REVERSE DISPLACEMENT ANALYSIS
FOR TENSEGRITY STRUCTURES

By

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A THESIS PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN PATIAL FULFILLMENT
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For this work I thank my parents for their infinite support and generosity.

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Abstract of Thesis Presented to the Graduate School
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A new type of parallel mechanism is introduced that is based on the principle of tensegrity. In tensegrity structures, ties are used for elements that are in tension and struts for elements that are in compression. In this research, three of the side ties of a tensegrity structure are composed of a compliant and a noncompliant segment that are in series. The length of the three noncompliant segments can be varied in order to control the shape and desired compliant characteristics of the tensegrity mechanism, provided that the structure is at a singular configuration. This study also shows how the lengths of the noncompliant segments are affected when an external wrench is applied to the structure. In addition, the research analyzes the new tensegrity structure when one or two side ties are replaced by struts; the tensegrity behavior remains unchanged if the struts are connected to each other by ball-and-socket joints. Software is created using the analysis to output the results into a file. The software can detect errors caused by invalid inputs.

CHAPTER 1
INTRODUCTION

Tensegrity structures consist of ties which are in tension and struts which are in compression. "Tensegrity" is a combination of the words "tension" and "integrity." Anthony Pugh [9] defined a tensegrity structure as "a set of discontinuous compressive components interacting with a set of continuous tensile components that define a stable volume in space." Because the forces in the members of a tensegrity structure are in pure compression and tension, no bending or twisting occurs; therefore, the design of the structure can be minimized. Figure 1-1 shows a family of tensegrity structures, which have also been called skew prisms, anti-prisms, and tensegrity prisms by Kenner [7], Gabriel [4], and Tobie [11]. Figure 1-2 depicts a simple tensegrity structure where the struts and ties are identified.

When the ties of the structure are elastic, it can be folded and when released will then self-deploy back to its original shape. The shape of the symmetric tensegrity structure is unique and was determined by Kenner to be

$$a = \frac{p}{2} - \frac{p}{n} \tag{1.1}$$

where n is the number of struts and a is the minimum rotation angle, whilst the maximum value is $\frac{p}{2}$, regardless of n (see Figure 1-1). Hence, the triangle, square, pentagon, and hexagon tensegrity prisms have a equal to $\frac{p}{6}$, $\frac{p}{4}$, $\frac{3p}{10}$, and $\frac{p}{3}$.

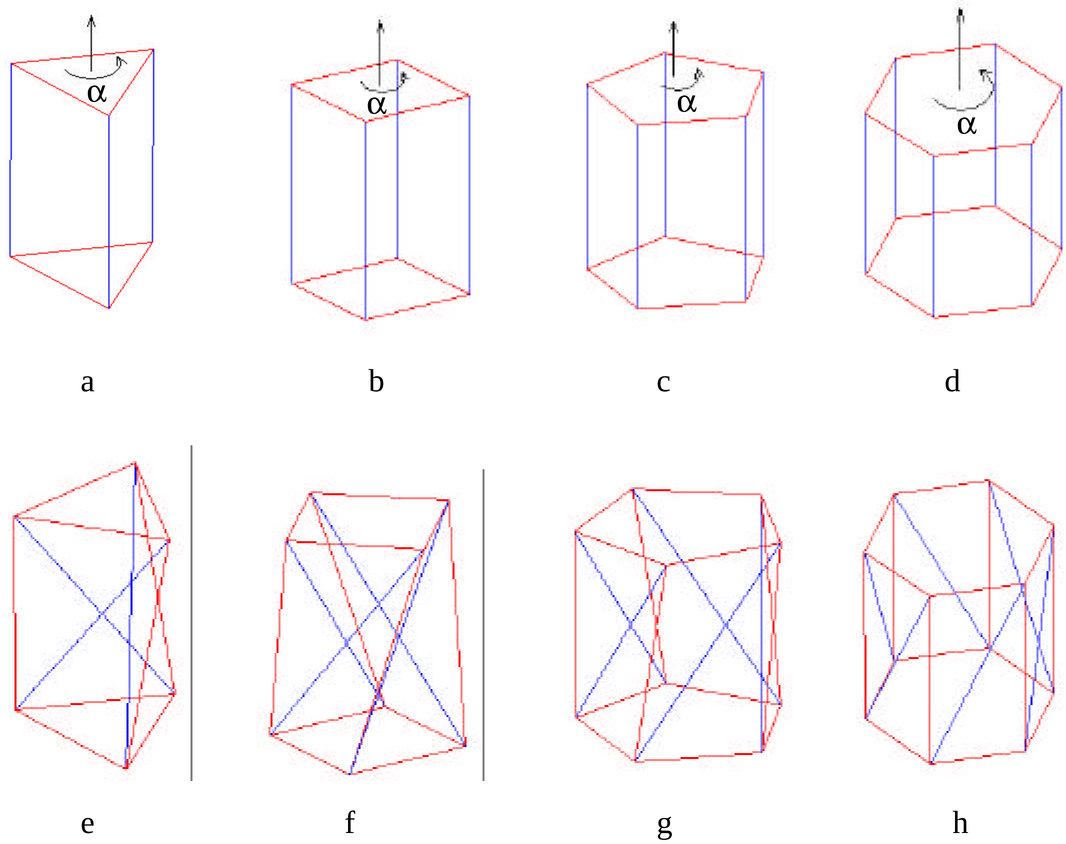

Figure 1-1. A family of tensegrity structures

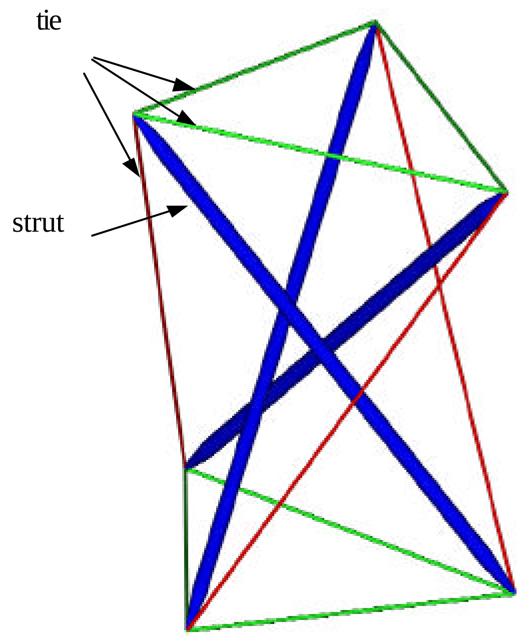

Figure 1-2. Tensegrity structure

The prisms in the first row (1-1a, b, c, and d) are called parallel prisms with all ties being parallel. When the top of each parallel prism is rotated counter-clockwise at an angle α relative to the base, a corresponding tensegrity prism is formed that is shown in the second row (1-1e, f, g, and h). These are also called right-handed tensegrity prisms. Left-handed ones are formed simply by rotating the tops clockwise with the same angle α. Much research has been done on tensegrity structures. Most recently at University of Florida, Correa [2] determined the equilibrium position of a tensegrity structure when external forces and external moments act on the structure. Prior to that, stability of tensegrity structures for deployable antenna was studied by Knight [8]. For each structure geometry, Stern [10] developed equations to find the lengths of its struts and ties. The relationship between the top and base of a tensegrity structure has been widely researched by Kenner [7], Tobie [11], and Yin [12]. The above studies have been done by assuming that the top and base of the structure remain parallel during the transformation. Often this is not the case. The top and base rarely stay parallel when an operation is performed. In order for this to be the case, each tie has to be tightened or loosened exactly the same amount of lengths, or the external forces or moments have to be applied to the structure equally at each strut end, which is not very practical. This study combines the concepts of tensegrity and parallel platforms into a structure which has elastic ties and variable strut lengths. The ties are composed of compliant and noncompliant parts. The noncompliant part is subjected to change in length. This thesis presents the reverse displacement analysis for this new type of tensegrity structure.

Nomenclature and problem statement. A three-three tensegrity structure is to be analyzed, as shown in Figure 1-3. Only three of the ties in this device, which can also be called side ties, are compliant. The three elastic ties that connect the top and base comprised noncompliant strings and springs in series. Since the top three and bottom three ties are noncompliant, the device is in effect a parallel mechanism and the reverse analysis of this mechanism is presented. For the reverse analysis, the desired position and orientation of the top platform (i.e. the transformation matrix ${}^{B}_{T}T$) together with the desired total potential energy U stored in the compliant elements is specified and the required strut lengths L_A, L_B, L_C and the variable noncompliant tie lengths l_A, l_B, l_C are determined. The innovative feature of the mechanism design is that the top platform can be moved to a desired position and orientation and the compliance characteristics of the mechanism can be simultaneously controlled.

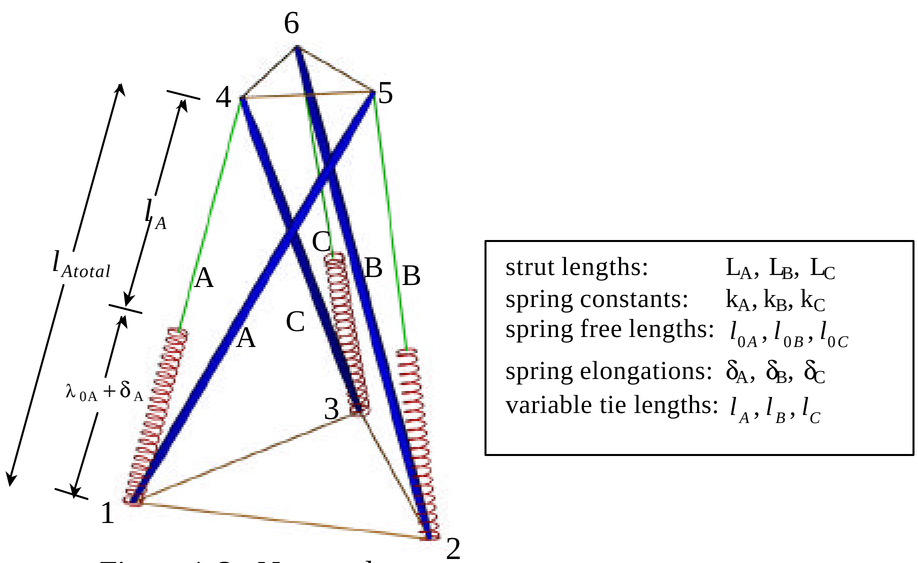

Figure 1-3. Nomenclature

Figure 1-3 depicts the nomenclature used in this paper. The six vertex points of the mechanism are numbered. Variable Strut A and variable Tie A both emanate from Point 1. Similarly, Strut and Tie B and Strut and Tie C emanate from Points 2 and 3, respectively. The following nomenclature is presented:

– The free lengths and spring constants of the three springs are written as l_{0A}, l_{0B}, l_{0C} and k_A, k_B, k_C.

- The elongations of the three springs are written as $\boldsymbol{d}_A$, $\boldsymbol{d}_B$, and $\boldsymbol{d}_C$.

- The top and base ties are written as d_T and $d_{\boldsymbol{B}}$.

- The lengths of the three variable length ties are written as l_A, l_B, and l_C.

- The lengths of the three variable length compressive struts are written as L_A, L_B, and L_C.
- The total length of ties A, B, and C are written as l_{Atotal}, l_{Btotal}, and l_{Ctotal}.

For the mechanism it is assumed that the coordinates of the Points *1, 2,* and *3* are known in terms of a coordinate system attached to the base triangle and that the coordinates of point *4, 5,* and *6* are known in terms of a coordinate system attached to the top triangle. For the reverse analysis, the desired position and orientation of the top platform will be specified relative to the base and from this specification the coordinates of Points *4, 5,* and *6* can readily be determined in terms of the coordinate system attached to the base. Thus, assuming that the coordinates of all the Points *1* through *6* are known, the reverse analysis problem statement can be stated as follows.

