# POSITION ANALYSIS OF PLANAR TENSEGRITY STRUCTURES

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
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To the scientific community.

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Tensegrity is an abbreviation of tension and integrity. Tensegrity structures are spatial structures formed by a combination of rigid elements in compression (struts) and connecting elements that are in tension (ties). In three-dimensional tensegrity structures no pair of struts touches, and the end of each strut is connected to non-coplanar ties, which are in tension. In two-dimensional tensegrity structures, struts still do not touch. A tensegrity structure stands by itself in its equilibrium position and maintains its form solely because of the arrangement of its struts and ties. The potential energy of the system stored in the springs is at a minimum in the equilibrium position when no external force or torque is applied. A closed-form solution of a two-spring, three-spring, and four-spring planar tensegrity mechanism was developed to determine all possible equilibrium configurations when no external force or moment is applied. Here "closed form" means that all solution equilibrium poses will be determined, although for each case a high-degree polynomial will have to be solved numerically.

# CHAPTER 1
# LITERATURE REVIEW

We examined the literature related to tensegrity systems, self-deployable tensegrity systems, and parallel mechanisms with compliant elements.

Knight et al. [1] reported on the line geometries of a family of tensegrity structures called skew prisms (anti-prisms, tensegrity prisms) with pairs of triangles, squares, pentagonals, hexagonals, and so on, located at tops and bases. They used the quality index with S-P-S connectors (S is a ball-and-socket joint, and P is a sliding joint). The quality index measures the geometric stability of in-parallel devices (which in three dimensions consist of a pair of rigid platforms connected by legs that are kinematical S-P-S connectors). Geometric stability depends on the geometry of the legs.

Figures 1-1 A and 1-1 C show a sequence of parallel prisms with parallel ties of fixed length.

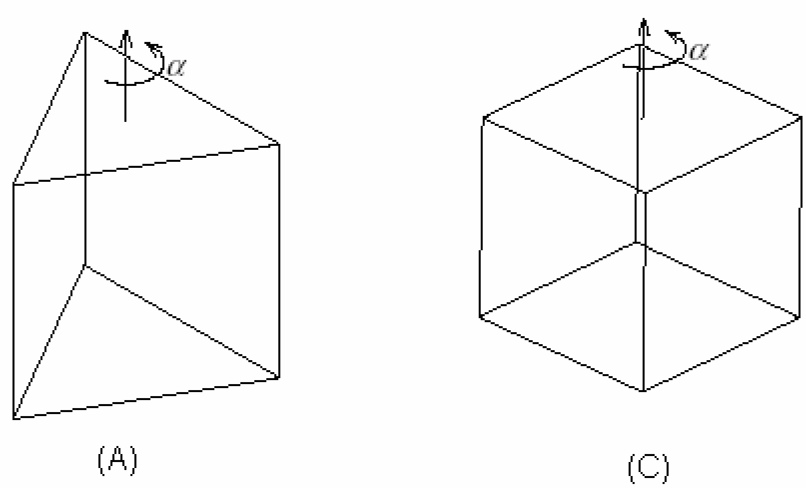


Figure 1-1. Parallel prisms.

A relative anti-clockwise rotation can be achieved by rotating the top of each prism through its vertical axes as shown in Figures 1-2 B and 1-2 D to yield a corresponding right-handed tensegrity prism that is completed by inserting struts on the diagonals of the skew quadrilaterals. Left-handed tensegrity prisms are mirror images of right-handed prisms and are obtained by rotating the tops in a clockwise direction and interchanging the ties and struts. The angle $\alpha$ is unique for each tensegrity prism ($\alpha = 90 - \frac{180}{n}$) where n is the total number of sides in the upper or lower polygons. The value of $\alpha$ (together with the size of the tops and bases and their distance apart) enables one to compute the length of the struts and ties that defines a unique configuration for the tensegrity prism.

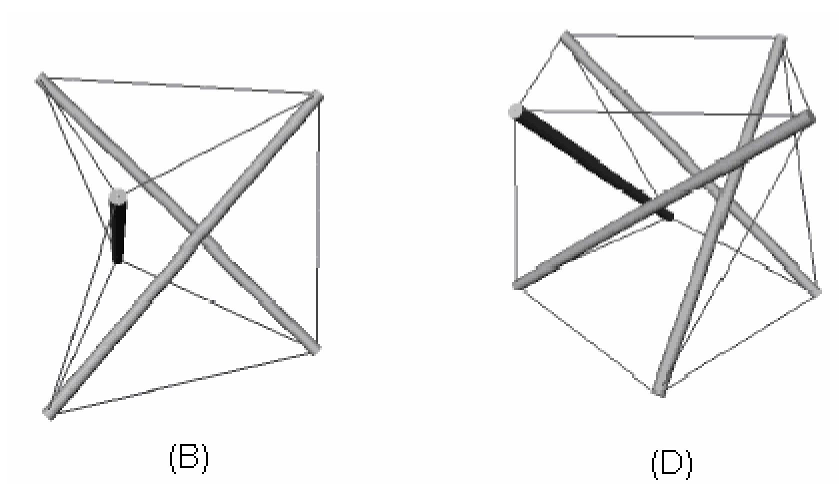


Figure 1-2. Tensegrity prisms.

The quality index for each skew prism is zero, which means that it has instantaneous mobility. Further examination revealed that all the sets of connector-lines for each prism belong to a linear complex of lines within a 5-system of screws. Each of the sets of lines is reciprocal to a single screw. Adding additional ties along the diagonals of the skew prism faces can easily form reinforced skew prisms.

Unloaded tensegrity prisms that are stable in the sense that they are configurations of minimum potential energy, but they have instantaneous mobility simply because the connector-lines belong to a linear complex. Reinforced tensegrity prisms where additional ties are inserted are completely stable and shown in Figure 1-3.

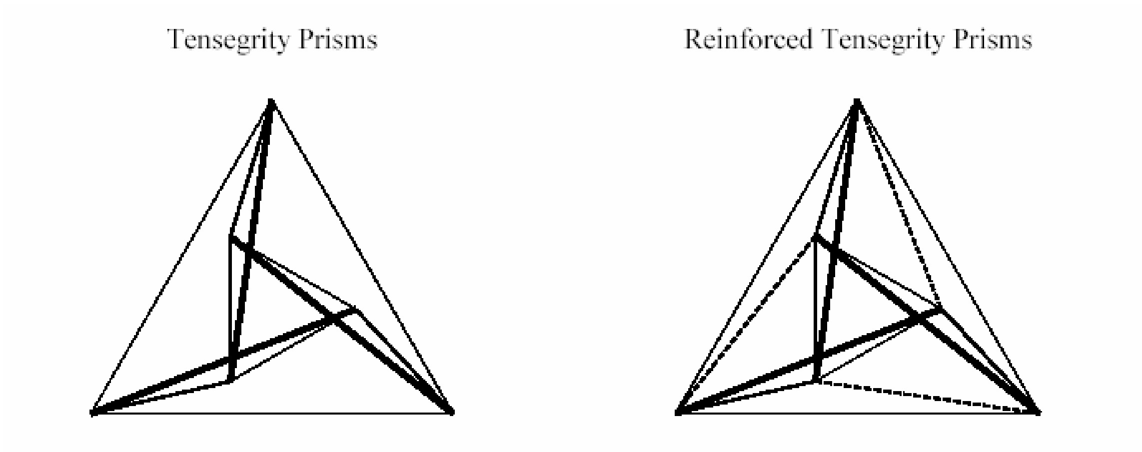


Figure 1-3. Plan view of tensegrity and corresponding reinforced prisms.

Pellegrino [2] shows how deployable structures are able to change their shape from a packed configuration to operational form. Usually, energy is stored in the structure when packed and released during unpacking when the operational configuration is required. Some simple examples are a spring loaded umbrella, a retractable roof of a car, and a space radio telescopic antenna. Deployable structures are used for ease of transportation and storage. The essential requirement is that the transformation process should be autonomous and reliable, and without causing any damage to nearby structures.

Examples of deployable structures that undergo large geometric transformations are coiled rods, flexible shells, lattice column, and membranes. A deployable structure using lattice columns with several longitudinal elements, called longerons, braced at regular intervals by short members perpendicular to the longerons and by diagonal members, has low dead weight but can support high load. This structure has a very small wind resistance in atmosphere and minimal particle meteoroid damage in space.

Tarnai [3] presents a brief analogy between compatibility and equilibrium of a finite linkage mechanism. Some linkages are known whose degree of freedom in some specific position is greater than the expected degree of freedom as calculated from the equilibrium equation. An example is a four bar linkage with equal length opposite bars. The linkage has two shapes: one associated with a parallelogram and the other with an anti parallelogram. Plotting these angles as a function of each other provides two curves with a common point. This point is when all four bars lie on a straight line. At this instant the mechanism may move from the parallelogram configuration to the anti-parallelogram configuration.

Duffy et al. [4-6] analyze a three dimensional tensegrity structure that is made up of elastic and rigid elements (Figure 1-4). In this assembly, the elastic elements are under tension and the rigid elements under compression. The papers present the static position analysis problem and determine the position assumed by the structure when external loads are applied, and when the system is presented by changing the free lengths of the compliant elements.

