{0.013 \cdot 12 \cdot 1.5}{5.1} = 0.0015 \text{ in}^2,$$

$$A_a = \frac{6\left(1.10 - 1\right)}{2 \cdot 8.55 \sin\left(\frac{\pi}{4}\right)} \frac{0.013 \cdot 12 \cdot 1.5}{5.1} = 0.0023 \text{ in}^2.$$

The lengths of the bottom and top ties are be calculated using

$$l_b = \frac{2\, L_t\, L_b\, A_b\, E_b \sin\left(\frac{\pi}{n}\right)}{2\, L_t\, A_b\, E_b \sin\left(\frac{\pi}{n}\right) + L_a\, A_t\, E_t \left(S_t - 1\right)}, \tag{4.21}$$

$$l_a = \frac{2\, L_t\, L_a\, A_a\, E_a \sin\left(\frac{\pi}{n}\right)}{2\, L_t\, A_a\, E_a \sin\left(\frac{\pi}{n}\right) + L_b\, A_t\, E_t \left(S_t - 1\right)}. \tag{4.22}$$

The initial lengths of the bottom and top ties of the example tensegrity system are

$$l_b = \frac{2 \cdot 10.35 \cdot 6 \cdot 0.0015 \cdot 12 \sin\left(\frac{\pi}{4}\right)}{2 \cdot 8.55 \cdot 0.0015 \cdot 12 \sin\left(\frac{\pi}{4}\right) + 4 \cdot 0.013 \cdot 12 \left(1.10 - 1\right)} = 5.65 \text{ ft.},$$

$$l_a = \frac{2 \cdot 10.35 \cdot 4 \cdot 0.0023 \cdot 12 \sin\left(\frac{\pi}{4}\right)}{2 \cdot 8.55 \cdot 0.0023 \cdot 12 \sin\left(\frac{\pi}{4}\right) + 6 \cdot 0.013 \cdot 12 \left(1.10 - 1\right)} = 3.78 \text{ ft.}$$

The geometric stiffness ratio of the top and bottom ties can be calculated using Eqs. (4.23) and (4.34). Using Eq. (4.20) will verify the values for S_b and S_a do not exceed the maximum geometric stiffness for the material.

These calculations will produce a self-deployable n-strut tensegrity system with bottom platform edges of length *b* (L_b), top platform edges of length *a* (L_a) and a height (distance between platforms) of *h*. The designed system is under no external force or weight. The next chapter will discuss how external forces and weight will effect the tensegrity system.

CHAPTER 5
WEIGHT AND EXTERNAL FORCES

The computations performed in this study considered weight to be negligible and disregarded all external forces. There are considerations that need to be taken when the weight of the tensegrity system is not negligible. The equation of static equilibrium at any joint on the bottom platform is altered to

$$\sum \underline{F}_i = F_b\, \underline{S}_1 + F_b\, \underline{S}_2 + F_t\, \underline{S}_3 + F_s\, \underline{S}_4 + \frac{W}{n}\hat{k} = 0, \qquad (5.1)$$

where W is the total weight of the tensegrity system and $\hat{k}$ is the unit vector in the Z-direction. The unit vectors of the members are denoted by $\underline{S}$. If external forces were acting on the system, W would be the weight plus the sum of the external forces.

The unit vectors of the members are expressed as a function of twist angle (α_o) of the top platform. Studying these calculations showed that the top platform must rotate past 90^O from the anti-prism or the internal forces of the members are infinite. As the top platform rotates past 90^O, the internal forces reduce. However, at some twist angle the potential energy of the system is at a minimum. Rotating past that point causes the potential energy of the system to increase, though the internal forces of some of the members continue to decrease.

One method of calculating the forces in the self-deployable n-strut tensegrity subjected to weigh and external forces is by using the theory of potential energy to calculate the twist angle and solving for the forces in the system using Eq. (5.1). The theory of potential energy claims that an object will go to the position of least potential energy, such that it will continually reduce its potential energy until it is forced to stop [1]. This concept is demonstrated by an object falling. As it falls, its potential energy is decreased until it is finally forced to stop falling by a ground. A rubber band displays this property when it is stretched and released. The rubber band will return to its unstretched size until an external force stops it or it is completely unstretched.