Given:

- The transformation matrix ${}^{B}_{T}T$ of the top platform with respect to the base platform (or coordinates of points *1* through *6* measured in the base coordinate system)
- Top and base tie lengths d_T and d_B (omitted if the coordinates of points 1 to 6 are given)
- Spring free lengths l_{0A}, l_{0B}, and l_{0C}

- Spring constants k_A, k_B, and k_C

- Desired total potential energy to be stored in the three springs, U

Find:

- Lengths of the three variable length struts, L_A, L_B, and L_C

- Lengths of the three variable length tie segments, l_A, l_B, and l_C

It is important to note that the position and orientation of the top platform relative to the base cannot be arbitrarily selected. In future chapters it will be shown that the total wrench that is acting on the top platform can be written as

$$\hat{\mathbf{w}} = \mathrm{f}_1\, B\hat{\mathbf{s}}_1 + \mathrm{f}_2\,\hat{\mathbf{s}}_2 + \mathrm{f}_3\,\hat{\mathbf{s}}_3 + \mathrm{f}_4\,\hat{\mathbf{s}}_4 + \mathrm{f}_5\,\hat{\mathbf{s}}_5 + \mathrm{f}_6\,\hat{\mathbf{s}}_6 \ . \qquad (1.1)$$

where f_i (i=1..6) are the force magnitudes in each of the leg connectors (struts or side ties) and $\hat{\mathbf{s}}_i$ (i=1..6) are the Plücker coordinates of the lines of action of each of the six legs. When there is no external wrench applied to the top platform, $\hat{\mathbf{w}} = \mathbf{0}$, and the above equation reduces to

$$\mathrm{f}_1\,\hat{\mathbf{s}}_1 + \mathrm{f}_2\,\hat{\mathbf{s}}_2 + \mathrm{f}_3\,\hat{\mathbf{s}}_3 + \mathrm{f}_4\,\hat{\mathbf{s}}_4 + \mathrm{f}_5\,\hat{\mathbf{s}}_5 + \mathrm{f}_6\,\hat{\mathbf{s}}_6 = \mathbf{0} \ . \qquad (1.2)$$

Ignoring the trivial solution of f_i=0, it is apparent that when there is no external wrench applied to the top platform that equilibrium can only occur if the Plücker coordinates of the six legs are linearly dependent. This implies that an equilibrium solution with no external wrench will exist only for certain positions and orientations of the top platform. This will be discussed in subsequent chapters.

CHAPTER 2
THREE-THREE TENSEGRITY PLATFORMS

In this chapter, the lengths l_A, l_B, l_C, L_A, L_B, and L_C are to be determined so as to position and orient the top platform as desired with a total potential energy as desired.

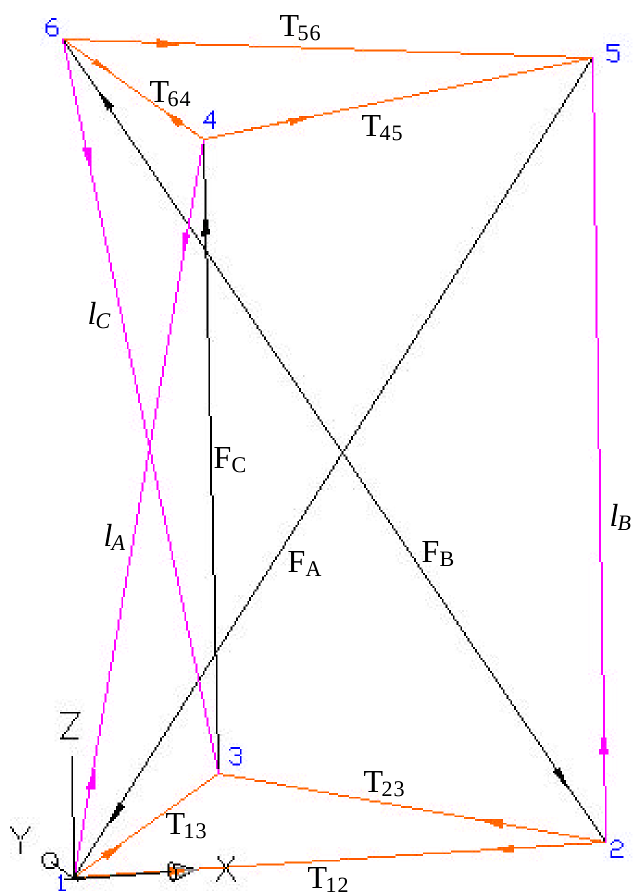

Figure 2-1. Force Notations

2.1 Transformation Matrix

A transformation matrix ${}^{B}_{T}T$ is a matrix which will transform all the points measured in the T (Top) system into the B (Base) system [1]. Therefore, a point P_1 measured in the T system, ${}^{T}P_1$, will have a coordinate of

$$ {}^{B}P_1 = {}^{B}_{T}T \, {}^{T}P_1 \qquad (2.1.1) $$

If the transformation matrix ${}^{B}_{T}T$ consists of one translation of $\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, one rotation of θ about its new x-axis, one rotation of β about its new y-axis, and one rotation of γ about its new z-axis, then [1]

$$ {}^{B}_{T}T = \begin{bmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 & 0 \\ \sin\gamma & \cos\gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.1.2) $$

As an example, consider that coordinate system T is initially aligned with coordinate system B. It is translated to the point $[5, 4, 1]^T$ and then rotated 30 degrees about its new x-axis. Then, the coordinate system is rotated 60 degrees about its new y-axis. Lastly, the system is rotated 50 degrees about its new z-axis. The task is to find the transformation ${}^{B}_{T}T$.

Equation (2.1.2) gives

$$ {}^{B}_{T}T = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 30 & -\sin 30 & 0 \\ 0 & \sin 30 & \cos 30 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 60 & 0 & \sin 60 & 0 \\ 0 & 1 & 0 & 0 \\ -\sin 60 & 0 & \cos 60 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 50 & -\sin 50 & 0 & 0 \\ \sin 50 & \cos 50 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

or $$ {}^{B}_{T}T = \begin{bmatrix} 0.3214 & -0.3830 & 0.8660 & 5 \\ 0.9417 & 0.2250 & -0.2500 & 4 \\ -0.0991 & 0.8959 & 0.4330 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

If the coordinates of a point P_1 is measured in the T coordinate system are $[25, 42, 80]^T$, then the coordinates of point P_1 in the B system is

$$ {}^{B}P_1 = {}^{B}_{T}T\,{}^{T}P_1 = \begin{bmatrix} 0.3214 & -0.3830 & 0.8660 & 5 \\ 0.9417 & 0.2250 & -0.2500 & 4 \\ -0.0991 & 0.8959 & 0.4330 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 25 \\ 42 \\ 80 \\ 1 \end{bmatrix} = \begin{bmatrix} 66.2299 \\ 16.9922 \\ 70.7932 \\ 1 \end{bmatrix}. $$

or $$ {}^{B}P_1 = \begin{bmatrix} 66.2299 \\ 16.9922 \\ 70.7932 \end{bmatrix}. $$

2.2 Coordinate Systems

Assuming there is a coordinate system B $\{X_B, Y_B, Z_B\}$ attached to the base platform such that its origin is at vertex *1*, its x-axis in the direction from vertex *1* to *2*, and its z-axis perpendicular to the base platform, as shown in Figure 2-2. Similarly, a coordinate system T $\{X_T, Y_T, Z_T\}$ is placed such that its origin is at vertex *4*, its x-axis in the direction from vertex *4* to *5*, and its z-axis perpendicular to the top platform. Given d_B and d_T as lengths of the bottom and top ties, respectively, the coordinates of vertices *1, 2,* and *3* in system B and vertices *4, 5,* and *6* in system T can be computed as follows

$$ {}^{B}\mathbf{P}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{B}\mathbf{P}_2 = \begin{bmatrix} d_B \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad {}^{B}\mathbf{P}_3 = \begin{bmatrix} \frac{d_B}{2} \\ d_B \sin(60^\circ) \\ 0 \end{bmatrix}, \qquad (2.2.1) $$

$$ {}^{T}\mathbf{P}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{T}\mathbf{P}_5 = \begin{bmatrix} d_T \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad {}^{T}\mathbf{P}_6 = \begin{bmatrix} \dfrac{d_T}{2} \\ d_T \sin(60^\circ) \\ 0 \end{bmatrix}. \qquad (2.2.2) $$

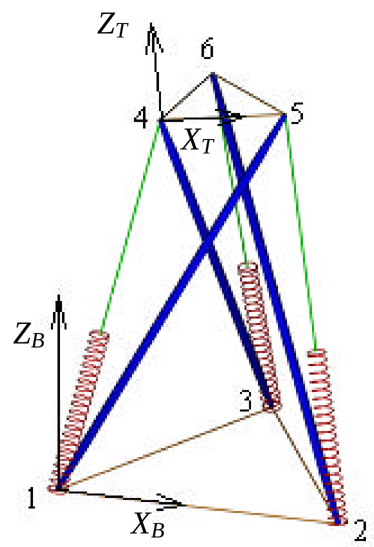

Figure 2-2. Coordinate systems

When the transformation matrix ${}^{B}_{T}T$ is given, the Points *4, 5,* and *6* can be obtained in the coordinate system B as

$$ {}^{B}\mathbf{P}_4 = {}^{B}_{T}T \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad {}^{B}\mathbf{P}_5 = {}^{B}_{T}T \begin{bmatrix} d_T \\ 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } {}^{B}\mathbf{P}_6 = {}^{B}_{T}T \begin{bmatrix} \dfrac{d_T}{2} \\ d_T \sin(60^\circ) \\ 0 \\ 1 \end{bmatrix} \qquad (2.2.3) $$

The lengths of the three Struts, L_A, L_B, and L_C are readily determined as the distance between Points *1* and *5, 2* and *6,* and *3* and *4,* respectively. Similarly, the total length of Ties *A, B,* and *C,* l_{Atotal}, l_{Btotal}, and l_{Ctotal}, are obtained as the distance between the Points 1 and 4, 2 and 5, and 3 and 6. Further, unit vectors along each of the lines defined by the ties and struts can readily be determined and will be written as $\boldsymbol{S}_{ij}$ where the subscripts i and j refer to the specific mechanism vertex points that define the vector from Point i to Point j. With this notation, it is apparent that

$$\boldsymbol{S_{ij}} = - \boldsymbol{S_{ji}}. \tag{2.2.4}$$

The analysis proceeds by performing a static force analysis. There are twelve unknown force magnitudes, i.e. the compressive (or tensile) forces of the three struts, the tensile forces of the three compliant ties, the tensile forces of the three bottom ties, and the tensile forces of the three top ties[1]. The compressive forces in the three struts are written as F_A, F_B, and F_C. The tensile forces in the three varying length ties are written as f_A, f_B, and f_C. The tensile forces in the base ties between the pairs of Points *1* and *2*, *2* and *3*, and *1* and *3* are written as T_{12}, T_{23}, and T_{13}. Similarly, the tensile forces in the top ties between the pairs of Points *4* and *5*, *5* and *6*, and *4* and *6* are written as T_{45}, T_{56}, and T_{46}.

2.3 Force Balance

Each vertex provides 3 independent equations, and the sum of these forces in the x, y, and z directions equals to zero. In vector form, writing force balance equation at points *1*, *2*, *4*, and *6* yields

$$\begin{aligned} &\boldsymbol{f_A} - \boldsymbol{F_A} + \boldsymbol{T_{12}} + \boldsymbol{T_{13}} = \boldsymbol{0} \\ &\boldsymbol{f_B} - \boldsymbol{F_B} + \boldsymbol{T_{12}} + \boldsymbol{T_{23}} = \boldsymbol{0} \\ &\boldsymbol{f_A} - \boldsymbol{F_C} + \boldsymbol{T_{45}} + \boldsymbol{T_{46}} = \boldsymbol{0} \\ &\boldsymbol{f_C} - \boldsymbol{F_B} + \boldsymbol{T_{56}} + \boldsymbol{T_{46}} = \boldsymbol{0} \end{aligned} \tag{2.3.1}$$

The reason that only 4 vertices (*1*, *2*, *4*, and *6*) are chosen is because only 12 equations are needed.

[1] An equivalent mechanism can be designed by replacing the three non-compliant ties in the top and base platforms by rigid bodies that can sustain compressive forces in addition to tensile forces.