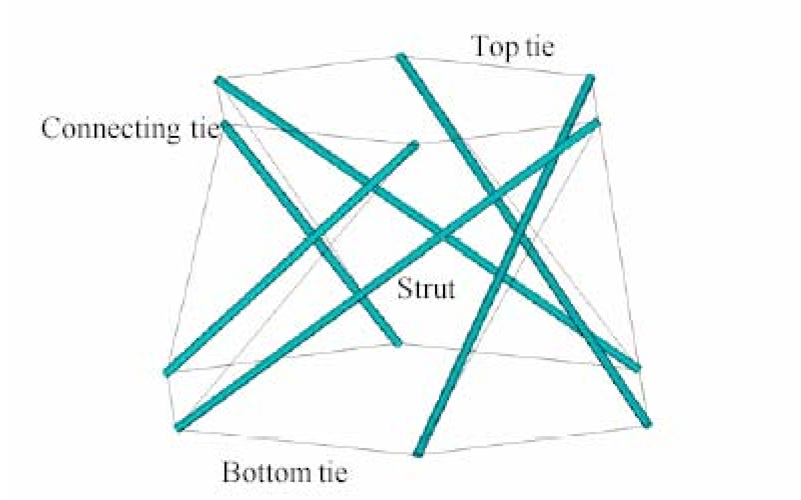


Figure 1-4. Component of a tensegrity structures.

The mathematical formulation to find the equilibrium positions of the structure is based on the virtual work principle. The obtained equations are solved using numerical methods. Some assumptions are made to simplify the derivation of the mathematical model. These assumptions are the absence of internal dissipative forces and the manner by which the external forces are applied. The numerical method determines the coordinates of the strut end points in the equilibrium position. A force balance is then conducted to validate the results.

Stern [7] presents the position analysis of a symmetric n-strut tensegrity system. A three dimensional n-struts system with two platforms, one at the top and the other at the bottom, is considered in the analysis. A static analysis of the internal forces is conducted on the top and bottom platforms. The relationship between the geometry of the structure and the internal forces are investigated. Tensegrity structures with different number of struts are analyzed. The results of each analysis are compared with the results of other analyses to obtain common patterns in all systems by relating the results to the number of struts in the systems. The patterns are formulated into equations based on the number of struts in the system.

The formulation is done for 3, 4, 5, and 6 struts tensegrity systems with solid top and bottom platforms. For example, a plan view of 4 strut tensegrity system is shown in Figure 1-5.

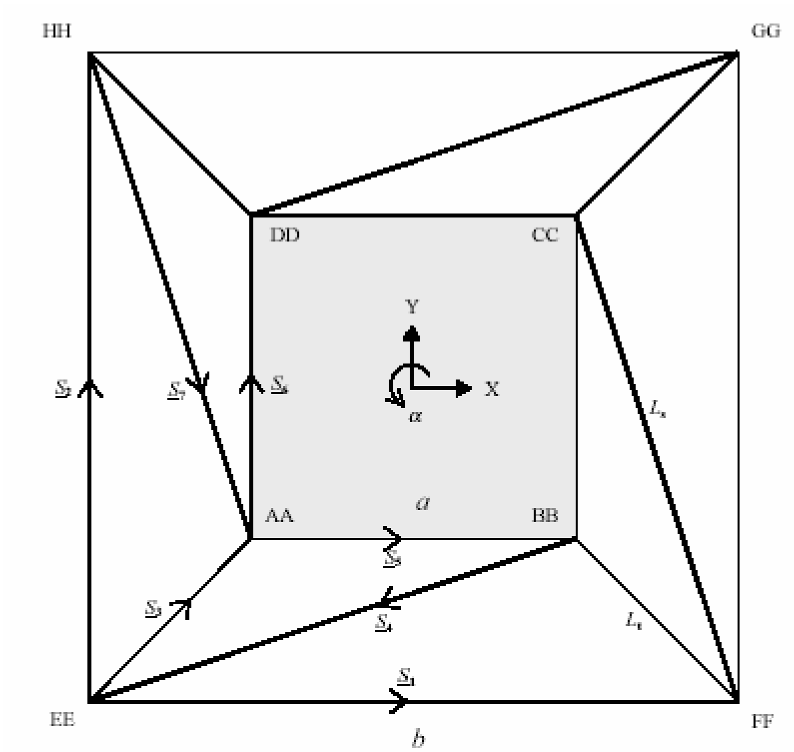


Figure 1-5. 4-strut tensegrity, 3D, top view.

Marshall [8] introduces a parallel platform device that incorporates tensegrity principles. The device, shown in Figure 1-5, replaces the struts, of a tensegrity structure with prismatic actuators, and each elastic member with a cable-spring combination in series. The length of the three prismatic actuators and the length of the cables that are in series with the spring are adjustable and thus the device has six degrees of freedoms. This study shows that in order to achieve an arbitrary position and orientation of the top platform, an external wrench must be applied to maintaining equilibrium. This study also shows that the device's compliance characteristics can be varied while maintaining its position and orientation.

A reverse analysis of the device is presented in which the desired pose and total potential energy are given and the lengths of the prismatic actuators and the cables are determined. The effect of a seventh leg, another prismatic actuator, is also analyzed and found to satisfactorily implement the needed external wrench (Figure 1-6).

Figure 1-6. Tensegrity platform.

Gantes and Konitopoulou [9] discusses bi-stable deployable structures. A bi-stable structure is self standing and stress free when fully closed or fully deployed. It exhibits incompatibilities between the member lengths at intermediate geometric configurations during the deployment process, which leads to the occurrence of second-order strains and stresses resulting in a snap-through phenomenon that "locks" the structures in their deployed configuration. Until now the geometric shapes that were possible in the deployed configuration were only flat or curved with constant curvature. This limitation is addressed in the paper by proposing a geometric design methodology for deployable arches of arbitrary curvature, accounting also for the discrete joint size, and applying it successfully for the geometric design of a semi-elliptical arch. The arch is then modeled with finite elements, and a geometrically nonlinear analysis is performed in order to verify the deploy ability feature. Further verification is provided by the construction of a small-scale physical model. A preliminary structural design indicates the overall feasibility of the arch for short to medium spans and light loads.

Deployable structures are prefabricated space frames consisting of straight bars linked together in the factory as a compact bundle, which can then be unfolded into large-span, load bearing structural shapes by simple articulation. Because of this feature they offer significant advantages in comparison to conventional, non-deployable structures for a wide spectrum of applications ranging from temporary structures to the aerospace industry, being mainly characterized by their feature of transforming and adapting to changing needs.

From a structural point of view, deployable structures have to be designed for two completely different loading conditions, under service loads in the deployed configuration, and during deployment. The structural design process is very complicated and requires successive iterations to achieve some balance between desired flexibility during deployment and desired stiffness in the deployed configuration  From a geometric point of view, the whole idea of this type of deployable structure is based on the so-called scissor-like elements, pairs of bars connected to each other at an intermediate point through a pivotal connection which allows them to rotate freely about an axis perpendicular to their common plane but restrains all other degrees of freedom, while, at the same time, their end points are hinged to the end points of other scissor-like elements.

Yin et al. [10] present a special planar three-spring mechanism that is designed to control contact forces. An energy function is defined to describe the behavior of this kind of mechanism and it can be used to perform the catastrophe analysis of this mechanism.

The analysis result can be used as a design and control tool. By comparing the three-spring system and a two-spring system, it was found that the three-spring mechanism has better stability than the two-spring system. A three-spring mechanism which can be used to control a general contact force in a plane is also analyzed.

Intuitively, Yin et al. [10] showed a catastrophe occurs whenever a smooth change of parameters gives rise to a discontinuous change in behavior. A well known example that can easily be made to demonstrate a catastrophe is Zeeman's catastrophe machine (see Figure1-7). Zeeman's machine can be constructed by attaching two linear springs to a single point C on a disk that can rotate about 0. One of the springs is attached to a fixed pivot at point A and the other spring is attached to point B which can be moved in the x, y plane. The position of point B is called the controlling parameter as it dictates the position of the disk which is defined by the angle between OA and OM.

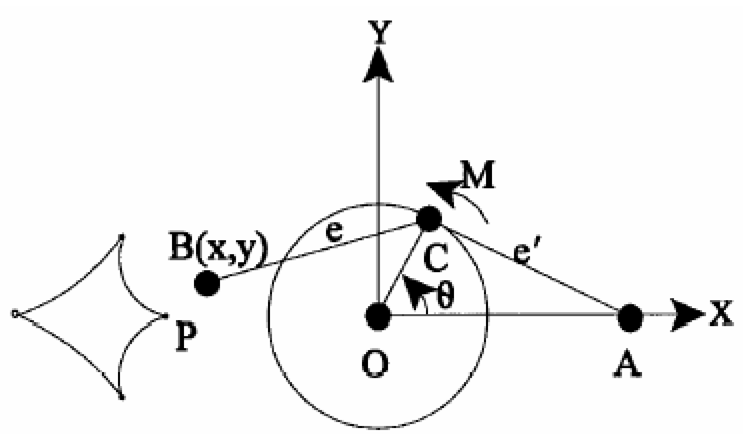


Figure 1-7. Zeeman's catastrophe machine.

Parallel compliant mechanisms are classically known for their role in mounting and suspension systems. Recently, however, they have been instrumental in the field of robotic force control. A new theory for the simultaneous control of displacement and force for a partially constrained end-effecter has been proposed in Yin et al. [10].

Figure 1-8 shows a planar compliant mechanism which is actually a planar two-spring system. It consists of a pair of linear springs connected at one end to a movable base and at the other end to a common pivot which is the axis of the wheel contacting to a surface. The base is connected to a planar two freedom P-P (P denotes prismatic pair ) manipulator . The contact force can be controlled by displacing the two prismatic joints of the manipulator. The required displacements can be calculated from the stiffness mapping. This kind of control was called kinestatic control by Griffis and Duffy [11]. In order to design the planar two-spring system it is necessary to compute a spring stiffness which will generate a range of displacements of the movable base which can be produced by the prismatic joints over a required range of change in contact force. Clearly if the system is over-designed and the spring stiffness is very high it is always possible to generate any necessary changes in contact force. However such a system will be too sensitive to errors because very small displacements of the platform will generate large changes in contact force. On the other hand if the springs are too soft there can be stability problems.