Because a tensegrity system is defined as super stable [4], the force exerted on the members is caused by the reaction to other member, regardless of whether there are any external forces. Due to this fact, a tensegrity will always assume the position of least potential energy. This position is found by equating the potential energy of the system and external forces to the twist angle. The position of least potential energy is where the derivative of the total potential energy equation, in terms of α_o, has the value of zero (0)

$$\frac{d\,PE}{d\alpha_o} = 0. \tag{5.2}$$

Two values for α_o will satisfy Eq. (5.2). One will be the position of least potential energy and the other will be the position of maximum potential energy. The greater the external force and weight exerted on the tensegrity is, the greater the value for α_o shall be.

The value of α_o at the position of least potential energy is substituted into Eq. (5.1). Equation (5.1) is resolved into three equations in the X, Y and Z-directions. Because external forces are acting on the tensegrity, the equations in X, Y and Z-directions are linearly independent. The three equations are used to solve for the value of the three unknown internal forces in the equation, F_b, F_t and F_s.

The equation of static equilibrium at any joint on the top platform is used to solve for the value of F_a. The theory of elasticity is used to solve for the dimensions of the ties and struts with the calculated value of internal forces.

Another method of analyzing the effects of weight and external loads is defined by Robert Burkhardt of Tensegrity Solutions. He states that the "structure is viewed as a flexibly-jointed set of elastic and fixed-length members, the [ties] being the elastic members, and the struts being the fixed-length members." The stresses in the unloaded system are calculated using Eq. (4.19). "The vectors representing the load[s] are then introduced, and the equation system is solved using Newton techniques to arrive at the values of the loaded relative stresses." [3:11]

CHAPTER 6
CONCLUSION

6.1 Summary of Study

The calculations in chapter 2 identify the geometry of the n-strut tensegrity system as an Archemedean anti-prism with the top platform rotated 90^{O}. This confirms Kenner's "twist angle" calculations [9:9]. Chapter 3 defines the relation between its internal forces and geometry. The lengths of the struts and the leg ties for a non-deployable tensegrity are

$$L_s = \sqrt{\frac{b^2 + 2ab\sin\left(\frac{\pi}{n}\right) + a^2}{4\sin^2\left(\frac{\pi}{n}\right)} + h^2}\,, \tag{3.7}$$

$$L_t = \sqrt{\frac{b^2 - 2ab\sin\left(\frac{\pi}{n}\right) + a^2}{4\sin^2\left(\frac{\pi}{n}\right)} + h^2}\,. \tag{3.8}$$

Equations (3.7) and (3.8) confirm Kenner's geometric calculations [9].

The relations between the internal forces of the members of the system are shown to be

$$\frac{F_t}{L_t} = \frac{F_s}{L_s}, \tag{3.1}$$

$$\frac{F_a}{b} = \frac{F_b}{a}, \tag{3.2}$$

$$\frac{F_t}{L_t} = \frac{2F_b}{a}\sin\left(\frac{\pi}{n}\right). \tag{3.4}$$

Equation (3.1) displays that when the length of the struts increases, the internal force in the struts increases. The same applies to the leg ties. Equation (3.2), however, reveals that when the length of the bottom ties increase, the internal force in the top ties increase. The inverse applies to the top ties.

The equations in chapter 4 design a self-deployable tensegrity to meet desired geometry and stiffness. This configuration requires elastic leg ties. The length of the leg ties before being stretched is defined as

$$l_t = \frac{1}{S_t}\sqrt{\frac{{L_b}^2 - 2L_a L_b \sin\left(\frac{\pi}{n}\right) + {L_a}^2}{4\sin^2\left(\frac{\pi}{n}\right)} + h^2}\ , \tag{4.29}$$

where S_t is the geometric stiffness ratio

$$S_t = \frac{L_t}{l_t}. \tag{4.17}$$

The stiffness ratio can be determine by the material properties of the leg ties

$$S_t = \left(\frac{\sigma_{Y\,t}}{G\,E_t} + 1 \right) \frac{L_t}{L_s} . \tag{4.28}$$

Note the safety of factor will be based on how close the desired stiffness of the tensegrity is to the material dependent maximum stiffness. The higher the safety of factor, the less stiff the system is.

Applications for self-deployable tensegrity systems are currently being discovered in engineering and science. The calculations conducted in this study help develop an understanding of the n-strut tensegrity system and the internal reactions of its members. They are the first step in creating functional tensegrity based systems for terrestrial and space employment.