Let $\boldsymbol{S_{ij}}$ be unit vector along i and j, i.e. along the ties and struts, where $j>i$ for i = 1,2,...,5 and j = 2,3,...,6. The system of equations (2.3.1) can be written as

$$f_A \boldsymbol{S_{14}} - F_A \boldsymbol{S_{15}} + T_{12} \boldsymbol{S_{12}} + T_{13} \boldsymbol{S_{13}} = \boldsymbol{0}$$

$$f_B \boldsymbol{S_{25}} - F_B \boldsymbol{S_{26}} - T_{12} \boldsymbol{S_{12}} + T_{23} \boldsymbol{S_{23}} = \boldsymbol{0}$$

$$- f_A \boldsymbol{S_{14}} + F_C \boldsymbol{S_{34}} + T_{45} \boldsymbol{S_{45}} + T_{46} \boldsymbol{S_{46}} = \boldsymbol{0}$$

$$- f_C \boldsymbol{S_{36}} + F_B \boldsymbol{S_{26}} - T_{56} \boldsymbol{S_{56}} - T_{46} \boldsymbol{S_{46}} = \boldsymbol{0} \tag{2.3.2}$$

Since the position of the structure is known, vectors $\boldsymbol{S_{ij}}$'s can be easily found. They are then broken down into x, y, and z components: $S_{ij}x$, $S_{ij}y$, and $S_{ij}z$. The Equations (2.3.2) can be rewritten as

$$f_A S_{14}x - F_A S_{15}x + T_{12} S_{12}x + T_{13} S_{13}x = 0$$

$$f_A S_{14}y - F_A S_{15}y + T_{12} S_{12}y + T_{13} S_{13}y = 0$$

$$f_A S_{14}z - F_A S_{15}z + T_{12} S_{12}z + T_{13} S_{13}z = 0$$

$$f_B S_{25}x - F_B S_{26}x - T_{12} S_{12}x + T_{23} S_{23}x = 0$$

$$f_B S_{25}y - F_B S_{26}y - T_{12} S_{12}y + T_{23} S_{23}y = 0$$

$$f_B S_{25}z - F_B S_{26}z - T_{12} S_{12}z + T_{23} S_{23}z = 0$$

$$- f_A S_{14}x + F_C S_{34}x + T_{45} S_{45}x + T_{46} S_{46}x = 0$$

$$- f_A S_{14}y + F_C S_{34}y + T_{45} S_{45}y + T_{46} S_{46}y = 0$$

$$- f_A S_{14}z + F_C S_{34}z + T_{45} S_{45}z + T_{46} S_{46}z = 0$$

$$- f_C S_{36}x + F_B S_{26}x - T_{56} S_{56}x - T_{46} S_{46}x = 0$$

$$- f_C S_{36}y + F_B S_{26}y - T_{56} S_{56}y - T_{46} S_{46}y = 0$$

$$- f_C S_{36}z + F_B S_{26}z - T_{56} S_{56}z - T_{46} S_{46}z = 0 \tag{2.3.4}$$

where $S_{ijx} = ||j_x - i_x||$, x component of unit vector $\boldsymbol{S_{ij}}$

$S_{ijy} = ||j_y - i_y||$, y component of unit vector $\boldsymbol{S_{ij}}$

$S_{ijz} = ||j_z - i_z||$, z component of unit vector $\boldsymbol{S_{ij}}$

i = coordinates of Points *1…5*

j = coordinates of Points *2…6*

These 12 equations of (2.3.4) can be written in the matrix form as

$$\mathbf{J}\ \mathbf{v} = \mathbf{0} \qquad (2.3.5)$$

where

$$\mathbf{J} = \begin{bmatrix} S_{14}x & 0 & 0 & -S_{15}x & 0 & 0 & S_{12}x & 0 & S_{13}x & 0 & 0 & 0 \\ S_{14}y & 0 & 0 & -S_{15}y & 0 & 0 & S_{12}y & 0 & S_{13}y & 0 & 0 & 0 \\ S_{14}z & 0 & 0 & -S_{15}z & 0 & 0 & S_{12}z & 0 & S_{13}z & 0 & 0 & 0 \\ 0 & S_{25}x & 0 & 0 & -S_{26}x & 0 & -S_{12}x & S_{23}x & 0 & 0 & 0 & 0 \\ 0 & S_{25}y & 0 & 0 & -S_{26}y & 0 & -S_{12}y & S_{23}y & 0 & 0 & 0 & 0 \\ 0 & S_{25}z & 0 & 0 & -S_{26}z & 0 & -S_{12}z & S_{23}z & 0 & 0 & 0 & 0 \\ -S_{14}x & 0 & 0 & 0 & 0 & S_{34}x & 0 & 0 & 0 & S_{45}x & 0 & S_{46}x \\ -S_{14}y & 0 & 0 & 0 & 0 & S_{34}y & 0 & 0 & 0 & S_{45}y & 0 & S_{46}y \\ -S_{14}z & 0 & 0 & 0 & 0 & S_{34}z & 0 & 0 & 0 & S_{45}z & 0 & S_{46}z \\ 0 & 0 & -S_{36}x & 0 & S_{26}x & 0 & 0 & 0 & 0 & 0 & -S_{56}x & -S_{46}x \\ 0 & 0 & -S_{36}y & 0 & S_{26}y & 0 & 0 & 0 & 0 & 0 & -S_{56}y & -S_{46}y \\ 0 & 0 & -S_{36}z & 0 & S_{26}z & 0 & 0 & 0 & 0 & 0 & -S_{56}z & -S_{46}z \end{bmatrix}$$

and $\mathbf{v} = [f_A, f_B, f_C, F_A, F_B, F_C, T_{12}, T_{23}, T_{13}, T_{45}, T_{56}, T_{46}]^T$

The matrix Equation (2.3.5) represents twelve homogeneous equations in twelve unknown force magnitudes. One solution is obviously the trivial solution where the forces in all the members are zero. To obtain a practical solution, the twelve equations in (2.3.4) will be rewritten by dividing all terms by the magnitude of the tensile force in the variable length tie A, i.e. f_A, to yield

$$S_{14}x - F_A' S_{15}x + T_{12}' S_{12}x + T_{13}' S_{13}x = 0$$

$$S_{14}y - F_A' S_{15}y + T_{12}' S_{12}y + T_{13}' S_{13}y = 0$$

$$S_{14}z - F_A' S_{15}z + T_{12}' S_{12}z + T_{13}' S_{13}z = 0$$

$$f_B' S_{25}x - F_B' S_{26}x - T_{12}' S_{12}x + T_{23}' S_{23}x = 0$$

$$f_B' S_{25}y - F_B' S_{26}y - T_{12}' S_{12}y + T_{23}' S_{23}y = 0$$

$$f_B' S_{25}z - F_B' S_{26}z - T_{12}' S_{12}z + T_{23}' S_{23}z = 0$$

$$- S_{14}x + F_C' S_{34}x + T_{45}' S_{45}x + T_{46}' S_{46}x = 0$$

$$- S_{14}y + F_C' S_{34}y + T_{45}' S_{45}y + T_{46}' S_{46}y = 0$$

$$- S_{14}z + F_C' S_{34}z + T_{45}' S_{45}z + T_{46}' S_{46}z = 0$$

$$- f_C' S_{36}x + F_B' S_{26}x - T_{56}' S_{56}x - T_{46}' S_{46}x = 0$$

$$- f_C' S_{36}y + F_B' S_{26}y - T_{56}' S_{56}y - T_{46}' S_{46}y = 0$$

$$- f_C' S_{36}z + F_B' S_{26}z - T_{56}' S_{56}z - T_{46}' S_{46}z = 0 \qquad (2.3.6)$$

where

$$f_B' = \frac{f_B}{f_A},$$

$$f_C' = \frac{f_C}{f_A},$$

.

.

.

$$T_6' = \frac{T_6}{f_A} \qquad (2.3.7)$$

Equation set (2.3.6) represents 12 equations in 11 unknowns. Any eleven of the equations can be used to solve for the unknowns. It is shown that it does not matter which equation is taken off; the results remain the same. For this analysis, the first equation is to be omitted. Putting all constants on the right and rewriting the matrix equation

$$\mathbf{J'\ v' = b} \qquad (2.3.8)$$

where

$$\mathbf{J}' = \begin{bmatrix} 0 & 0 & -S_{15}y & 0 & 0 & S_{12}y & 0 & S_{13}y & 0 & 0 & 0 \\ 0 & 0 & -S_{15}z & 0 & 0 & S_{12}z & 0 & S_{13}z & 0 & 0 & 0 \\ S_{25}x & 0 & 0 & -S_{26}x & 0 & -S_{12}x & S_{23}x & 0 & 0 & 0 & 0 \\ S_{25}y & 0 & 0 & -S_{26}y & 0 & -S_{12}y & S_{23}y & 0 & 0 & 0 & 0 \\ S_{25}z & 0 & 0 & -S_{26}z & 0 & -S_{12}z & S_{23}z & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{34}x & 0 & 0 & 0 & S_{45}x & 0 & S_{46}x \\ 0 & 0 & 0 & 0 & S_{34}y & 0 & 0 & 0 & S_{45}y & 0 & S_{46}y \\ 0 & 0 & 0 & 0 & S_{34}z & 0 & 0 & 0 & S_{45}z & 0 & S_{46}z \\ 0 & -S_{36}x & 0 & S_{26}x & 0 & 0 & 0 & 0 & 0 & -S_{56}x & -S_{46}x \\ 0 & -S_{36}y & 0 & S_{26}y & 0 & 0 & 0 & 0 & 0 & -S_{56}y & -S_{46}y \\ 0 & -S_{36}z & 0 & S_{26}z & 0 & 0 & 0 & 0 & 0 & -S_{56}z & -S_{46}z \end{bmatrix}$$

$$\mathbf{b} = [-S_{14}y, -S_{14}z, 0, 0, 0, S_{14}x, S_{14}y, S_{14}z, 0, 0, 0]^T$$

and $$\mathbf{v}' = [f_B', f_C', F_A', F_B', F_C', T_{12}', T_{23}', T_{13}', T_{45}', T_{56}', T_{46}']^T$$

The result **v¢** can be computed by

$$\mathbf{v}' = (\mathbf{J}')^{-1}\,\mathbf{b} \qquad (2.3.9)$$

The matrix **J'** will be invertible when the line coordinates of the struts and compliant ties are not linearly dependent. At this point of the solution the magnitudes of the components of **v'** must be checked to ensure that f_B' and f_C' are positive in magnitude. A negative value would indicate that the mechanism is not in tensegrity and would collapse.

2.4 Potential Energy

The potential energy, U, of the structure is given by

$$U = \frac{1}{2}\,(k_A d_A^2 + k_B d_B^2 + k_C d_C^2) \qquad (2.4.1)$$

where d_A, d_B, d_C are the elongations of the springs and k_A, k_B, and k_C are the spring constants.

The tensile force in the ties is

$$f_i = k_i \boldsymbol{d}_i \qquad \text{for } i = A, B, C \tag{2.4.2}$$

Combining the two above equations yields

$$U = \frac{1}{2}\left(\frac{f_A^2}{k_A} + \frac{f_B^2}{k_B} + \frac{f_C^2}{k_C} \right) \tag{2.4.3}$$

From Equations (2.3.7) and (2.4.3), we have

$$U = \frac{1}{2}\left(\frac{f_A^2}{k_A} + \frac{f_A^2 f_B'^2}{k_B} + \frac{f_A^2 f_C'^2}{k_C} \right) \tag{2.4.4}$$

Solving this equation for f_A yields

$$f_A = \sqrt{\frac{2Uk_A k_B k_C}{k_B k_C + k_A k_C f_B'^2 + k_A k_B f_C'^2}} \tag{2.4.5}$$

The force in the compliant ties B and C can now be determined from (2.3.7) as

$$f_B = f_A f_B'$$

$$f_C = f_A f_C'$$

$$F_A = f_A F_A'$$

$$F_B = f_A F_B'$$

$$F_C = f_A F_C' \tag{2.4.6}$$

From Equation (2.4.2), we have

$$\boldsymbol{d}_A = \frac{f_A}{k_A} \tag{2.4.7}$$

$$\boldsymbol{d}_B = \frac{f_B}{k_B} \tag{2.4.8}$$

$$\boldsymbol{d}_C = \frac{f_C}{k_C} \tag{3.4.9}$$

Finally the lengths of the three variable length tie segments l_A, l_B, and l_C can be determined from

$$l_A = l_{Atotal} - l_{0A} - \boldsymbol{d}_A \quad (2.4.10)$$

$$l_B = l_{Btotal} - l_{0B} - \boldsymbol{d}_B \quad (2.4.11)$$

$$l_C = l_{Ctotal} - l_{0C} - \boldsymbol{d}_C \quad (2.4.12)$$

where $l_{Atotal} = ||{}^{\mathbf{B}}\mathbf{P_4} - {}^{\mathbf{B}}\mathbf{P_1}||$, the distance from Points *4* and *1*

$l_{Btotal} = ||{}^{\mathbf{B}}\mathbf{P_5} - {}^{\mathbf{B}}\mathbf{P_2}||$, the distance from Points *5* and *2*

$l_{Ctotal} = ||{}^{\mathbf{B}}\mathbf{P_6} - {}^{\mathbf{B}}\mathbf{P_3}||$, the distance from Points *6* and *3*

Different values of potential energy give different lengths of the three variable length tie segments. Therefore, in order to increase the stiffness of the structure, the lengths of these tie segments have to be shortened [9].

2.5 Numerical Examples

2.5.1 Example 1

In this example, the values of the constants and inputs were given such that the top and base platforms were parallel; when three spring constants were the same and free lengths of the three springs were equal, the expected result was $l_A = l_B = l_C$. This also meant the side ties must have experienced the same amount of force. These conditions will result in a situation where the Plücker line coordinates of the six legs (struts and side ties) are linearly dependent and it will be shown that the desired position and orientation can be achieved with no external wrench applied.