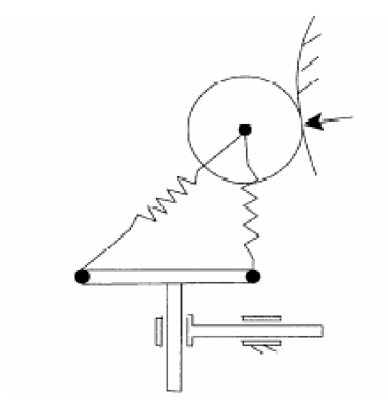

Figure 1-8. Planar two-spring system.

Yin et al. [10] are also considered another compliant mechanism. That is shown in Figure 1-9. It consists of three linear springs joined to the triangular frame fixed points and connected at the axis of the wheel. The triangular frame is connected to the planar two freedom P-P manipulator. This mechanism is a special three-spring system. The catastrophe analysis of this system was presented. Comparing the results demonstrates that three-spring system has better stability characteristics than the two-spring system.

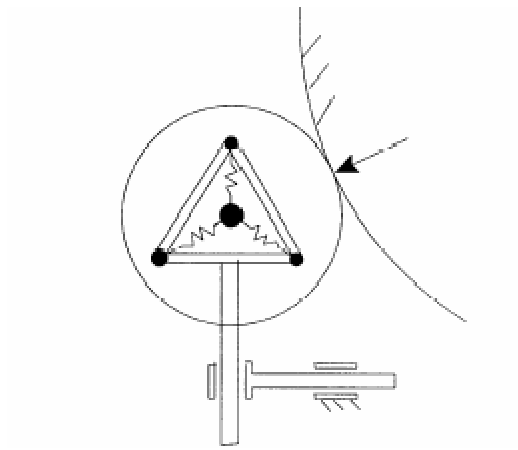

Figure 1-9. Special three-spring system.

A more general planar three-spring compliant mechanism is shown in Figure 1-10. This mechanism is connected to the planar three freedom R-R-R ( R denotes revolute pair) manipulator. It can be used to control both force and moment. The catastrophe analysis of this mechanism was performed in Yin et al. [10].

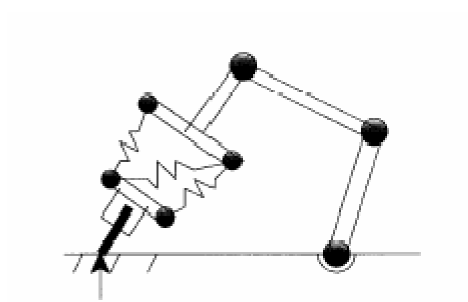

Figure 1-10. General planar three-spring compliant mechanism.

Crane and Duffy [12] studied kinematics of robot manipulators and Crane and Bayat [13] presented cases of two dimensional tensegrity structures that proved to be mathematically challenging in spite of their simple configurations. Different methods were tried to solve these problems. It was expected the final closed form solution be a simple mathematical expression. In each case the intermediate mathematical expressions become complicated to solve and the relevant matrices become large to manipulate.

Roth and Whiteley [14] propose a technology based on tensegrity for tough, rigid, large-scale domes that are also economical to construct. The development of a structural technology to economically cover large areas would be useful for warehouses, permanent or temporary protection for archaeological and other vulnerable sites, large-scale electrical or electromagnetic shielding and exclusion or containment of flying animals or other objects. Structures based on such a technology can serve as frameworks in which environmental control, energy transformation and food production facilities could be embedded. The space application is also possible by using self-deployed structures. Summary Advantages are improved rigidity, ethereal, resilient, equal-length struts, simple Joints.

The resulting list represents an initial attempt to identify applications for which the technology would be suitable. As the technology develops and is tested against them, some or all of the applications may be winnowed out and other suitable applications not in the list may become apparent. The current list is as follows:

Superstructures for embedded substructures allow the substructures to escape terrestrial confines where this is useful (e.g. in congested or dangerous areas, urban areas, flood plains or irregular, delicate or rugged terrains).

1. Economic large-scale protection of storage, archaeological, agricultural, construction or other sites.
2. Refugee or hiking shelters.
3. Frames over cities for environmental control, energy transformation and food production.

4. Large-scale electrical or electromagnetic shielding.

5. Exclusion or containment of flying animals or other objects.

6. Spherical superstructures for space stations.

7. Earthquake-resistant applications.

These structures are extremely resilient and testing would very likely show they could withstand large structural shocks like earthquakes. Thus, they would likely be desirable in areas where earthquakes are a problem. A tensegrities, pneumatic, structure performed very well during recent earthquakes in Japan.

- Low-environmental-impact shells for musical performances.
- Indoor/outdoor pavilions for trade shows etc.
- Supports to hold sunscreen protection for vulnerable amphibians.
- Watersheds to keep rain water from percolating through contaminated soils into groundwater, perhaps temporarily during in-situ remediation.
- Frames for hanging plants or other objects to dry.
- Pergola, trellis, or topiary framework.
- Micro-meteorite protection, sun-shielding for Martian colonies.

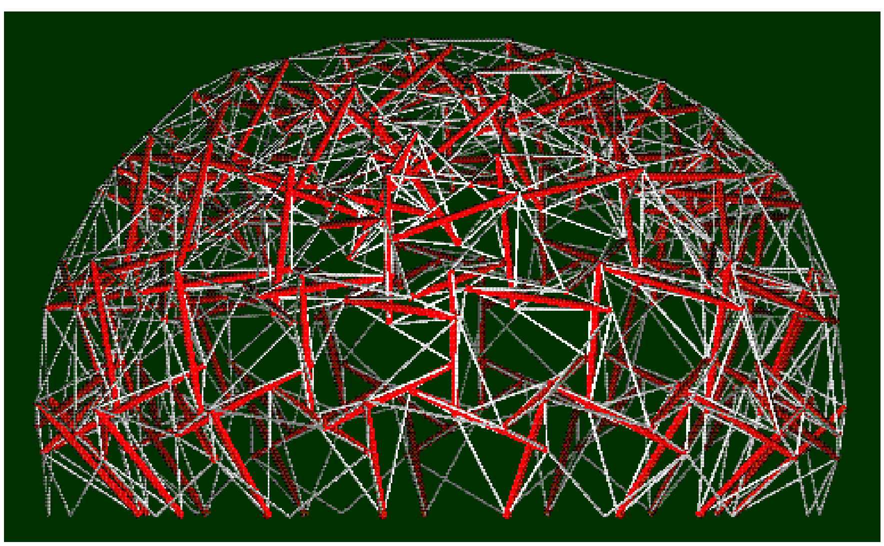

Figure 1-11. A representation of a dome which utilizes tensegrity solutions' technology

Skelton et al. [15] present a solution so the tensegrity structures are reduced to linear algebra problems, after first characterizing the problem in a vector space where direction cosines are not needed. That is, we describe the components of all members by vectors as opposed to the usual practice of characterizing it as static problem in terms of the magnitude of tension and compression. While this approach transfer the problem into vector space to describe the problem, the advantage is that the vector space makes the mathematical structure of the problem amenable to linear algebra treatment for both two and three dimensional tensegrity structures.

This paper characterizes the static equilibrium some tensegrity structures. Furthermore, it uses vectors to describe each element (springs and studs), eliminating the need to use direction cosines and the subsequent transcendental functions that followthe use of trigonometry equations.

In this paper, the authors choose to represent a tensegrity structure as an oriented graph in real three dimensional space R3 defined in terms of nodes and directed branches which are all represented as vectors in R3. A loop is any closed path in the graph.

The advantage of this approach is that the both the magnitude and the direction cosines of the forces are contained in vectors which can be solved using linear algebra. Thus linear algebra plays a larger role in this approach compared to the usual approach in mechanics and finite element methods using direction cosines.

In this oriented graph, the nodes consist of the ends of the bars. Hence if there are n bars, then there are 2n nodes. There are two types of directed branches; the string branch (in vectors) and the bar branches (in vectors).

Geometric connectivity of the structure maintained such that each directed branch can undergo a displacement in reaching its equilibrium state. String vectors can change both their length and orientation while bar vectors can only change their orientation. Node vectors can change both their length and orientation but subject to a Law of Geometric Connectivity is stated as follows: The vector sum of all branch vectors in any loop is zero. These loop equations are in the form of a set of linear algebraic equations in the branch vectors.

In this study of tensegrity structures, force equilibrium are such that spring can only take tension and bars sustain compressive forces. We therefore choose to distinguish between the string (or tensile) forces and the bar (or compressive) forces which are defined in terms of the string and bar vectors respectively.

This paper reduces the study of the tensegrity equilibrium to a series of linear algebra problems using directed graph theory. Of course the existence conditions for the linear algebra problems are nonlinear in the design variables. The presented procedure and formulation give some insight to solution of tensegrity structures and identifies the free parameters that may be used to achieve desired structural shapes.

Tibert [16] presents adjustable tensegrity structure. A tensegrity structure is a lightweight consisting of compression members surrounded by a network of tension members. They can be easily dismantled providing possibilities for reusable and modular structures. Tensegrities adapt their shape by self adjusting their tension and compression in their members. They can adapt to changing environment when they are equipped with sensors and actuators.

A full-scale prototype of an adjustable tensegrity is built and tested at Swiss Federal Institute of Technology. This paper begins with a description of important aspects of the design, assembly, and static testing. Tests show that the structure behaves linearly when subjected to vertical loads applied to a single joint. Nonlinearities are detected for small displacements, for loads applied to several joints and for adjusting combinations of telescoping compression members. To predict behavior, dynamic relaxation—a nonlinear method—has been found to be reliable. Appropriate strut adjustments found by a stochastic search algorithm are identified for the control goal of constant roof slope and for the load conditions studied. When adjusting struts, an excessive number of adjustable members does not necessarily lead to improved performance.