6.2 Suggestions for Future Study

There are three main areas of study that must be conducted on n-strut tensegrity systems to develop operational self-deployable tensegrity based systems. They are stiffness, weight/external forces, and vibrations.

The stiffness of a tensegrity is a function of the material of the ties, cross-sectional area of the ties and the ratio of the initial length of the ties to the final length of the ties. The geometric stiffness ratio, S, defined in Chapter 4, quantifies only one variable. The stiffness of a tensegrity must be quantified, so that there is a method of comparing two separate systems.

Forward and reverse analysis of the reaction to weight and external forces should be conducted. The method defined by Burkhardt in Chapter 5 can be studied to identify how external forces alter the geometry of the system. Equations derived from the potential energy theory and static analysis will calculate design equations for self-deployable tensegrity systems reacting to weight and vertical forces that meet desired geometric criteria.

The tensegrity system must be analyzed to identify velocity and vibrations of deployment. This study should include damping and control of deployment.

With research on these three areas, the design and development of self-deployable n-strut tensegrity systems will be conducted with relative ease. Application concepts will continue to be discovered with the continual study of the tensegrity system.

REFERENCES

[1] Boresi, A., Schmidt, R., Sidebottom, O., Advanced Mechanics of Materials, 5th Ed., John Wiley and Sons, Inc., New York, 1993.

[2] Bruno, E., "Design Tools for Tensegrity Structures," M.S. Thesis, Pennsylvania State University, 1997.

[3] Burkhardt, R., "A Technology for Designing Tensegrity Domes and Spheres," Tensegrity Solutions, Cambridge, MA, 1999.

[4] Connelly, R., Back, A., "Mathematics and Tensegrity," American Scientist, Vol. 86, March-April 1998, pp. 142-151.

[5] Duffy, J., Rooney, J., Knight, B., Crane, C., "An Analysis of the Deployment of Tensegrity Structures Using Screw Theory," Center for Intelligent Machines and Robots, University of Florida, 1998.

[6] Edmondson, A., A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller, Birkhauser, Boston, 1987.

[7] Fuller, R., Synergetics: The Geometry of Thinking, MacMillan Publishing Co., Inc., New York, 1975.

[8] Fuller, R., Synergetics 2: Explorations in the Geometry of Thinking, MacMillan Publishing Co., Inc., New York, 1979.

[9] Kenner, H., Geodesic Math and How to Use It, University of California Press, Berkeley and Los Angeles, California, 1976.

[10] Lee, J., Duffy, J., Hunt, K., "A Practical Quality Index Based on the Octahedral Manipulator," The International Journal of Robotics Research, Vol. 17, No. 10, Oct. 1998, pp. 1081-1090.

[11] Pugh, A., An Introduction to Tensegrity, University of California Press, Berkeley and Los Angeles, California, 1976.

[12] Rooney, J., Duffy, J., Lee, J., "Tensegrity and Compegrity Configurations in Anti-Prism Manipulator Platforms," Tenth World Congress on the Theory of Machines and Mechanisms, Vol. 3, Jun. 1999, pp. 1278-1287.

[13] Shames, I., Engineering Mechanics: Statics, Prentice-Hall, Inc., Englewood, New Jersey, 1958.

[14] Spiegel, M., Vector Analysis: and an Introduction to Tensor Analysis, Schaum Publishing Company, New York, 1959.

BIOGRAPHICAL SKETCH

Ian P. Stern earned a commission in the US Army as a Second Lieutenant at New Mexico Military Institute, May of 1992. After receiving a Bachelor of Science in aerospace engineering from Embry-Riddle Aeronautical University in May of 1995, he entered active duty for a three-year tour. Mr. Stern was promoted to Captain on October 1, 1997.

In August of 1998, CPT Ian Stern transferred to the Florida National Guard and enrolled as a graduate student at the University of Florida in August of 1998.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

____

Joseph Duffy, Chairman
Graduate Research Professor of
Mechanical Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

____

Carl D. Crane, III
Professor of Mechanical
Engineering

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

____

Paul A. C. Mason
Assistant Professor of
Mechanical Engineering

This thesis was submitted to the Graduate Faculty of the College of Engineering and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Master of Science.

December 1999

____

M. Jack Ohanian
Dean, College of
Engineering

____

Winfred M. Phillips
Dean, Graduate School