Assuming that a set of values were given as:

$$ {}^{B}_{T}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 21.906 \\ 0.5 & 0.866 & 0 & -14.019 \\ 0 & 0 & 1 & 135.171 \\ 0 & 0 & 0 & 1 \end{bmatrix}, $$

d_T = 80 mm (top tie length),

d_B = 90 mm (base tie length),

$k_A = k_B = k_C = k = 1.5$ N/mm (spring constants),

$l_{0A} = l_{0B} = l_{0C}$ = 25 mm (spring free lengths),

U = 1200 N mm (desired potential energy).

Equation 2.2.1 gives the coordinates of Vertices 1, 2, and 3 in the B system as

$^{\mathbf{B}}\mathbf{P_1}$ = [0,0,0] mm, $^{\mathbf{B}}\mathbf{P_2}$ = [90,0,0] mm, and $^{\mathbf{B}}\mathbf{P_3}$ = [45,77.943,0] mm.

The coordinates of Vertices 4, 5, and 6 in the T system are obtained from Equation (2.2.2)

$^{\mathbf{T}}\mathbf{P_4}$ = [0,0,0] mm, $^{\mathbf{T}}\mathbf{P_5}$ = [80,0,0] mm, and $^{\mathbf{T}}\mathbf{P_6}$ = [40,69.282,0] mm

These are then transformed to the B coordinate system using Equation (2.2.3) as

$^{\mathbf{B}}\mathbf{P_4}$ = [21.906,-14.019,135.171] mm,

$^{\mathbf{B}}\mathbf{P_5}$ = [91.188,25.981,135.171] mm,

$^{\mathbf{B}}\mathbf{P_6}$ = [21.906,65.981,135.171] mm.

From this data, the length of the three struts was determined to be

$\underline{L_A = L_B = L_C = 165.11 \text{ mm.}}$

Equation (2.3.9) is solved to yield

v' = [1.000, 1.000, 1.200, 1.200, 1.200, 0.336, 0.336, 0.336, 0.378, 0.378, 0.378]^T N

Equation (2.4.5) gives

$$ f_A = \sqrt{\frac{2Uk}{3}} = \sqrt{\frac{2*1200*1.5}{3}} = 34.641 \text{ N}. $$

Now from Equations (2.4.6) yielded

$$f_A = f_B = f_C = 34.641 \text{ N.}$$

Equations (2.4.6), (2.4.7), and (2.4.9) give

$$\boldsymbol{d}_A = \boldsymbol{d}_B = \boldsymbol{d}_C = 23.094 \text{ mm.}$$

The total length of Ties A, B, and C are calculated to be

$$l_{Atotal} = l_{Btotal} = l_{Ctotal} = 137.65 \text{ mm.}$$

Therefore using Equations (2.4.9), (2.4.10), and (2.4.11), the lengths of the variable ties are determined to be

$$\underline{l_A = l_B = l_C = 89.56 \text{ mm.}}$$

Table 1 and 2 summarize the given data and results of Example 1

Table 1. Data for Example 1

Coordinate *1* (mm)	**[0,0,0]**	Spring constant k_A (N/mm)	**1.5**
Coordinate *2* (mm)	**[90,0,0]**	Spring constant k_B (N/mm)	**1.5**
Coordinate *3* (mm)	**[45,77.943,0]**	Spring constant k_C (N/mm)	**1.5**
Coordinate *4* (mm)	**[21.906,-14.019,135.171]**	Spring free length l_{0A} (mm)	**25**
Coordinate *5* (mm)	**[91.188,25.981,135.171]**	Spring free length l_{0B} (mm)	**25**
Coordinate *6* (mm)	**[21.906,65.981,135.171]**	Spring free length l_{0C} (mm)	**25**
		Potential energy *U* (N.mm)	**1200**

Table 2. Results of Example 1

Length of Strut A L_A (mm)	**165.11**	Elongation A $\ddot{a}_A$ (mm)	**23.094**
Length of Strut B L_B (mm)	**165.11**	Elongation B $\ddot{a}_B$ (mm)	**23.094**
Length of Strut C L_C (mm)	**165.11**	Elongation C $\ddot{a}_C$ (mm)	**23.094**
Force in Tie A f_A (N)	**34.641**	Length of Tie A l_A (mm)	**89.560**
Force in Tie B f_B (N)	**34.641**	Length of Tie B l_B (mm)	**89.560**
Force in Tie C f_C (N)	**34.641**	Length of Tie C l_C (mm)	**89.560**

Figure 2-3 illustrates the final shape of the structure in Example 1.

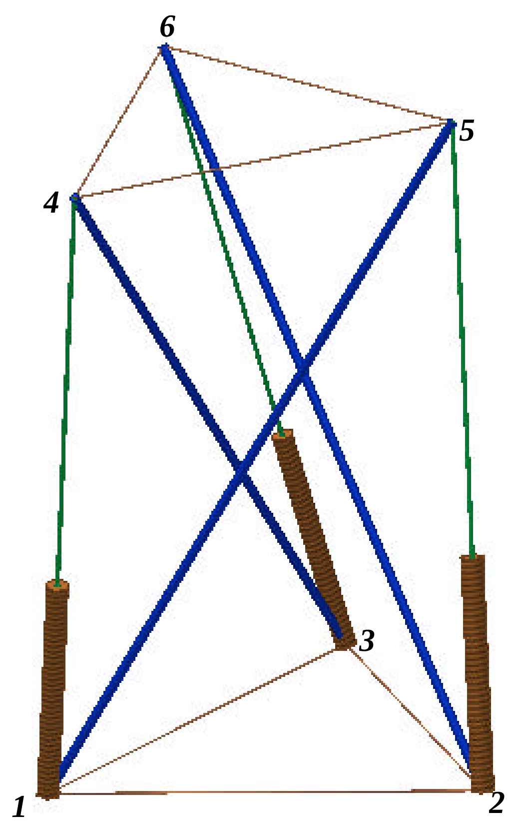

Figure 2-3. Example 1

2.5.2 Example 2

In this example, the coordinates of Vertices *1* to *6* are given, as measured in the base coordinate system, instead of the transformation matrix. The values of the constants and inputs were given such that the top and base platforms were not parallel; this means that the structure is not symmetrical, and hence the Plücker line coordinates of the six legs (struts and side ties) may not be linearly dependent. This would mean that an external wrench would have to be applied to the top platform to maintain equilibrium at the desired position and orientation and with the desired potential energy. This will be discussed in Section 3.2.

Assuming that a set of values are given as

$^{B}\mathbf{P_1}$ = [0,0,0] cm (Coordinate1),

$^{B}\mathbf{P_2}$ = [15,0,0] cm (Coordinate 2),

$^{B}\mathbf{P_3}$ = [7.5,13,0] cm (Coordinate 3),

$^{B}\mathbf{P_4}$ = [6.865,2.473,28.073] cm (Coordinate 4),

$^{B}\mathbf{P_5}$ = [18.291,9.778,21.677] cm (Coordinate 5),

$^{B}\mathbf{P_6}$ = [4.915,16.425,22.985] cm (Coordinate 6),

$k_A = k_B = k_C = k = 20$ N/cm (spring constant),

$l_{0A} = l_{0B} = l_{0C} = 8$ cm (spring free lengths),

$U = 40$ N cm (desired potential energy).

From this data, the length of the three struts was determined to be

$$\underline{L_A = L_B = L_C = 30 \text{ cm.}}$$

Equation (2.3.9) was solved and obtained

$\mathbf{v'}$ = [0.953, 0.988, 1.339, 1.123, 1.364, 0.377, 0.262, 0.406, 0.415, 0.244, 0.389]^T N.

Equation (2.4.3) gave

$$f_A = \sqrt{\frac{2Uk}{1 + f_B'^2 + f_C'^2}} = \sqrt{\frac{2*40*20}{1+0.953^2+0.988^2}} = 23.552 \text{ N.}$$

From Equations (2.4.6) yielded

$f_B = f_A f_B' = 22.440$ N,

$f_C = f_A f_C' = 23.270$ N.

Equations (2.4.7), (2.4.8), and (2.4.9) gave

$\boldsymbol{d}_A = 1.178$ cm,

$\boldsymbol{d}_B = 1.122$ cm,

$\boldsymbol{d}_C = 1.163$ cm.

The total length of Ties A, B, and C were calculated to be

$l_{Atotal} = \| {}^{\mathbf{B}}\mathbf{P_4} - {}^{\mathbf{B}}\mathbf{P_1} \| = 29.006$ cm,

$l_{Btotal} = \| {}^{\mathbf{B}}\mathbf{P_5} - {}^{\mathbf{B}}\mathbf{P_2} \| = 24.007$ cm ,

$l_{Ctotal} = \| {}^{\mathbf{B}}\mathbf{P_6} - {}^{\mathbf{B}}\mathbf{P_3} \| = 23.383$ cm.

Therefore using Equations (2.4.10), (2.4.11), and (2.4.12), the lengths of the variable ties were determined to be

$\underline{l_A = 19.844 \text{ cm,}}$

$\underline{l_B = 14.876 \text{ cm,}}$

$\underline{l_C = 14.218 \text{ cm.}}$

Table 3 and 4 summarize the given data and results of Example 2

Table 3. Data for Example 2

Coordinate *1* (cm)	**[0,0,0]**	Spring constant k_A (N/cm)	**20**
Coordinate *2* (cm)	**[15,0,0]**	Spring constant k_B (N/cm)	**20**
Coordinate *3* (cm)	**[7.5,13,0]**	Spring constant k_C (N/cm)	**20**
Coordinate *4* (cm)	**[6.865,2.473,28.073]**	Spring free length l_{0A} (cm)	**8**
Coordinate *5* (cm)	**[18.291,9.778,21.677]**	Spring free length l_{0B} (cm)	**8**
Coordinate *6* (cm)	**[4.915,16.425,22.985]**	Spring free length l_{0C} (cm)	**8**
		Potential energy U (N.cm)	**40**

Table 4. Results of Example 2

Length of Strut A L_A (cm)	**30**	Elongation A $\ddot{a}_A$ (cm)	**1.178**
Length of Strut B L_B (cm)	**30**	Elongation B $\ddot{a}_B$ (cm)	**1.122**
Length of Strut C L_C (cm)	**30**	Elongation C $\ddot{a}_C$ (cm)	**1.163**
Force in Tie A f_A (N)	**23.552**	Length of Tie A l_A (cm)	**19.828**
Force in Tie A f_B (N)	**22.440**	Length of Tie B l_B (cm)	**14.885**
Force in Tie A f_C (N)	**23.270**	Length of Tie C l_C (cm)	**14.219**

Figure 2-4 shows the resulting structure.

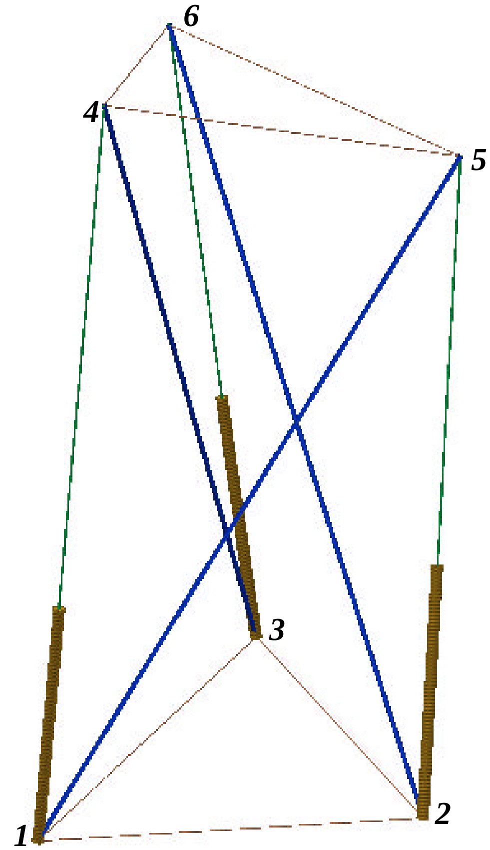

Figure 2-4. Example 2

2.5 Verification of Results

The analysis was done using summation of forces at the vertices. It is necessary to verify the results using a different method, i.e. an analysis of summation of moments. The summation of the moments at the vertices must equal to zero.