Tensegrity is an abbreviation of tensile–integrity as termed by one R. Buckminster Fuller. He described tensigrity as ‘‘small islands of compression in a sea of tension.’’ This is a description of a network of light tension members that provide rigidity to a limited number of discontinuous compression members. Some researchers have found that discontinuous compressions are not a necessity for creating a tensegrity and more efficient structures can be built if compression elements are allowed to join. Defining a tensegrity structure as any self-stressing structure composed of struts and cables would include structures such as a bicycle wheel. In order to add precision terminology, a definition has been proposed by Motro and Raducanu. They describe a tensegrity as a stable system that contains a discontinuous set of components in compression inside a network of components in tension.

Several independent efforts have contributed to the invention and development of tensegrity structures. Le Ricolais designed many unusual lightweight cable–strut structures. The sculptor Kenneth Snelson started in 1948 with sculptures employing members in continuous tension and discontinuous compression. Applications of the tensegrity principle in nature have recently been found through studies that use a tensegrity model to describe observations of how cells respond to stress.

Transforming tensegrity from sculptures into practical structural has been a challenge that began with several studies of their geometric, nonlinear behavior and states of self stress. Reasons for the lack of test data are related to the fabrication assembly process as well as to the lack of rigidity unless pre-stressed members are used.

Joint design is the biggest challenge in constructing a full-scale tensegrity. Although it is the problem for all space structures, tensegrities present particular challenges. Joints need to be pin jointed, modular, and light in order to take advantage of tensegrities ease of dismantling and potential reuse.

In addition to the complexities of constructing the structure itself, simulation of the behavior is not straightforward. Tensegrity structures exhibit geometric nonlinear behavior. In addition, as it is common with other pin-jointed structures, nodal friction should be included in the simulation of a truss structure for precision control. Finally, for full-scale construction, assembly sequences, fabrication tolerances, and construction lack-of-fit need to be considered.

Fest et al. [17] present a study of adjustable tensegrity structures. A tensegrity structure is consisting of compression members surrounded by a network of tension members. A tensegrity structure is a lightweight structure. Tensegrity structures can be easily dismantled providing possibilities for reusable and modular structures. Tensegrities adapt their shape by self adjusting their tension and compression in their members. They can adapt to changing environment when they are equipped with sensors and actuators.

A full-scale prototype of an adjustable tensegrity is built and tested at Swiss Federal Institute of Technology. This paper presents important aspects of the design, assembly, and static testing. Tests show that the structure behaves linearly when it is subjected to vertical loads applied to a single joint. Nonlinearities are observed when loads applied to several joints and for adjusting combinations of telescoping compression members. Appropriate strut adjustments found by a search algorithm for the load conditions studied. When adjusting struts, an excessive number of adjustable members do not necessarily lead to equilibrium.

Active tensegrity structures have the potential to widen the scope for innovative, lightweight, and reusable structural systems. Lessons learn through the construction and testing of an adjustable full-scale prototype can be used in construction and design of practical tensegrity structures. This work expected to contribute to the development of tensigrity and improve in their design and performance during service.

CHAPTER 2
TWO SPRING PLANAR TENSEGRITY SYSTEM

## 2.1 Introduction

This Chapter presents two approaches to solve the forward position analysis of a two-strut tensegrity system with two compliant ties and two non-compliant ties (Figure 2-1).

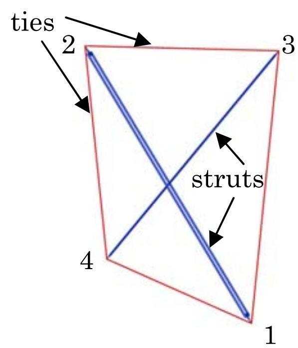


Figure 2-1. Two dimensional tensegrity structure.

Here it is assumed that the lengths of the struts and noncompliant ties are known together with the spring constants and free lengths of the two compliant ties. The objective is to determine all possible equilibrium poses for the device when no external loads are applied. Gravity loads are neglected.

The objective of this effort is to determine, in closed-form, all possible equilibrium deployed positions in stable condition (minimum potential energy) of a planar tensegrity system wherein two of the ties are compliant. Figure 2-2 shows the system which is comprised of two struts (compression members $a_{12}$ and $a_{34}$), two non-compliant ties (tension members $a_{41}$ and $a_{23}$), and two elastic tensile members (springs), one connected between points 1 and 3 and one between points 2 and 4. It should be noted in Figure 2-2 that strut $a_{34}$ passes through a slit cut in strut $a_{12}$ and as such the two struts do not intersect or collide.

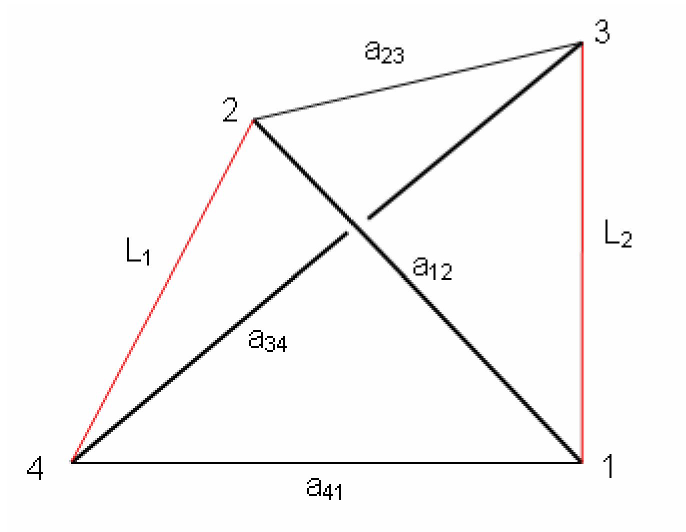


Figure 2-2. Two-spring planar tensegrity system.

## 2.2 Problem Statement

The problem statement for the two-spring plane tensegrity system can be explicitly written as

given:

$a_{12}$, $a_{34}$ lengths of struts

$a_{23}$, $a_{41}$ lengths of non-compliant ties

$k_1$, $L_{01}$ spring constant and free length of compliant tie between points 4 and 2

$k_2$, $L_{02}$ spring constant and free length of compliant tie between points 3 and 1

find:

$L_1$ length of spring 1 at equilibrium position

$L_2$ length of spring 2 at equilibrium position

It should be noted that the problem statement could be formulated in a variety of ways, that is, a different variable (such as the relative angle between strut $a_{34}$ and tie $a_{41}$)

could have been selected as the generalized parameter for this problem. Two solution approaches are presented.

## 2.3 Approach 1: Determine ($L_1$, $L_2$) to Minimize Potential Energy

### 2.3.1 Development of Geometric Equations

Figure 2-3 shows the three angles $\theta_4$, $\theta'_4$, and $\theta''_4$ which must satisfy the relation

$$\theta_4 + \theta_4' = \pi + \theta_4'' \quad . \tag{2-1}$$

Figure 2-4 shows the triangle formed by side $a_{34}$, $a_{23}$, and $L_1$. A cosine law for this triangle can be written as

$$\frac{L_1^{\,2}}{2} + \frac{a_{34}^{\,2}}{2} + L_1\, a_{34} \cos\theta_4' = \frac{a_{23}^{\,2}}{2} \quad . \tag{2-2}$$

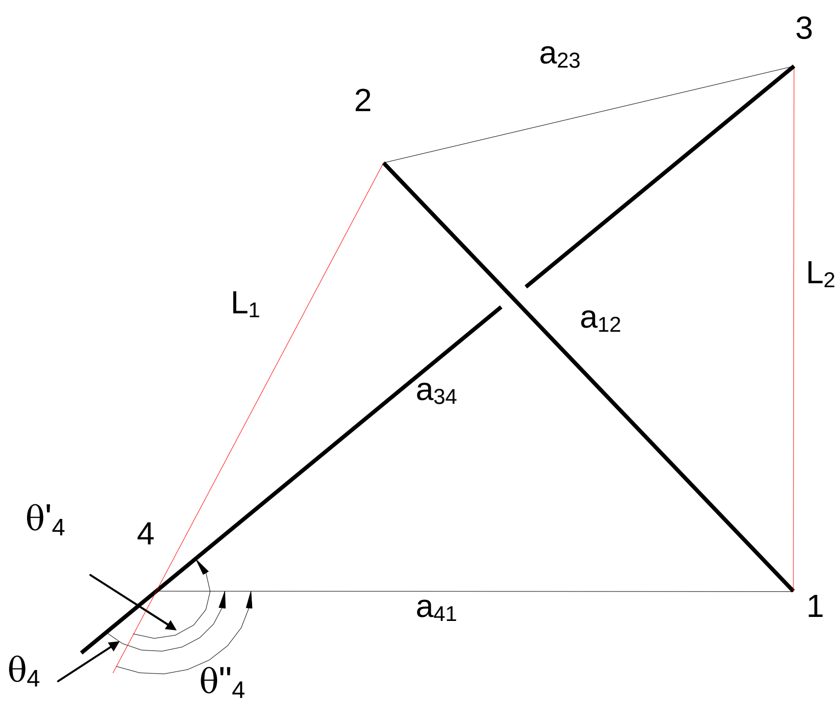


Figure 2-3. Identification of angles $\theta_4$, $\theta'_4$, and $\theta''_4$.

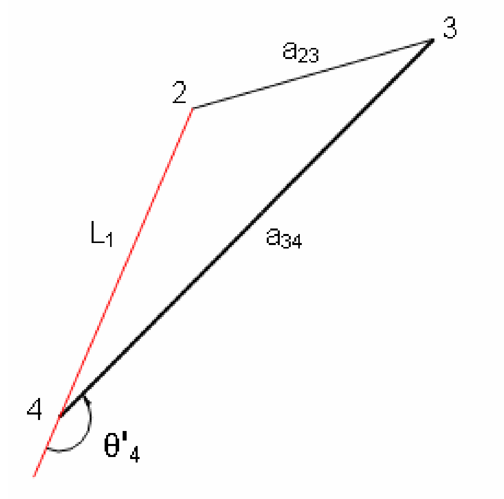


Figure 2-4. Triangle 4-1-3.

Solving for $\cos\theta_4'$ yields

$$\cos\theta_4' = \frac{a_{23}^2 - L_1^2 - a_{34}^2}{2 L_1 a_{34}}. \tag{2-3}$$

Figure 2-5 shows the triangle formed with sides $a_{41}$, $a_{12}$, and $L_1$. A cosine law for this triangle can be written as

$$\frac{L_1^2}{2} + \frac{a_{41}^2}{2} + L_1 a_{41} \cos\theta_4'' = \frac{a_{12}^2}{2}. \tag{2-4}$$

Solving for $\cos\theta_4''$ yields

$$\cos\theta_4'' = \frac{a_{12}^2 - L_1^2 - a_{41}^2}{2 L_1 a_{41}}. \tag{2-5}$$

Figure 2-6 shows the triangle formed by sides $a_{41}$, $a_{34}$ and $L_2$. A cosine law for this triangle can be written as

$$\frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} \cos\theta_4 = \frac{L_2^2}{2} \tag{2-6}$$

Solving for $\cos\theta_4$ yields

$$\cos\theta_4 = \frac{L_2^2 - a_{34}^2 - a_{41}^2}{2 a_{34} a_{41}} \tag{2-7}$$

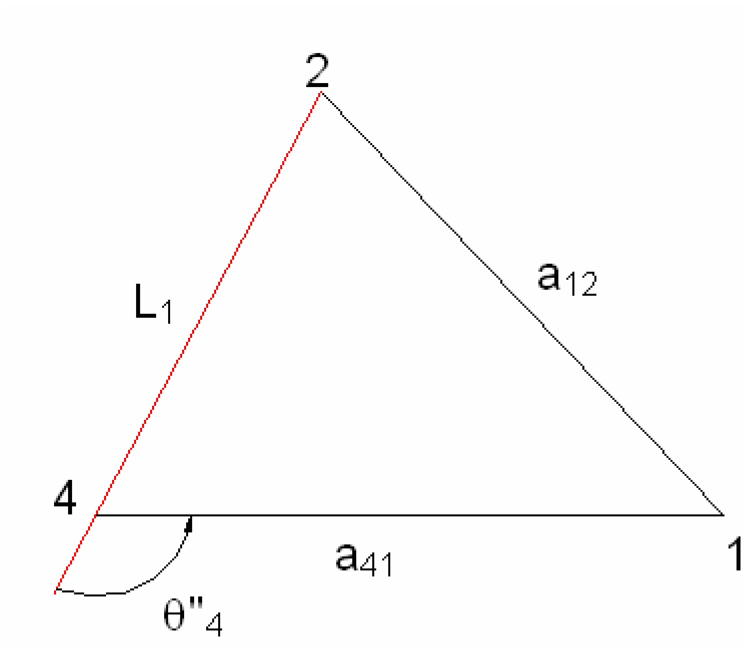


Figure 2-5. Triangles 4-1-2.

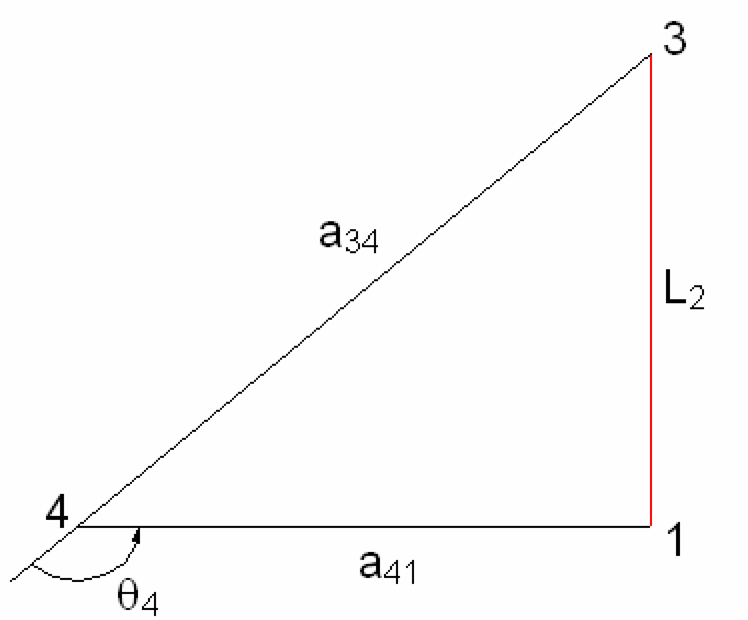


Figure 2-6. Triangles 4-1-3.

Equating the cosine of the left and right sides of 2-1 yields

$$\cos(\theta_4 + \theta_4') = \cos(\pi + \theta_4'') \quad . \tag{2-8}$$

Expanding this equation yields

$$\cos\theta_4 \cos\theta_4' - \sin\theta_4 \sin\theta_4' = -\cos\theta_4'' \quad . \tag{2-9}$$

Rearranging 2-9 yields

$$\cos\theta_4 \cos\theta_4' + \cos\theta_4'' = \sin\theta_4 \sin\theta_4' \tag{2-10}$$

Squaring both sides of 2-10 gives

$$(\cos\theta_4)^2 (\cos\theta_4')^2 + 2\cos\theta_4 \cos\theta_4' \cos\theta_4'' + (\cos\theta_4'')^2 = (\sin\theta_4)^2 (\sin\theta_4')^2 \quad . \tag{2-11}$$

Substituting for $(\sin\theta_4)^2$ and $(\sin\theta_4')^2$ in terms of $\cos\theta_4$ and $\cos\theta_4'$ gives

$$(\cos\theta_4)^2 (\cos\theta_4')^2 + 2\cos\theta_4 \cos\theta_4' \cos\theta_4'' + (\cos\theta_4'')^2 =$$

$$(1-\cos^2\theta_4)(1-\cos^2\theta_4') \quad . \tag{2-12}$$

Equations 2-3, 2-5, and 2-7 are substituted into 2-12 to yield a single equation in the parameters $L_1$ and $L_2$ which can be written as

$$A\,L_2^4 + B\,L_2^2 + C = 0 \tag{2-13}$$

where

$$A = L_1^2, \quad B = L_1^4 + B_2 L_1^2 + B_0, \quad C = C_2 L_1^2 + C_0 \tag{2-14}$$

and

$$B_2 = -(a_{23}^2 + a_{34}^2 + a_{41}^2 + a_{12}^2),$$

$$B_0 = (a_{12} - a_{41})(a_{12} + a_{41})(a_{23} - a_{34})(a_{23} + a_{34}) \ ,$$

$$C_2 = (a_{34} - a_{41})(a_{34} + a_{41})(a_{23} - a_{12})(a_{23} + a_{12}) \ ,$$

$$C_0 = (a_{41}a_{23} + a_{34}a_{12})(a_{41}a_{23} - a_{34}a_{12})(a_{41}^2 + a_{23}^2 - a_{12}^2 - a_{34}^2) \tag{2-15}$$

Equation 2-13 expresses length $L_2$ as a function of the length $L_1$. The function is quadratic with respect to $(L_2)^2$ and $(L_1)^2$ and thus there will be two possible values for $(L_2)^2$ for each value of $(L_1)^2$.

As a verification of Equation 2-13 a numerical example is presented to show how four values of $L_2$ can occur for a given value of $L_1$. For this example, assume $a_{41}$ = 1.0 m, $a_{12}$ = 2.0 m, $a_{23}$ = 2.0 m, and $a_{34}$ = 2.5 m. Further, assume the value of $L_1$ is given as $L_1$ = 2.25 m. Evaluating the coefficients in Equations 2-15 and 2-14 and substituting into Equation 2-13 gives

$$(5.0625)\, L_2^{4} + (-58.3242)\, L_2^{2} + (110.2500) = 0\ . \qquad (2\text{-}16)$$

Solving this equation for $L_2$ yields four answers;

$$L_{2a} = -3.0228,\ L_{2b} = 3.0228,\ L_{2c} = -1.5438,\ L_{2d} = 1.5438\ . \qquad (2\text{-}17)$$

Figure 2-7 shows the mechanism in four configurations where for each of these configurations $L_2$ can have a positive or negative value, thereby giving eight possible states of the mechanism. However, due to the symmetry associated with the reflected solutions about $a_{41}$, only four values of $L_2$ (or two values of $L_2^{2}$) exist. This example was presented to show that the degree of the polynomial which relates $L_1$ and $L_2$ is indeed fourth order.