Verification of Example1. Figure 2-5 shows the free body diagram of Strut B and its forces acting on Vertex *6*. The directions of the forces are in the opposite of the unit vectors $\boldsymbol{S}$'s. Hence, $\boldsymbol{f_C}$, $\boldsymbol{T_{56}}$, and $\boldsymbol{T_{46}}$ all have negative values. If the system is in an equilibrium position, then the summation of moments with respect to Vertex *2* must be zero.

$$-\mathbf{L}_B \times \mathbf{f}_C - \mathbf{L}_B \times \mathbf{T}_{46} - \mathbf{L}_B \times \mathbf{T}_{56} = 0 \tag{2.5.1}$$

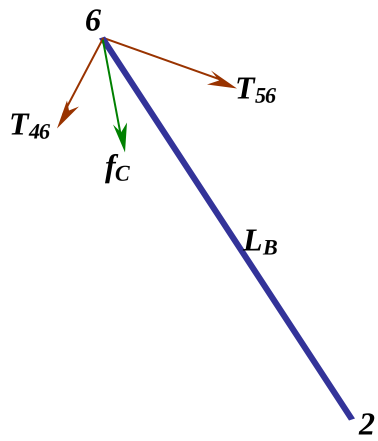

Figure 2-5. Free body diagram of Strut B of Example 1

The Vector $\boldsymbol{L_B}$ is given by

$$\boldsymbol{L_B} = \mathbf{P}_6 - \mathbf{P}_2 = \begin{bmatrix} 21.906 \\ 65.981 \\ 135.171 \end{bmatrix} mm - \begin{bmatrix} 90 \\ 0 \\ 0 \end{bmatrix} mm = \begin{bmatrix} -68.094 \\ 65.981 \\ 135.171 \end{bmatrix} mm \,. \tag{2.5.2}$$

The Vector $\boldsymbol{f_C}$ is the product of the force magnitude and the Unit vector $\boldsymbol{S_{36}}$ of the force

$$\mathbf{f}_C = f_C \mathbf{S}_{36} = 34.641 \frac{\begin{bmatrix} 21.906 \\ 65.981 \\ 135.171 \end{bmatrix} - \begin{bmatrix} 45 \\ 77.943 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 21.906 \\ 65.981 \\ 135.171 \end{bmatrix} - \begin{bmatrix} 45 \\ 77.943 \\ 0 \end{bmatrix} \right\|} N = \begin{bmatrix} -5.813 \\ -3.01 \\ 34.017 \end{bmatrix} N \,. \tag{2.5.3}$$

The Vector tension $\boldsymbol{T_{46}}$ is also found by multiplying the magnitude and its Unit vector $\boldsymbol{S_{46}}$

$$\mathbf{T}_{46} = T_{46}\mathbf{S}_{46} = 13.094\frac{\begin{bmatrix}21.906\\65.981\\135.171\end{bmatrix}-\begin{bmatrix}21.906\\-14.019\\135.171\end{bmatrix}}{\left\|\begin{bmatrix}21.906\\65.981\\135.171\end{bmatrix}-\begin{bmatrix}21.906\\-14.019\\135.171\end{bmatrix}\right\|}N = \begin{bmatrix}0\\13.077\\0\end{bmatrix}N. \quad (2.5.4)$$

And similarly, $\boldsymbol{T_{56}}$ is given by

$$\mathbf{T}_{56} = T_{56}\mathbf{S}_{56} = 13.094\frac{\begin{bmatrix}21.906\\65.981\\135.171\end{bmatrix}-\begin{bmatrix}91.188\\25.981\\135.171\end{bmatrix}}{\left\|\begin{bmatrix}21.906\\65.981\\135.171\end{bmatrix}-\begin{bmatrix}91.188\\25.981\\135.171\end{bmatrix}\right\|}N = \begin{bmatrix}-11.324\\6.537\\0\end{bmatrix}N. \quad (2.5.5)$$

Now the summation of moments can be evaluated by substituting (2.5.2), (2.5.3), (2.5.4), and (2.5.5) into (2.5.1) to yield

$$-\begin{bmatrix}265.14\\153.08\\58.84\end{bmatrix}N\cdot mm - \begin{bmatrix}-176.76\\0\\-89.04\end{bmatrix}N\cdot mm - \begin{bmatrix}-88.38\\-153.08\\30.20\end{bmatrix}N\cdot mm = \begin{bmatrix}0\\0\\0\end{bmatrix}N\cdot mm.$$

Since the summation of moments with respect to vertex *2* is zero, it confirms that the strut *B* is in equilibrium. The same results were obtained for strut *A* and *C*. Therefore, the structure is in equilibrium. Different approach will also be used to verify this in Section 3.2.

For Example 2, it was found that the summation of moments at Vertices *3* and *5* are not equaled to zero. This means that the structure is not in equilibrium for it is not at a singular configuration. This will be explained in more details in the next chapter.

CHAPTER 3
THREE-THREE TENSEGRITY PLATFORMS
WITH AN APPLIED EXTERNAL WRENCH

3.1 Plücker Coordinates [1]

A line is determined by joining two distinct Points $\mathbf{p_1}(x_1, y_1, z_1)$ and $\mathbf{p_2}(x_2, y_2, z_2)$. The Vector **S** along the line is written as

$$\mathbf{S} = (\mathbf{p_2} - \mathbf{p_1}) \tag{3.1.1}$$

Alternatively,

$$\mathbf{S} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k} \tag{3.1.2}$$

If $\mathbf{S_0}$ is the moment of the line about a reference Point O, it is given by

$$\mathbf{S_0} = \mathbf{p_1} \times \mathbf{S} \tag{3.1.3}$$

which is

$$\mathbf{S_0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & y_1 & z_1 \\ L & M & N \end{vmatrix} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k} \tag{3.1.4}$$

where

$$P = y_1N - z_1M$$

$$Q = z_1L - x_1N$$

$$R = x_1M - y_1L \tag{3.1.5}$$

The coordinates of a line is written as $\$ = \{\mathbf{S};\mathbf{S_0}\} = \{L,M,N;P,Q,R\}$ and is called the Plücker coordinates of the line [1]. Given the coordinates of the two Points $\mathbf{p_1}$ and $\mathbf{p_2}$, the Plücker coordinates for the line can be expressed by the six 2×2 determinants of the array [6]

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \end{bmatrix} \tag{3.1.6}$$

as

$$L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, \quad M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}, \quad N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix},$$

$$P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \quad R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{3.1.7}$$

It is more convenient to express the Plücker in the unitized form by dividing **$** by |**S**|, which is $\sqrt{L^2 + M^2 + N^2}$

$$\hat{\mathbf{s}} = \frac{1}{\sqrt{L^2 + M^2 + N^2}} \begin{bmatrix} \mathbf{S} \\ \mathbf{S_0} \end{bmatrix} = \begin{bmatrix} \mathbf{s} \\ \mathbf{s_0} \end{bmatrix}. \tag{3.1.8}$$

A Force **f** of magnitude *f* acting on a Line $\hat{\mathbf{s}}$ = {**s;s₀**} where |**s**| = 1 can be expressed as a scalar multiple *f***s** of the Unit vector **s** which is bound to the Line $\hat{\mathbf{s}}$. The moment of the Force **f** about the reference Point O, i.e. $\mathbf{m_0}$ can be expressed as a scalar multiple $f\mathbf{s_0}$. The action of the force upon the body can therefore be elegantly expressed as a scalar multiple of the unit line vector, and the its coordinates are given by

$$\hat{\mathbf{w}} = f\hat{\mathbf{s}} = f\{\mathbf{s};\mathbf{s_0}\}, \tag{3.1.9}$$

or

$$\hat{\mathbf{w}} = f\hat{\mathbf{s}} = \{\mathbf{f};\mathbf{m_0}\}. \tag{3.1.10}$$

where $\hat{\mathbf{w}}$ may be called a wrench.

3.2 Tensegrity Platforms with a Wrench

3.2.1 Forward and Reverse Static Analysis

The forward static analysis consists of evaluating the resultant Wrench $\hat{\mathbf{w}}$ = {**f**;$\mathbf{m_0}$}due to three forces in the compliant Ties f_A, f_B, f_C and three forces in the Struts F_A, F_B, F_C, which are obtained from Chapter 2. The resultant Wrench $\hat{\mathbf{w}}$ can be expressed in the form

$$\hat{\mathbf{w}} = \{\mathbf{f_A};\mathbf{m_{0A}}\} + \{\mathbf{f_B};\mathbf{m_{0B}}\} + \{\mathbf{f_C};\mathbf{m_{0C}}\} +$$

$$+ \{\mathbf{F_A};\mathbf{m_{0FA}}\} + \{\mathbf{F_B};\mathbf{m_{0FB}}\} + \{\mathbf{F_C};\mathbf{m_{0FC}}\}, \qquad (3.2.1)$$

or

$$\hat{\mathbf{w}} = -f_A\{\mathbf{s_A};\mathbf{s_{0A}}\} - f_B\{\mathbf{s_B};\mathbf{s_{0B}}\} - f_C\{\mathbf{s_C};\mathbf{s_{0C}}\}+$$

$$+ F_A\{\mathbf{s_{FA}};\mathbf{s_{0FA}}\} + F_B\{\mathbf{s_{FB}};\mathbf{s_{0FB}}\} + F_C\{\mathbf{s_{FC}};\mathbf{s_{0FC}}\}, \qquad (3.2.2)$$

where all the $\mathbf{s}$'s and $\mathbf{s_0}$'s are evaluated from Equations (3.1.7) and (3.1.8). Note that f_A, f_B, and f_C have negative values since they are in tension.

Figure 3-1 illustrates a three-three tensegrity platform with the six forces and their corresponding lines of action.

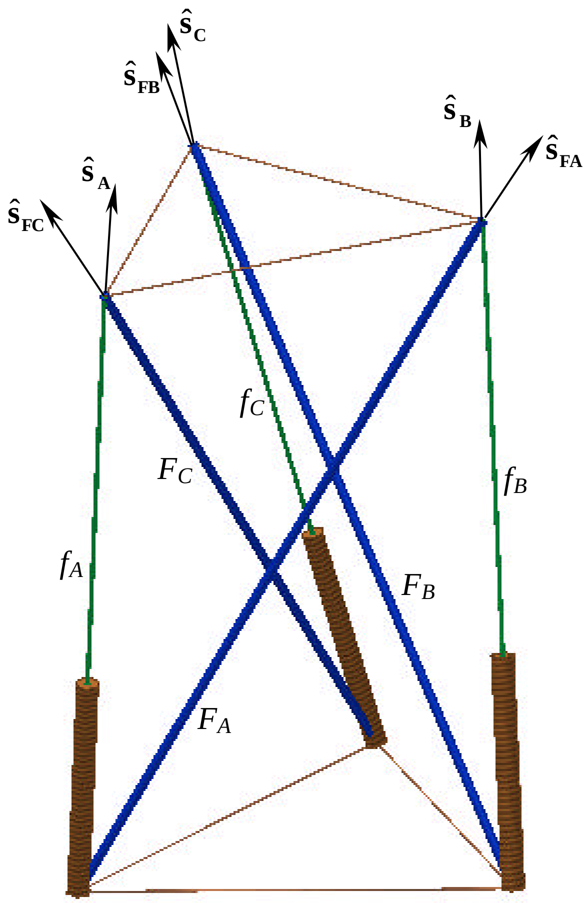

Figure 3-1. Three-Three Tensegrity Platform with the Forces

It is convenient to express Equation (3.2.2) in the matrix form

$$\hat{\mathbf{w}} = -f_A\begin{bmatrix}\mathbf{s_A}\\ \mathbf{s_{0A}}\end{bmatrix} - f_B\begin{bmatrix}\mathbf{s_B}\\ \mathbf{s_{0B}}\end{bmatrix} - f_C\begin{bmatrix}\mathbf{s_C}\\ \mathbf{s_{0C}}\end{bmatrix} + F_A\begin{bmatrix}\mathbf{s_{FA}}\\ \mathbf{s_{0FA}}\end{bmatrix} + F_B\begin{bmatrix}\mathbf{s_{FB}}\\ \mathbf{s_{0FB}}\end{bmatrix} + F_C\begin{bmatrix}\mathbf{s_{FC}}\\ \mathbf{s_{0FC}}\end{bmatrix}. \qquad (3.2.3)$$

Equation (3.2.3) may be written as

$$\hat{\mathbf{w}} = \boldsymbol{j}\,\boldsymbol{q}, \tag{3.2.4}$$

where $\boldsymbol{j}$ is a 6x6 matrix

$$\boldsymbol{j} = \begin{bmatrix} \mathbf{s_A} & \mathbf{s_B} & \mathbf{s_C} & \mathbf{s_{FA}} & \mathbf{s_{FB}} & \mathbf{s_{FC}} \\ \mathbf{s_{0A}} & \mathbf{s_{0B}} & \mathbf{s_{0C}} & \mathbf{s_{0FA}} & \mathbf{s_{0FB}} & \mathbf{s_{0FB}} \end{bmatrix}, \tag{3.2.5}$$

and $\boldsymbol{q}$ is a 6×1 column vector

$$\boldsymbol{q} = \begin{bmatrix} -f_A \\ -f_B \\ -f_C \\ F_A \\ F_B \\ F_C \end{bmatrix}. \tag{3.2.6}$$

Since the six columns of $\boldsymbol{j}$ and magnitude of the forces in $\boldsymbol{q}$ are readily determined, the coordinates of the resultant wrench can be evaluated from Equation (3.2.4) and hence

$$\hat{\mathbf{w}} = \begin{bmatrix} \mathbf{f} \\ \mathbf{m_0} \end{bmatrix} = \begin{bmatrix} L \\ M \\ N \\ P \\ Q \\ R \end{bmatrix}. \tag{3.2.7}$$

The magnitude of the resultant wrench is given by

$$f = \sqrt{L^2 + M^2 + N^2}\,. \tag{3.2.8}$$

The pitch h is determined as

$$h = \frac{\mathbf{f}\cdot\mathbf{m_0}}{\mathbf{f}\cdot\mathbf{f}} = \frac{LP + MQ + NR}{L^2 + M^2 + N^2}. \tag{3.2.9}$$

And finally, the coordinates for the line of which the resultant wrench acts on are given by

$$\mathbf{s} = \left(\frac{L}{f}, \frac{M}{f}, \frac{N}{f} \right), \tag{3.2.10}$$

$$\mathbf{s_0} = \left(\frac{P - hL}{f}, \frac{Q - hM}{f}, \frac{R - hN}{f} \right). \tag{3.2.11}$$

Therefore, for a given position and orientation of the top platform of a tensegrity structure along with a desired potential energy, the screw theory based summation of the forces that the struts and compliant ties experience is equivalent to the resultant wrench $\hat{\mathbf{w}}$ acting on the top platform. When the structure is in equilibrium, the resultant wrench must equal to zero.

For the resultant Wrench $\hat{\mathbf{w}}$ is zero, Equation (3.2.4) becomes

$$\mathbf{0} = \boldsymbol{jq}. \tag{3.2.12}$$

To avoid having trivial solution, i.e. the Force vector $\boldsymbol{q}$ equals to zero, the columns of the $\boldsymbol{j}$ matrix have to be linearly dependent, which results in zero for its determinant (or less than 6 for its rank). This configuration is said to be at singularity [1]. The next section will show that the symmetric tensegrity structure (Example 1 in Section 2.4) is singular when it is in equilibrium. This can be considered as addition verification for Example 1, where it was claimed that for the forces in the struts and side ties, the structure was in equilibrium.

When $\boldsymbol{j}$ has a non-zero determinant (or a rank of 6), the left-hand side of Equation (3.2.12) is no longer **0**. This implies that the structure is not in equilibrium position. This is the case of Example 2 in Section 2.4, which will be clarified in the following section.

More often times, it is desired to perform the reverse static analysis for a structure with an applied wrench. The reverse static analysis is when the positions of the vertices of the structure is known, and there is an external wrench or force applied to the top platform, the task is to determine the magnitudes of Forces f_A, f_B, f_C, F_A, F_B, and F_C. This can be accomplished by solving (3.2.4) for $\boldsymbol{q}$ as

$$\boldsymbol{q} = \boldsymbol{j}^{-1}\ \hat{\mathbf{w}} \tag{3.2.13}$$

where $\boldsymbol{j}^{-1}$ is the inverse of $\boldsymbol{j}$, provided the structure is not at singularity.

3.2.2 Examples

Example 1

Assuming the same data as in Example 1 of Section 2.4 is used, it is to determine the equivalent resultant wrench acting on the structure.

Using the same procedure as in Example 1, the force ratio vector was found to be **v' =** [1.000, 1.000, 1.200, 1.200, 1.200, 0.336, 0.336, 0.336, 0.378, 0.378, 0.378]^T N.

Then Equation (2.4.3) gives

$$f_A = \sqrt{\frac{2Uk}{1 + f_B'^2 + f_C'^2}} = \sqrt{\frac{2 * 1200 * 1.5}{1 + 1^2 + 1^2}} = 34.641 \text{ N.}$$

Applying Equations (2.4.5) yields

$f_B = f_A f_B' = 34.641$ N,

$f_C = f_A f_C' = 34.641$ N,

$F_A = f_A F_A' = 41.552$ N,

$F_B = f_A F_B' = 41.552$ N,

$F_C = f_A F_C' = 41.552$ N.

Hence Equation (3.2.6) gives

$$\boldsymbol{q} = \begin{bmatrix} -34.641 \\ -34.641 \\ -34.641 \\ 41.552 \\ 41.552 \\ 41.552 \end{bmatrix} \text{N},$$

and Equation (3.2.5) gives

$$\boldsymbol{j} = \begin{bmatrix} 0.159 & 0.0086 & -0.168 & 0.552 & -0.412 & -0.140 \\ -0.102 & 0.189 & -0.0869 & 0.157 & 0.400 & -0.557 \\ 0.982 & 0.982 & 0.982 & 0.819 & 0.819 & 0.819 \\ 0 & 0 & 76.538 & 0 & 0 & 63.809 \\ 0 & -88.379 & -44.190 & 0 & -73.680 & -36.840 \\ 0 & 16.987 & 9.166 & 0 & 35.966 & -14.162 \end{bmatrix}.$$

Finally, the resultant wrench is found using Equation (3.2.4)

$$\hat{\mathbf{w}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{N}.$$

This is the expected result. Since the structure is in equilibrium, the resultant wrench must equal to zero.

Example 2

For the structure in Example 2 of Section 2.4, it is to determine the resultant wrench of the six forces in the struts and side ties. The force ratio was found to be

$\mathbf{v'} = [0.953, 0.988, 1.339, 1.123, 1.364, 0.377, 0.262, 0.406, 0.415, 0.244, 0.389]^T$ N.

and $f_A = 23.552$ N

Applying Equations (2.4.5) yields

$$f_B = f_A f_B' = 22.445 \text{ N},$$

$f_C = f_A f_C' = 23.269$ N,

$F_A = f_A F_A' = 31.536$ N,

$F_B = f_A F_B' = 26.449$ N,

$F_C = f_A F_C' = 32.125$ N.

Hence Equation (3.2.6) gives

$$\boldsymbol{q} = \begin{bmatrix} -23.552 \\ -22.445 \\ -22.269 \\ 31.536 \\ 26.449 \\ 32.125 \end{bmatrix} \text{N},$$

and Equation (3.2.5) gives

$$\boldsymbol{j} = \begin{bmatrix} 0.237 & 0.110 & -0.111 & 0.610 & -0.336 & -0.0212 \\ 0.0853 & 0.327 & 0.147 & 0.326 & 0.548 & -0.351 \\ 0.968 & 0.939 & 0.983 & 0.723 & 0.766 & 0.936 \\ 0 & 0 & 12.769 & 0 & 0 & 12.162 \\ 0 & -10.872 & -7.372 & 0 & -11.494 & -7.022 \\ 0 & 4.904 & 2.538 & 0 & 8.213 & -2.356 \end{bmatrix}.$$

The rank of this $\boldsymbol{j}$ matrix is 6, and its determinant is –32.968. Therefore, using Equation (3.2.4), the resultant wrench is

$$\hat{\mathbf{w}} = \begin{bmatrix} 4.194 \\ 0.736 \\ 6.405 \\ 93.599 \\ -114.023 \\ -27.545 \end{bmatrix} \text{N}.$$

This non-zero resultant wrench shows that the structure is not in equilibrium unless an external wrench of $-\hat{\mathbf{w}}$ is applied to the system.

Therefore, to keep structure at the desired position and orientation as in Example 2, an external wrench $\hat{\mathbf{w}}_e$ should be applied to the structure, where

$$\hat{\mathbf{w}}_e = -\begin{bmatrix} 4.194 \\ 0.736 \\ 6.405 \\ 93.599 \\ -114.023 \\ -27.545 \end{bmatrix} \text{N}$$

With this external wrench, Equations (3.2.7) to (3.2.11) give

The magnitude of the force is

$$f = \sqrt{L^2 + M^2 + N^2} \quad = 7.691 \text{ N},$$

The pitch is

$$h = \frac{\mathbf{f} \cdot \mathbf{m_0}}{\mathbf{f} \cdot \mathbf{f}} = \frac{LP + MQ + NR}{L^2 + M^2 + N^2} = 2.235 \text{ cm},$$

The unitized Plücker coordinates are

$$\mathbf{s} = \left(\frac{L}{f}, \frac{M}{f}, \frac{N}{f} \right) = \text{(-0.545, -0.0957, -0.833)},$$

$$\mathbf{s_0} = \left(\frac{P - hL}{f}, \frac{Q - hM}{f}, \frac{R - hN}{f} \right) = \text{(10.951, -15.039, -5.443) cm}.$$

Example 3

If there is a different external wrench applied to the structure in Example 2, it is expected that the struts and side ties will experience different forces, and hence the variable lengths will be different.

Assuming that an external wrench with a force magnitude *f* of 10 N, bounded to a Line $\hat{\mathbf{s}}$ of {-0.477, -0.191, -0.858; 15.5, -20, -10}. The last three terms have unit of cm.

The given wrench can be rewritten as

$$\hat{\mathbf{w}} = f\hat{\mathbf{s}} = \{-4.77, -1.91, -8.58; 155, -200, -100\}$$

where the first three components have units of N and the last three components have units of N.cm.

The 6×6 matrix remains the same as in the previous example and is written as

$$\boldsymbol{j} = \begin{bmatrix} 0.237 & 0.110 & -0.111 & 0.610 & -0.336 & -0.0212 \\ 0.0853 & 0.327 & 0.147 & 0.326 & 0.548 & -0.351 \\ 0.968 & 0.939 & 0.983 & 0.723 & 0.766 & 0.936 \\ 0 & 0 & 12.769 & 0 & 0 & 12.162 \\ 0 & -10.872 & -7.372 & 0 & -11.494 & -7.022 \\ 0 & 4.904 & 2.538 & 0 & 8.213 & -2.356 \end{bmatrix}.$$

Equation (3.2.12) then gives

$$\boldsymbol{q} = \boldsymbol{j}^{-1}\,\hat{\mathbf{w}} = \begin{bmatrix} -74.069 \\ -118.352 \\ -97.392 \\ 95.658 \\ 121.560 \\ 115.0 \end{bmatrix} \text{N}.$$

Hence,

f_A = 74.069 N, f_B = 118.352 N, and f_C = 97.392 N

The new elongations can be computed from Equations (2.4.6), (2.4.7), and (2.4.8) as

$$\boldsymbol{d}_A = 3.703 \text{ cm},$$

$$\boldsymbol{d}_B = 5.918 \text{ cm},$$

$$\boldsymbol{d}_C = 4.870 \text{ cm}.$$

The total lengths of Ties A, B, and C are still the same

$l_{Atotal} = \| {}^{\mathbf{B}}\mathbf{P_4} - {}^{\mathbf{B}}\mathbf{P_1} \| = 29.006$ cm,

$l_{Btotal} = \| {}^{B}\mathbf{P}_5 - {}^{B}\mathbf{P}_2 \| = 24.007$ cm,

$l_{Ctotal} = \| {}^{B}\mathbf{P}_6 - {}^{B}\mathbf{P}_3 \| = 23.383$ cm.

Therefore using Equations (2.4.9), (2.4.10), and (2.4.11), the new lengths of the variable ties are determined to be

$$\underline{l_A = 15.088 \text{ cm},}$$

$$\underline{l_B = 8.304 \text{ cm},}$$

$$\underline{l_C = 10.513 \text{ cm.}}$$

The new lengths of the variable ties are now shorter than when they were in Example 2 due to this new wrench. The potential energy of the system then has to change; in this case, it increases to 724.5 N.cm.

CHAPTER 4
MODIFIED TENSEGRITY PLATFORMS

A regular three-three tensegrity structure consists of three struts and three compliant ties. In this chapter, the same three-three geometry will be used, i.e. there are three connection points on the top platform and three on the base platform. Now, however, two new cases will be considered. One will consist of four struts and two compliant ties, and the other will consist of five struts and one compliant tie. Because of the nature of the tensegrity structure, it is not possible to have one strut and five compliant ties or two struts and four compliant ties. The struts are always in the position that they must take compressive forces; hence, replacing any strut with a tie will cause the structure to collapse. As was the case with the three-strut three-tie platform discussed in the previous chapters, an external wrench will have to be applied to the top platform to maintain equilibrium except for those positions and orientations where the Plücker line coordinates of the six legs become linearly dependent.

4.1 Four Struts-Two Compliant Ties

For this analysis, it is assumed that the tie connecting Vertices *3* and *6* is substituted by a strut as seen in Figure 4-1.