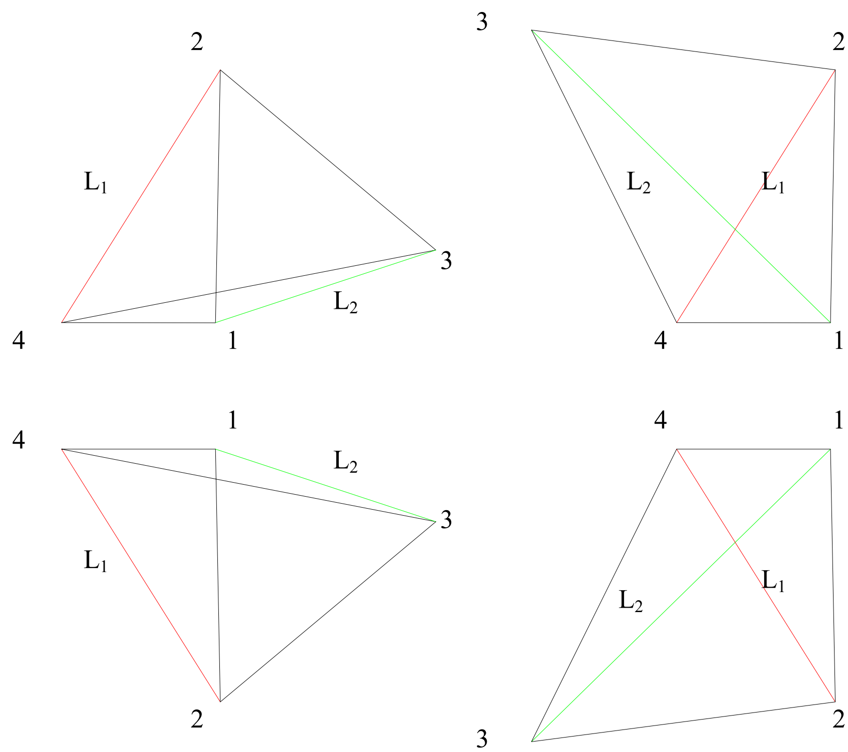


Figure 2-7. Possible configurations for numerical example.

The potential energy of the system can be evaluated as

$$U = \frac{1}{2}k_1(L_1 - L_{01})^2 + \frac{1}{2}k_2(L_2 - L_{02})^2 \quad . \tag{2-18}$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the length $L_1$ equals zero, i.e.

$$\frac{dU}{dL_1} = k_1(L_1 - L_{01}) + k_2(L_2 - L_{02})\frac{dL_2}{dL_1} = 0 \quad . \tag{2-19}$$

The derivative $dL_2/dL_1$ can be determined via implicit differentiation from Equation 2-13 as

$$\frac{dL_2}{dL_1} = \frac{-L_1[L_2^2(L_2^2 + 2L_1^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{23}^2)(a_{41}^2 - a_{34}^2)]}{L_2[L_1^2(L_1^2 + 2L_2^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{41}^2)(a_{23}^2 - a_{34}^2))]} \quad . \tag{2-20}$$

Substituting Equation 2-20 into Equation 2-19 and regrouping gives

$$D\,L_2^5 + E\,L_2^4 + F\,L_2^3 + G\,L_2^2 + H\,L_2 + J = 0 \qquad (2\text{-}21)$$

where

$$D = D_1\,L_1,\;\; E = E_1\,L_1,\;\; F = F_3\,L_1^3 + F_2\,L_1^2 + F_1\,L_1,\;\; G = G_3\,L_1^3 + G_1\,L_1,$$

$$H = H_5\,L_1^5 + H_4\,L_1^4 + H_3\,L_1^3 + H_2\,L_1^2 + H_1\,L_1 + H_0,\;\; J = J_1\,L_1 \qquad (2\text{-}22)$$

and where,

$$D_1 = k_2,\;\; E_1 = -k_2\,L_{02},$$

$$F_3 = 2\,(k_2 - k_1),\; F_2 = 2\,k_1\,L_{01},\; F_1 = -\,k_2\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$G_3 = -2\,k_2\,L_{02},\; G_1 = k_2\,L_{02}\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$H_5 = -\,k_1,\; H_4 = k_1\,L_{01},\; H_3 = k_1\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$H_2 = -\,k_1\,L_{01}\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$

$$H_1 = -\,k1\,(a_{34}^2 - a_{23}^2)\,(a_{41}^2 - a_{12}^2) + k_2\,(a_{34}^2 - a_{41}^2)\,(a_{23}^2 - a_{12}^2),$$

$$H_0 = k_1\,L_{01}\,(a_{34}^2 - a_{23}^2)\,(a_{41}^2 - a_{12}^2),$$

$$J_1 = k_2\,L_{02}\,(a_{34}^2 - a_{41}^2)\,(a_{12}^2 - a_{23}^2) \qquad (2\text{-}23)$$

Equations 2-13 and 2-21 represent two equations in the two variables $L_1$ and $L_2$. The simultaneous solution of these equations is a necessary condition that the system is in equilibrium.

### 2.3.2 Sylvester's Solution Method

A system of *m* equations of order *n* in the variable Y may be written as (Gantes and Konitopoulou [9])

$$A_{1,n}\,Y^n + A_{1,n-1}\,Y^{n-1} + A_{1,n-2}\,Y^{n-2} + \ldots + A_{1,0} = 0$$

$$A_{2,n}\,Y^n + A_{2,n-1}\,Y^{n-1} + A_{2,n-2}\,Y^{n-2} + \ldots + A_{2,0} = 0\;,$$

$$A_{m,n}\,Y^n + A_{m,n-1}\,Y^{n-1} + A_{m,n-2}Y^{n-2} + \ldots + A_{m,0} = 0\;. \qquad (2\text{-}24)$$

The terms $U_n = Y^n$ , $U_{n-1} = Y^{n-1}$ , . . . , $U_1 = Y$, $U_0 = 1$ may be substituted into Equation 2-24 to yield

$$A_{1,n} U_n + A_{1,n-1} U_{n-1} + A_{1,n-2} U_{n-2} + \ldots + A_{1,0} U_0 = 0$$

$$A_{2,n} U_n + A_{2,n-1} U_{n-1} + A_{2,n-2} U_{n-2} + \ldots + A_{2,0} U_0 = 0 ,$$

$$A_{m,n} U_n + A_{m,n-1} U_{n-1} + A_{m,n-2} U_{n-2} + \ldots + A_{m,0} U_0 = 0 . \qquad (2\text{-}25)$$

Equation 2-25 represents a system of m homogeneous equations with n+1 unknowns (the term $U_0$ will be treated as an unknown for the time being) and can be written in matrix form as

$$\begin{bmatrix} A_{1,n} & A_{1,n-1} & \cdots & A_{1,0} \\ A_{2,n} & A_{2,n-1} & \cdots & A_{2,0} \\ \vdots & \vdots & \vdots & \vdots \\ A_{m,n} & A_{m,n-1} & \cdots & A_{m,0} \end{bmatrix} \begin{bmatrix} U_n \\ U_{n-1} \\ \cdots \\ U_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix} . \qquad (2\text{-}26)$$

If the number of equations is less than the number of unknowns, additional equations can be created by multiplying selected equations from set of Equation 2-24 by powers of Y to obtain a new system. For example, multiplying the first equation from the set of Equation 2-24 by Y will introduce one new equation and one new unknown, $Y^{n+1}$ (which will be represented by $U_{n+1}$). Now, multiplying the second equation of the set of Equation 2-24 by Y will yield another equation, but no new variables are introduced. Ultimately, the system will be formed where the new total number of homogeneous equations equals the new total number of unknowns. In other words, the coefficient matrix will be a square matrix. Now, one solution to the set of 'homogeneous' equations is the trivial solution where all the $U_i$ terms equal zero. However, this cannot occur since $U_0$=1. Thus it must be the case that the set of equations are linearly dependent which means that the determinant of the coefficient matrix must equal zero.

Now consider that the terms $A_{i,j}$ of the coefficient matrix are polynomial functions of order q of another variable called X, that is

$$A_i = a_i X^q + b_i X^{q-1} + c_i X^{q-2} + \ldots + z_i = 0 \tag{2-27}$$

where $a_i$ , $b_i$ , $c_i$ , … $z_i$ are constants. Evaluation of the determinant of the coefficient matrix will result in a polynomial in the variable X that must equal zero so that the set of equations in the variables $U_i$ will be linearly dependent. This solution approach will now be applied to Equations 2-13 and 2-21 which represent two equations in the two unknowns $L_1$ and $L_2$.

### 2.3.3 Solution of Geometry and Energy Equations

Equations 2-13 and 2-21 can be solved by using Sylvester's variable elimination procedure by multiplying Equation 2-13 by $L_2$, $L_2^2$, $L_2^3$, and $L_2^4$ and Equation 2-21 by $L_2$, $L_2^2$, $L_2^3$ to yield a total of nine equations that can be written in matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & D & E & F & G & H & J \\ 0 & 0 & 0 & 0 & A & 0 & B & 0 & C \\ 0 & 0 & 0 & A & 0 & B & 0 & C & 0 \\ 0 & 0 & D & E & F & G & H & J & 0 \\ 0 & 0 & A & 0 & B & 0 & C & 0 & 0 \\ 0 & D & E & F & G & H & J & 0 & 0 \\ 0 & A & 0 & B & 0 & C & 0 & 0 & 0 \\ D & E & F & G & H & J & 0 & 0 & 0 \\ A & 0 & B & 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_2^8 \\ L_2^7 \\ L_2^6 \\ L_2^5 \\ L_2^4 \\ L_2^3 \\ L_2^2 \\ L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{2-28}$$

This set of equations can only be solved if the determinant of the 9×9 coefficient matrix is equal to zero. Expansion of this determinant yields a 30th degree polynomial in the variable $L_1$. When the determinant was expanded symbolically, it was seen that the two lowest order coefficients were identically zero. Thus the polynomial can be divided throughout by $L_1^2$ to yield a 28th degree polynomial. The coefficients of the 28th degree polynomial were obtained symbolically in terms of the given quantities. They are not presented here due to their length and complexity.

Values for $L_2$ that correspond to each value of $L_1$ can be determined by first solving Equation 2-13 for four possible values of $L_2$. Only one of these four values also satisfies Equation 2-21.