The force balance analysis in Section 2.3 is still valid to find the forces in each member. Once **v'** is found from Equation (2.3.9), the ratios in Equations (2.3.7) can be obtained. The second term of **v'** can be negative since the substituting strut can take either tensile or compressive force. Similar to Equation (2.4.2) but with $\boldsymbol{d}_C = 0$, the potential energy of the structure is

$$U = \frac{1}{2}\left(k_A\delta_A^{\,2} + k_B\delta_B^{\,2}\right) \qquad (4.1.1)$$

where δ_A, δ_B are the elongations of the springs and k_A, k_B are the spring constants of the compliant Ties A and B, respectively.

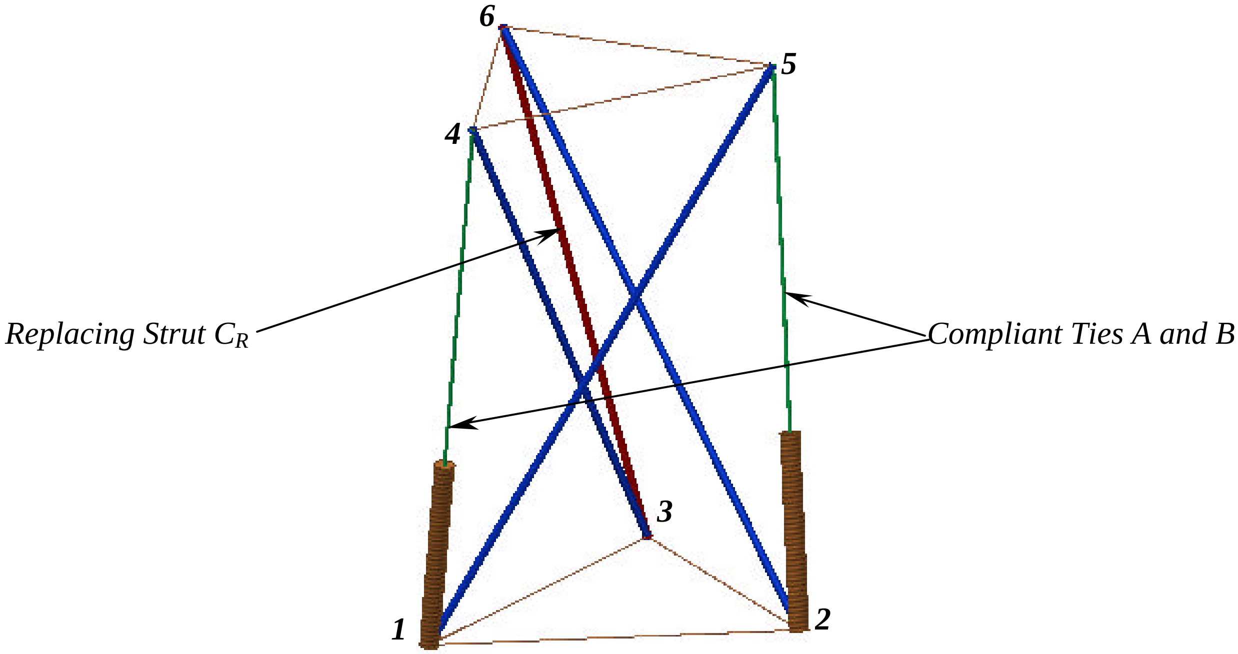

Figure 4-1. Platforms with Replacing Strut C_R

The tensile force in the Ties A and B are

$$f_A = k_A \boldsymbol{d}_A,$$

$$f_B = k_B \boldsymbol{d}_B. \qquad (4.1.2)$$

Combining Equations (4.1.2) and (4.1.3) yie lds

$$U = \frac{1}{2}\left(\frac{f_A^2}{k_A} + \frac{f_B^2}{k_B}\right). \qquad (4.1.3)$$

Substituting Equations (2.3.7) into (4.1.4) gives

$$U = \frac{1}{2}\left(\frac{f_A^2}{k_A} + \frac{f_A^2 f_B^{'2}}{k_B}\right).$$

Solving this equation for f_A yields

$$f_A = \sqrt{\frac{2Uk_A k_B}{k_B + k_A f_B'^2}}\ . \qquad (4.1.4)$$

The force in the compliant tie B can now be determined from (2.3.7) as

$$f_B = f_A f_B'. \qquad (4.1.5)$$

From Equation (4.1.3)

$$\boldsymbol{d}_A = \frac{f_A}{k_A}\ , \qquad (4.1.6)$$

$$\boldsymbol{d}_B = \frac{f_B}{k_B}\ . \qquad (4.1.7)$$

Finally the lengths of the three variable length tie segments l_A, l_B, and l_C can be determined from:

$$l_A = l_{Atotal} - l_{0A} - \boldsymbol{d}_A, \qquad (4.1.8)$$

$$l_B = l_{Btotal} - l_{0B} - \boldsymbol{d}_B. \qquad (4.1.9)$$

where $l_{Atotal} = ||{}^{\mathbf{B}}\mathbf{P}_4 - {}^{\mathbf{B}}\mathbf{P}_1||$, the distance from Points *4* and *1*

$l_{Btotal} = ||{}^{\mathbf{B}}\mathbf{P}_5 - {}^{\mathbf{B}}\mathbf{P}_2||$, the distance from Points *5* and *2*.

Example

This example uses the same data in Example1 of Section 2.5. Tie C has been replaced by a Strut C_R which can take either tension and compression force. The problem statement is written as

Given:

$${}^{B}_{T}T = \begin{bmatrix} 0.866 & -0.5 & 0 & 21.906 \\ 0.5 & 0.866 & 0 & -14.019 \\ 0 & 0 & 1 & 135.171 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

d_T = 80 mm (top tie length),

d_B = 90 mm (base tie length),

$k_A = k_B = k_C = k$ = 1.5 N/mm (spring constants),

$l_{0A} = l_{0B} = l_{0C}$ = 25 mm (spring free lengths),

U = 1200 N mm (desired potential energy).

Find:

Strut lengths L_A, L_B, and L_C,

Replacing strut length L_{CR},

Variable length ties l_A, l_B, and l_C.

Similar to Example 1, the coordinates of the vertices in the B system are

$^{\mathbf{B}}\mathbf{P}_1$ = [0,0,0] mm,

$^{\mathbf{B}}\mathbf{P}_2$ = [90,0,0] mm,

$^{\mathbf{B}}\mathbf{P}_3$ = [45,77.943,0] mm,

$^{\mathbf{B}}\mathbf{P}_4$ = [21.906,-14.019,135.171] mm,

$^{\mathbf{B}}\mathbf{P}_5$ = [91.188,25.981,135.171] mm,

$^{\mathbf{B}}\mathbf{P}_6$ = [21.906,65.981,135.171] mm.

From this data, the length of the three struts was determined to be

$$\underline{L_A = L_B = L_C = 165.11 \text{ mm.}}$$

And the length of the replacing strut is

$$\underline{L_{CR} = \| {}^{\mathbf{B}}\mathbf{P}_6 - {}^{\mathbf{B}}\mathbf{P}_3 \| = 137.65 \text{ mm.}}$$

Equation (2.3.9) was solved and obtained

$$\mathbf{v'} = [1.000,\ 1.000,\ 1.200,\ 1.200,\ 1.200,\ 0.336,\ 0.336,\ 0.336,\ 0.378,\ 0.378,\ 0.378]^T \text{ N}$$

Equation (4.1.5) gives

$$f_A = \sqrt{\frac{2Uk}{2}} = \sqrt{\frac{2*1200*1.5}{2}} = 42.426 \text{ N.}$$

Applying to Equations (4.1.5) yields

$$f_A = f_B = 42.426 \text{ N.}$$

Equations (4.1.6) and (4.1.7) give

$$\delta_A = \delta_B = 28.284 \text{ mm.}$$

The total length of Ties A, B, and C are calculated to be

$$l_{Atotal} = l_{Btotal} = 137.65 \text{ mm.}$$

Therefore using Equations (4.1.8) and (4.1.9), the lengths of the variable ties are determined to be

$$\underline{l_A = l_B = 84.36 \text{ mm.}}$$

4.2 Five Struts-One Compliant Tie

Figure 4-2 depicts a tensegrity structure which has compliant Ties *A* and *C* replaced by two Struts A_R and C_R. Since there is only one compliant tie which can be adjusted, the rigidity of the structure depends only on it. The potential energy of the structure determines directly the length of the Tie l_B, which is

$$l_B = l_{Btotal} - l_{0B} - \delta_B \qquad (4.2.1)$$

where

$$\delta_B = \sqrt{\frac{2U}{k_B}}\,. \qquad (4.2.2)$$

As long as the tie is in tension, the shape of the structure remains unchanged since the five strut lengths stay the same.

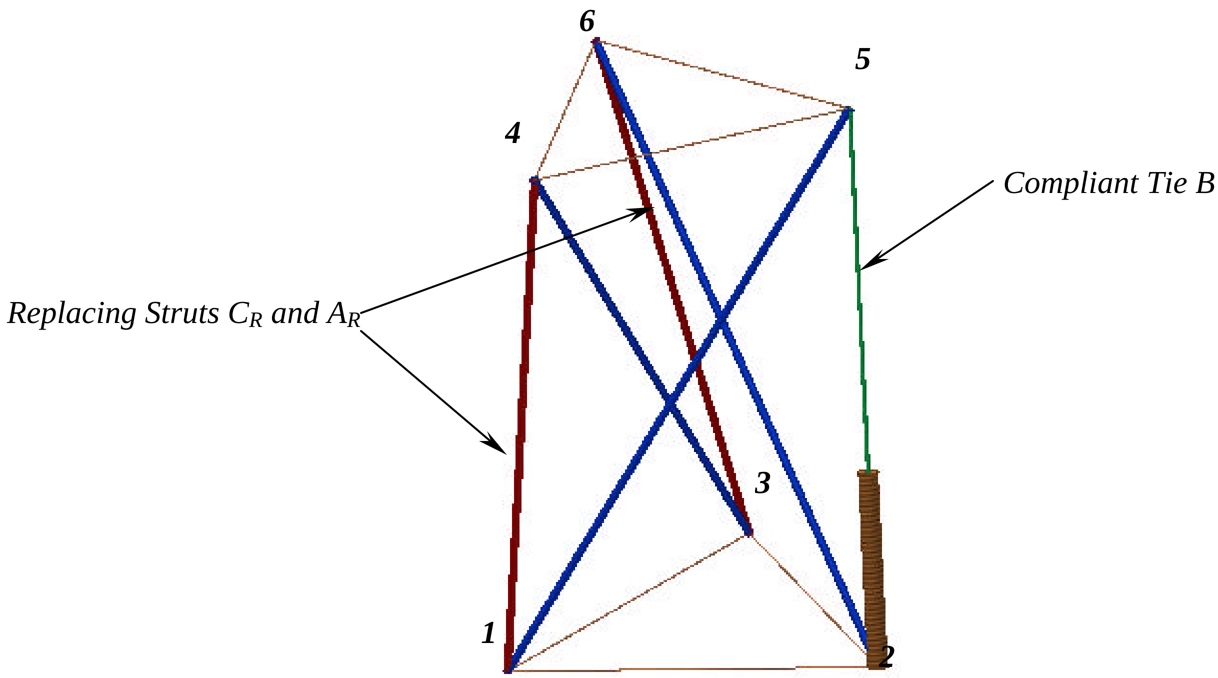

Figure 4-2. Platforms with Replacing Strut C_R and A_R

CHAPTER 5
CONCLUSIONS

The principal of tensegrity is used to develop a new kind of structure which has the innovative feature of the mechanism design that the top platform can be kept at a certain position and orientation and the compliance characteristics of the mechanism can be simultaneously controlled. The position and orientation of the structure (or the transformation matrix) cannot be arbitrarily chosen. It has to be in such a way that the lines in the struts and the side ties are singular.

When the structure is not at singularity, the structure is not in equilibrium unless there is a correct wrench applied to it. The wrench is the opposite of the resultant wrench obtained from the forces in the struts and side ties. If a different wrench is applied to the system, different variable tie lengths and potential energy are resulted.

The analysis in this paper is only applied to the tensegrity structure that is singular. Hence, the given coordinates or the transformation matrix is not arbitrary, but they have to be in a 'singular zone' so that the six Plücker coordinates are linearly dependent.

Modified structure is examined in Chapter 4. One or two side ties of the structure are replaced by struts. The struts are assumed to connect to each other by ball-and-socket joints so none of the struts experience any force other than compression or tension. The four struts-two compliant ties platforms have some degree of mobility, while the five struts-one compliant tie platforms stay the same shape when the variable tie length is changed. It just increases or decreases the potential energy of the system.

Future Work. This research opens up to new challenges. One of them is to study the four-four tensegrity structures on how to retain their desired position and orientation by controlling the non-compliant segments of the ties. Next is to replace the ball-and-socket joints by simple spherical joints at the struts connections. This can be accomplished by offsetting the struts on the platforms, which requires a different analysis. In addition, it would be interesting and helpful if the 'singular zone' is clearly defined.

APPENDIX
MATLAB CODES

Below is the Matlab code for the main program for the Three-Three Reverse Analysis. The program prints out the results in a file under the directory that the Matlab's path is. The outputs are the lengths of the struts, the lengths of the variable ties, and the resultant wrench. The Output Filename and the Message String require user to enter in a string format, such as: 'Test.txt' and 'Test Results'. This program will call two functions: *force* and *length_*. If there is a message displayed at the end of the program, it implies that there is at least a negative value for the forces. This indicates at least one of the tensile forces is not in tension and/or one of the compressive forces is not in compression.

```
% This is the REVERSE DISPLACEMENT ANALYSIS program.

% It will ask for all constants, and input variables
% and output equivalent wrench, lengths LA, LB, LC, lA, lB, and lC.
% Please enter coordinates in the form [x,y,z]

filename = input('Enter Output Filename: ') ;
mystring = input('Enter Message String: ') ;
fp = fopen(filename, 'w+') ;
fprintf(fp, '%s\n', mystring) ;

U=input('Please Use Consistent Units \n \nEnter Value of Potential
Energy U: ');
kA=input('Enter Value of Spring Constant kA: ');
kB=input('Enter Value of Spring Constant kB: ');
kC=input('Enter Value of Spring Constant kC: ');
```

```
loA=input('Enter Value of Spring Free Length loA: ');
loB=input('Enter Value of Spring Free Length loB: ');
loC=input('Enter Value of Spring Free Length loC: ');
A=input('\nTransfomation Matrix Availble, Enter 1 \nCoordinates
Available, Enter 2: ');
if A==1
   T=input('Enter the Transformation Matrix T: ');
   d1=input('Enter Value of Top Tie Length dT: ');
   d2=input('Enter Value of Base Tie Length dB: ');
   a=[0,0,0];
   b=[d2,0,0];
   c=[d2/2,d2*sin(pi/3),0];
   d=(T*[0,0,0,1]')';
   d=d(1:3);
   e=(T*[d1,0,0,1]')';
   e=e(1:3);
   f=(T*[d1/2,d1*sin(pi/3),0,1]')';
   f=f(1:3);
else if A==2
      a=input('Enter Value of Coordinate 1 [x,y,z]: ');
      b=input('Enter Value of Coordinate 2 [x,y,z]: ');
      c=input('Enter Value of Coordinate 3 [x,y,z]: ');
      d=input('Enter Value of Coordinate 4 [x,y,z]: ');
      e=input('Enter Value of Coordinate 5 [x,y,z]: ');
      f=input('Enter Value of Coordinate 6 [x,y,z]: ');
   end
end

LA=sqrt(sum((e-a).^2));
LB=sqrt(sum((f-b).^2));
LC=sqrt(sum((d-c).^2));

F=force(a,b,c,d,e,f);
[w,l]=length_(U,kA,kB,kC,F,a,b,c,d,e,f,loA,loB,loC);

for i=1:11
   if F(i)<=0
      if i==1
```

```
         disp('The Structure May Not Be in Tensegrity! fB is Not in
Tension!!!')
      else if i==2
         disp('The Structure May Not Be in Tensegrity! fC is Not in
Tension!!!')
      else if i==3
         disp('The Structure May Not Be in Tensegrity! FA is Not in
Compression!!!')
      else if i==4
         disp('The Structure May Not Be in Tensegrity! FB is Not in
Compression!!!')
      else if i==5
         disp('The Structure May Not Be in Tensegrity! FC is Not in
Compression!!!')
      else if i==6
         disp('The Structure May Not Be in Tensegrity! T12 is Not in
Tension!!!')
      else if i==7
         disp('The Structure May Not Be in Tensegrity! T23 is Not in
Tension!!!')
      else if i==8
         disp('The Structure May Not Be in Tensegrity! T13 is Not in
Tension!!!')
      else if i==9
         disp('The Structure May Not Be in Tensegrity! T45 is Not in
Tension!!!')
      else if i==10
         disp('The Structure May Not Be in Tensegrity! T56 is Not in
Tension!!!')
      else if i==11
         disp('The Structure May Not Be in Tensegrity! T46 is Not in
Tension!!!')
      end
      end
      end
      end
      end
      end
```

```
      end
      end
      end
      end
      end
   end
end

fprintf (fp, '\nFinal Results:\n') ;
fprintf (fp,  '\tLength of Strut A =  %8.5f\n', LA) ;
fprintf (fp,  '\tLength of Strut B =  %8.5f\n', LB) ;
fprintf (fp,  '\tLength of Strut C =  %8.5f\n', LC) ;
fprintf (fp,  '\tLength of Tie A =  %8.5f\n', l(1)) ;
fprintf (fp,  '\tLength of Tie B =  %8.5f\n', l(2)) ;
fprintf (fp,  '\tLength of Tie C =  %8.5f\n', l(3)) ;
fprintf (fp,  '\tThe Equivalent Wrench =  %8.5f\n', w(1)) ;
fprintf (fp,  '\t                         %8.5f\n', w(2)) ;
fprintf (fp,  '\t                         %8.5f\n', w(3)) ;
fprintf (fp,  '\t                         %8.5f\n', w(4)) ;
fprintf (fp,  '\t                         %8.5f\n', w(5)) ;
fprintf (fp,  '\t                         %8.5f\n', w(6)) ;

fclose(fp);
```

Following is the two functions needed for the main program.

```
function F=force(a,b,c,d,e,f)
% F=force will compute the forces in the ties and rods of a
% tensegrity structure.

% a,b,c,d,e,f are corresponding to the corners 1,2,3,4,5,6.
```

```
S14=(d-a)/sqrt(sum((d-a).^2));
S25=(e-b)/sqrt(sum((e-b).^2));
S36=(f-c)/sqrt(sum((f-c).^2));
S15=(e-a)/sqrt(sum((e-a).^2));
S26=(f-b)/sqrt(sum((f-b).^2));
S12=(b-a)/sqrt(sum((b-a).^2));
S23=(c-b)/sqrt(sum((c-b).^2));
S34=(d-c)/sqrt(sum((d-c).^2));
S45=(e-d)/sqrt(sum((e-d).^2));
S56=(f-e)/sqrt(sum((f-e).^2));
S46=(f-d)/sqrt(sum((f-d).^2));
S13=(c-a)/sqrt(sum((c-a).^2));

% corner 1
% T1+T3+fA-FA=0
% or s12+s13+s14-s15=0

% corner 2
% T1+T2+fB-FB=0
% or -s12+s23+s25-s26=0

% corner 4
% T4+T6+fA-FC=0
% or s45+s46-s14+s34=0

% corner 6
% T5+T6+fC-FB=0
% or -s56-s46-s36+s26=0

% Jacobian Matrix
J=[0,0,-S15(2),0,0,S12(2),0,S13(2),0,0,0;
   0,0,-S15(3),0,0,S12(3),0,S13(3),0,0,0;
   S25(1),0,0,-S26(1),0,-S12(1),S23(1),0,0,0,0;
   S25(2),0,0,-S26(2),0,-S12(2),S23(2),0,0,0,0;
   S25(3),0,0,-S26(3),0,-S12(3),S23(3),0,0,0,0;
   0,0,0,0,S34(1),0,0,0,S45(1),0,S46(1);
   0,0,0,0,S34(2),0,0,0,S45(2),0,S46(2);
```

```
   0,0,0,0,S34(3),0,0,0,S45(3),0,S46(3);
   0,-S36(1),0,S26(1),0,0,0,0,0,-S56(1),-S46(1);
   0,-S36(2),0,S26(2),0,0,0,0,0,-S56(2),-S46(2);
   0,-S36(3),0,S26(3),0,0,0,0,0,-S56(3),-S46(3)];

% The inverse of the Jacobian Matrix
J_inv=inv(J);

% The b vector
RS=[-S14(2);-S14(3);0;0;0;S14(1);S14(2);S14(3);0;0;0];

% The v vector
F=J_inv*RS;

% Check for forces and moments = zero
forces_sum=-F(2)*S36-F(10)*S56-F(11)*S46+F(4)*S26;
moment=cross((f-b),-F(10)*S56)+cross((f-b),-F(11)*S46)+cross((f-b),-
F(2)*S36);
% End of function force.m %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
function [w,l]=length_(U,kA,kB,kC,F,a,b,c,d,e,f,loA,loB,loC)

% L will compute the lengths lA, lB, lC of the system.

fA=sqrt(2*U*kA*kB*kC/(kB*kC+kA*kC*F(1)^2+kA*kB*F(2)^2));
fB=F(1)*fA;
fC=F(2)*fA;
FA=F(3)*fA;
FB=F(4)*fA;
FC=F(5)*fA;

% Calculate the equivalent wrench
v=[-fA;-fB;-fC;FA;FB;FC];

det1=det([1 a(1);1 d(1)]);
det2=det([1 a(2);1 d(2)]);
det3=det([1 a(3);1 d(3)]);
```

```
det4=det([a(2) a(3);d(2) d(3)]);
det5=det([a(3) a(1);d(3) d(1)]);
det6=det([a(1) a(2);d(1) d(2)]);
col1=[det1;det2;det3;det4;det5;det6]/sqrt(det1^2+det2^2+det3^2);

det7=det([1 b(1);1 e(1)]);
det8=det([1 b(2);1 e(2)]);
det9=det([1 b(3);1 d(3)]);
det10=det([b(2) b(3);e(2) e(3)]);
det11=det([b(3) b(1);e(3) e(1)]);
det12=det([b(1) b(2);e(1) e(2)]);
col2=[det7;det8;det9;det10;det11;det12]/sqrt(det7^2+det8^2+det9^2);

det13=det([1 c(1);1 f(1)]);
det14=det([1 c(2);1 f(2)]);
det15=det([1 c(3);1 f(3)]);
det16=det([c(2) c(3);f(2) f(3)]);
det17=det([c(3) c(1);f(3) f(1)]);
det18=det([c(1) c(2);f(1) f(2)]);
col3=[det13;det14;det15;det16;det17;det18]/sqrt(det13^2+det14^2+det1
5^2);

det19=det([1 a(1);1 e(1)]);
det20=det([1 a(2);1 e(2)]);
det21=det([1 a(3);1 e(3)]);
det22=det([a(2) a(3);e(2) e(3)]);
det23=det([a(3) a(1);e(3) e(1)]);
det24=det([a(1) a(2);e(1) e(2)]);
col4=[det19;det20;det21;det22;det23;det24]/sqrt(det19^2+det20^2+det2
1^2);

det25=det([1 b(1);1 f(1)]);
det26=det([1 b(2);1 f(2)]);
det27=det([1 b(3);1 f(3)]);
det28=det([b(2) b(3);f(2) f(3)]);
det29=det([b(3) b(1);f(3) f(1)]);
det30=det([b(1) b(2);f(1) f(2)]);
```

```
col5=[det25;det26;det27;det28;det29;det30]/sqrt(det25^2+det26^2+det2
7^2);

det31=det([1 c(1);1 d(1)]);
det32=det([1 c(2);1 d(2)]);
det33=det([1 c(3);1 d(3)]);
det34=det([c(2) c(3);d(2) d(3)]);
det35=det([c(3) c(1);d(3) d(1)]);
det36=det([c(1) c(2);d(1) d(2)]);
col6=[det31;det32;det33;det34;det35;det36]/sqrt(det31^2+det32^2+det3
3^2);

% The 6x6 line matrix
j=[col1 col2 col3 col4 col5 col6];

w=j*v;  % The equivalent wrench

% Elongations of the three springs:
delA=fA/kA;
delB=fB/kB;
delC=fC/kC;
% Total length of ties A, B, and C:
lA_total=sqrt(sum((d-a).^2));
lB_total=sqrt(sum((e-b).^2));
lC_total=sqrt(sum((f-c).^2));
% Lengths of the three variable length ties:
lA=lA_total-loA-delA;
lB=lB_total-loB-delB;
lC=lC_total-loC-delC;

l=[lA;lB;lC];

% End of function force.m %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

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BIOGRAPHICAL SKETCH

Mr. Tung M. Tran was born in Bac Lieu, Southern Vietnam on January 19, 1975. Two years after he graduated from high school, his family moved to Jacksonville, Florida in August 1994. He then pursued his undergraduate studies at the University of Florida, where he received his Bachelor of Science degree in Mechanical Engineering. He worked for Vistakon, a division of Johnson and Johnson, as a designing engineer intern. In August 2000, he came back to the University of Florida to pursue a Master of Science in Mechanical Engineering with a minor in Mathematics.