### 2.3.4 Numerical Examples

Example 1: The following parameters were selected for this numerical example:

| | | |
|---|---|---|
| Strut lengths | $a_{12}$ = 3 in. | $a_{34}$ = 3.5 in. |
| Non-compliant tie lengths | $a_{41}$ = 4 in. | $a_{23}$ = 2 in. |
| Spring 1 free length & spring constant | $L_{01}$ = 0.5 in. | $k_1$ = 4 lbf/in. |
| Spring 2 free length & spring constant | $L_{02}$ = 1 in. | $k_2$ = 2.5 lbf/in. |

Eight real and twenty complex roots were obtained for the 28th degree polynomial in $L_1$. Real values for $L_1$ and the corresponding values of $L_2$ are shown in Table 2-1.

Table 2-1. Eight real solutions.

| Case | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| L1, in. | -5.485 | -5.322 | -1.741 | -1.576 | 1.628 | 1.863 | 5.129 | 5.476 |
| L2, in. | 2.333 | -2.901 | -1.495 | 1.870 | 1.709 | -1.354 | -3.288 | 2.394 |

The values of $L_1$ and $L_2$ listed in Table 2-1 satisfy the geometric constraints defined by Equation 2-13 and the energy condition defined by Equation 2-21. Each of these eight cases was analyzed to determine whether it represented a minimum or maximum potential energy condition and cases 3, 4, 5, and 6 were found to be minimum states. A free body analysis of struts $a_{12}$ and $a_{34}$ was performed to show that these bodies were indeed in equilibrium for each of these four cases. Figure 2-8 shows the four static equilibrium configurations of the system. Note that there are several cases where the

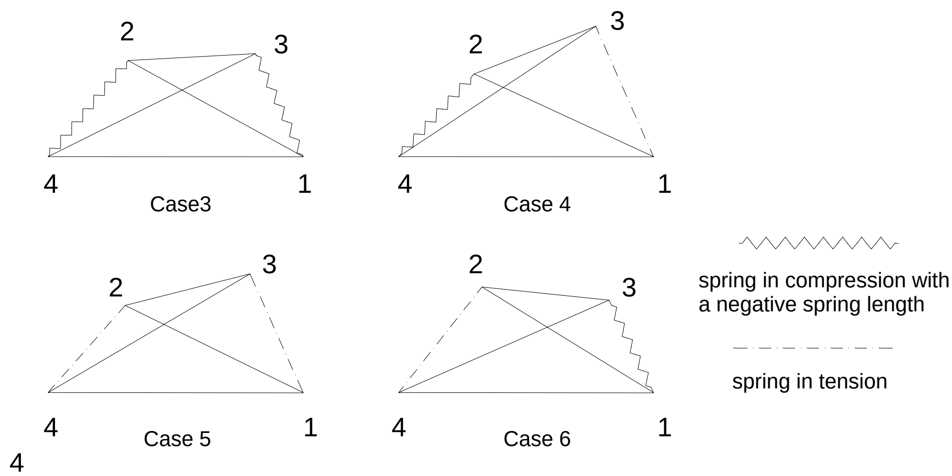


Figure 2-8. Four equilibrium configurations.

spring lengths are negative, such as for example the spring between points 2 and 4 in case 4. In this case the spring is acting to pull point 2 towards point 4 which is counter intuitive for the normal spring that is in compression.

There are twenty complex roots for $L_1$ in this example. Unique corresponding values for $L_2$ were obtained for each case (Table 2-2). In fact there are corresponding values for $L_2$ such that the complex pair of $L_1$ and $L_2$ satisfies both Equations 2-13 and 2-21. This result means that no extraneous terms were introduced in the elimination procedure. These complex roots were not analyzed further.

Table 2-2. Complex solutions.

| Case | $L_1$, inches | $L_2$, inches |
|---|---|---|
| 1 | -7.0285 – 0.0648i | -0.3225 + 2.5486i |
| 2 | -7.0285 + 0.0648i | -0.3224 – 2.5486i |
| 3 | -2.0919 – 2.4639i | -6.5995 + 0.3617i |
| 4 | -2.0919 + 2.4639i | -6.5995 – 0.3617i |
| 5 | -1.7904 – 2.0044i | 6.5232 + 0.0738i |
| 6 | -1.7904 + 2.0044i | 6.5232 – 0.0738i |
| 7 | -0.7211 – 0.1827i | 0.2411 – 1.2909i |
| 8 | -0.7211 + 0.1827i | 0.2411 + 1.2909i |
| 9 | 0.005823 – 1.7651i | -0.05803 – 0.01682i |
| 10 | 0.005823 + 1.7651i | -0.05803 + 0.01682i |
| 11 | 0.3062 – 0.4101i | -0.1220 – 0.9493i |
| 12 | 0.3062 + 0.4101i | -0.1220 + 0.9493i |
| 13 | 0.8140 – 0.5553i | 0.5055 +0.9541i |

Table 2-2. Continued.

| Case | $L_1$, inches | $L_2$, inches |
|---|---|---|
| 14 | 0.8140 + 0.5553i | 0.5055 +0.9541i |
| 15 | 2.3264 – 1.9808i | 6.1894 + 0.2416i |
| 16 | 2.3264 + 1.9808i | 6.1894 – 0.2416i |
| 17 | 2.9910 – 2.5413i | -6.2069 – 0.9310i |
| 18 | 2.9910 + 2.5413i | -6.2069 + 0.9310i |
| 19 | 7.0351 – 0.1067i | -0.4518 – 2.6442i |
| 20 | 7.0351 + 0.1067i | -0.4518 + 2.6442i |

Example 2: For this case, a C language program was written that would randomly select values for the parameters $a_{12}$, $a_{23}$, $a_{34}$, $a_{41}$, $k_1$, $L_{01}$, $k_2$, and $L_{02}$ in attempt to find a case where the number of real roots for $L_1$ was twenty eight. Such a case was not found, but a set of inputs resulting in twenty four real values for $L_1$ is presented here.

The input parameters for this case were selected as:

| | | |
|---|---|---|
| Strut lengths | $a_{12}$ = 3.309848 in. | $a_{34}$ = 3.002692 in. |
| Non-compliant tie lengths | $a_{41}$ = 7.335484 in. | $a_{23}$ = 7.978210 in. |
| Spring 1 free length & spring constant | $L_{01}$ = 0.953703 in. | $k_1$ = 0.193487 lbf/in. |
| Spring 2 free length & spring constant | $L_{02}$ = 0.607318 in. | $k_2$ = 9.108249 lbf/in. |

Twenty four real and four complex roots were obtained for $L_1$. The real values for $L_1$ and the corresponding values of $L_2$ are shown in Table 2-3. The four complex values for $L_1$ were 1.2818 ± 18.2760i and -0.8019 ± 18.2940i.

Table 2-3. Twenty-four real solutions.

| Case | $L_1$, inches | $L_2$, inches |
|---|---|---|
| 1 | -19.0226 | -2.5540 |
| 2 | -16.9651 | 2.8637 |
| 3 | -11.7045 | 0.6215 |
| 4 | -11.0691 | 4.3323 |
| 5 | -11.0649 | -4.3325 |
| 6 | -10.5742 | -4.6685 |

Table 2-3. Continued.

| Case | $L_1$, inches | $L_2$, inches |
|---|---|---|
| 7 | -10.5716 | 4.6687 |
| 8 | -5.1348 | 10.3382 |
| 9 | -5.1343 | -10.3382 |
| 10 | -3.7712 | -11.2881 |
| 11 | -3.7704 | 11.2881 |
| 12 | -0.02315 | 11.7686 |
| 13 | -0.02315 | -11.7686 |
| 14 | 3.7733 | 11.2881 |
| 15 | 3.7737 | -11.2881 |
| 16 | 5.1330 | -10.3382 |
| 17 | 5.1334 | 10.3382 |
| 18 | 10.5735 | 4.6686 |
| 19 | 10.5755 | -4.6685 |
| 20 | 11.0631 | -4.3326 |
| 21 | 11.0664 | 4.3324 |
| 22 | 11.7046 | 0.6193 |
| 23 | 17.4121 | 2.7902 |
| 24 | 19.5242 | -2.4884 |

At first glance, it appears that a set of input parameters have been found whereby most of the roots of the resulting polynomial equation in $L_1$ and the corresponding values of $L_2$ are real. However, from Figure 2-2, it is apparent that the system will be realizable only if

$$|a_{41} - a_{12}| \leq |L_1| \leq a_{41} + a_{12} ,$$

$$|a_{34} - a_{23}| \leq |L_1| \leq a_{34} + a_{23} ,$$

$$|a_{23} - a_{12}| \leq |L_2| \leq a_{23} + a_{12} ,$$

$$|a_{41} - a_{34}| \leq |L_2| \leq a_{41} + a_{34} . \qquad (2\text{-}29)$$

For this numerical case, the system will be realizable only if

$$4.9755 \text{ in.} \leq |L_1| \leq 10.6453 \text{ in.} ,$$

$$4.6684 \text{ in.} \leq |L_2| \leq 10.3382 \text{ in.} \qquad (2\text{-}30)$$

Table 2-4 lists the cases which satisfy these conditions.

Table 2-4. Eight feasible real solutions.

| Case | $L_1$, inches | $L_2$, inches |
|---|---|---|
| 6 | -10.5742 | -4.6685 |
| 7 | -10.5716 | 4.6687 |
| 8 | -5.1348 | 10.3382 |
| 9 | -5.1343 | -10.3382 |
| 16 | 5.1330 | -10.3382 |
| 17 | 5.1334 | 10.3382 |
| 18 | 10.5735 | 4.6686 |
| 19 | 10.5755 | -4.6685 |

The spring $L_2$ is at an extreme limit value in every one of the cases listed in Table 2-4. When $|L_2|$ = 10.3382 in., strut $a_{34}$, tie $a_{41}$, and the spring $L_2$ are collinear. When $|L_2|$ = 4.668, strut $a_{12}$, tie $a_{23}$, and spring $L_2$ are collinear. Both configurations are shown in Figure 2-9.

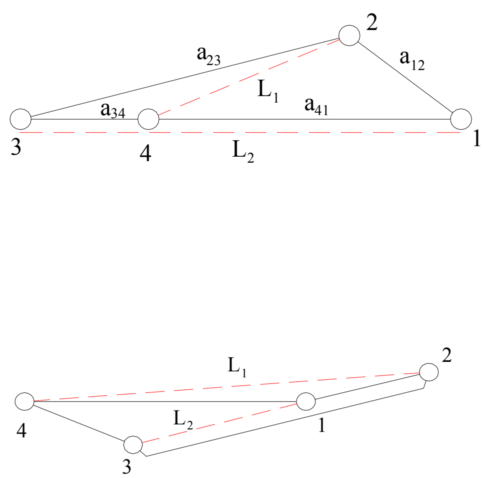


Figure 2-9. Realizable configurations for case 2.

### 2.3.5 Discussion of Results

It is apparent that obtaining values of $L_1$ and $L_2$ that simultaneously satisfy Equations 2-13 and 2-21 is a necessary condition for equilibrium. Satisfying these equations is not sufficient, however, to guarantee that the system is physically realizable.

The real values for $L_1$ and $L_2$ that were calculated in the second example, but which violate the conditions of Equation 2-29 are an interesting case. Here some or all the angles $\theta_4$, $\theta_4'$ and $\theta_4''$ will be complex, yet the condition of Equation 2-1 can still be satisfied.

Presented above was the technique to obtain all possible equilibrium positions of a planar tensegrity system that incorporates two compliant members. The approach of satisfying geometric constraints while simultaneously finding positions where the derivative of the total potential energy with respect to the generalized coordinate equaled zero resulted in a 28$^{th}$ degree polynomial in a single variable. Although the resulting polynomial was of higher degree than anticipated, an analysis of the real and complex solutions indicates that no extraneous solutions were introduced during the variable elimination procedure. Complex solutions satisfy the geometry equation but they can not be used to construct the structure, hence they are not the solution.

## 2.4 Approach 2: Determine (cos $\theta_4$ and cos $\theta_1$) to Minimize Potential Energy

The objective of this approach is to investigate, in closed-form, possible equilibrium positions using the cosine of two angles, $\theta_4$ and $\theta_1$, as the descriptive parameters for the system.

Figure 2-10 shows the tensegrity system which is comprised of two struts (compression members $a_{12}$ and $a_{34}$), two non-compliant ties (tension members $a_{41}$ and $a_{23}$), and two elastic tensile members (springs), one connected between points 1 and 3 and one between points 2 and 4. It should be noted in Figure 2-10 that strut $a_{34}$ passes through a slit cut in strut $a_{12}$ and as such the two struts do not intersect or collide.

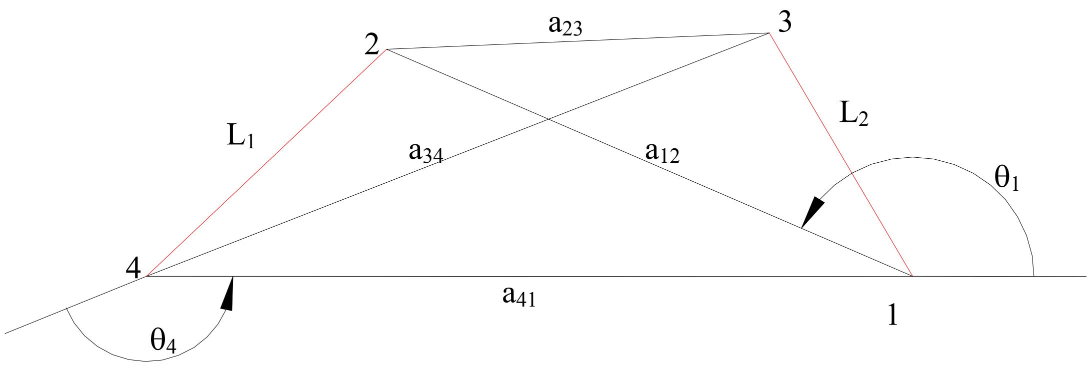


Figure 2-10. Tensegrity system with two struts, two ties and two springs.

The problem statement is written as

given $a_{41}$, $a_{12}$, $a_{23}$, $a_{34}$

$k_1$, $k_2$, $L_{01}$, $L_{02}$

find $\cos\theta_4$ (and corresponding value of $\cos\theta_1$) when the system is in equilibrium

The solution approach is outlined as follows:

1. Obtain expression for $\cos\theta_1$ in terms of $\cos\theta_4$
2. Write $L_1$ in terms of $\cos\theta_1$ and $L_2$ in terms of $\cos\theta_4$
3. Write the potential energy equation
4. Obtain an equation in the variables $\cos\theta_4$ and $\cos\theta_1$ which corresponds to the condition $dU/d\cos\theta_4 = 0$
5. Utilize Sylvester' method to obtain values for $\cos\theta_4$ and $\cos\theta_1$ that satisfy the equation from step 1 and step 4.

### 2.4.1 Obtain Expression for $\cos\theta_1$ in terms of $\cos\theta_4$

The cosine law for the quadrilateral 1-2-3-4 is written as

$$Z_{41} = \frac{a_{23}^2}{2} \tag{2-31}$$

where by definition

$$Z_{41} = -a_{12}(X_4 s_1 + Y_4 c_1) + Z_4 + \frac{a_{12}^2}{2} \tag{2-32}$$

$$X_4 = a_{34}\, s_4 \tag{2-33}$$

$$Y_4 = -\,(a_{41} + a_{34}\, c_4) \tag{2-34}$$

$$Z_4 = \frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} c_4\ . \tag{2-35}$$

Substituting Equation 2-31 into 2-32 and rearranging gives

$$a_{12} X_4 s_1 = -a_{12} Y_4 c_1 + Z_4 + \frac{a_{12}^2}{2} - \frac{a_{23}^2}{2} \tag{2-36}$$

Substituting for $X_4$, $Y_4$, and $Z_4$ and then squaring this equation and substituting $s_4^2=1-c_4^2$ and $s_1^2=1-c_1^2$ yields

$$A\, c_1^2 + B\, c_1 + D = 0 \tag{2-37}$$

where

$$A = A_1\, c_4 + A_2$$

$$A_1 = -2\, a_{12}^2\, a_{34}\, a_{41}$$

$$A_2 = -a_{12}^2\, a_{34}^2 - a_{12}^2\, a_{41}^2$$

$$B = B_1\, c_4^2 + B_2\, c_4 + B_3$$

$$B_1 = -\,2\, a_{12}\, a_{41}\, a_{34}^2$$

$$B_2 = a_{12}\, a_{34}\, (-a_{34}^2 - a_{12}^2 - 3\, a_{41}^2 + a_{23}^2)$$

$$B_3 = a_{12}\, a_{41}\, (-a_{41}^2 - a_{12}^2 - a_{34}^2 + a_{23}^2)$$

$D = D_1\, c_4^2 + D_2\, c_4 + D_3$

$D_1 = -\, a_{34}^2\, a_{41}^2 - a_{12}^2\, a_{34}^2$

$D_2 = a_{34}\, a_{41}\, (-a_{41}^2 - a_{12}^2 - a_{34}^2 + a_{23}^2)$

$D_3 = (-a_{12}^2 + a_{23}^2 - 2\, a_{12}\, a_{34} - a_{34}^2 - a_{41}^2)\, (-a_{12}^2 + a_{23}^2 + 2\, a_{12}\, a_{34} - a_{34}^2 - a_{41}^2) / 4$

(2.38)

Equation (2-37) provides a relationship between $c_1$ and $c_4$.

### 2.4.2: Write $L_1$ in terms of $\cos\theta_1$ and $L_2$ in terms of $\cos\theta_4$

A cosine law for triangle 4-1-2 that expresses $L_1$ in terms of $c_1$ may be written as

$$\frac{a_{12}^2}{2} + \frac{a_{41}^2}{2} + a_{12}a_{41}c_1 = \frac{L_1^2}{2} \quad . \tag{2-39}$$

A cosine law for the triangle 3-4-1 that expresses $L_2$ in terms of $c_4$ may be written as

$$\frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34}a_{41}c_4 = \frac{L_2^2}{2} \quad . \tag{2-40}$$

### 2.4.3: Write the potential energy equation

The energy stored in the springs is given by

$$U = ½\, k_1\, (L_1 - L_{01})^2 + ½\, k_2\, (L_2 - L_{02})^2 \tag{2-41}$$

### 2.4.4: Express the equation $dU/d\cos\theta_4 = 0$ in terms of the variables $c_4$ and $c_1$

The derivative of the potential energy U with respect to $c_4$ may be written as

$$\frac{dU}{dc_4} = \frac{d\left(\frac{1}{2}k_1(L_1 - L_{01})^2\right)}{dL_1}\frac{dL_1}{dc_1}\frac{dc_1}{dc_4} + \frac{d\left(\frac{1}{2}k_2(L_2 - L_{02})^2\right)}{dL_2}\frac{dL_2}{dc_4} \tag{2-42}$$

$$\frac{dU}{dc_4} = k_1(L_1 - L_{01})\frac{dL_1}{dc_1}\frac{dc_1}{dc_4} + k_2(L_2 - L_{02})\frac{dL_2}{dc_4} \tag{2-43}$$

From Equations 2-39 and 2-40

$$\frac{dL_1}{dc_1} = \frac{a_{12}\, a_{41}}{L_1} \tag{2-44}$$

$$\frac{dL_2}{dc_4} = \frac{a_{34}\, a_{41}}{L_2} \tag{2-45}$$

From Equation 2-37

$$\frac{dc_1}{dc_4} = \frac{A_1 c_1^{\,2} + 2B_1 c_1 c_4 + B_2 c_1 + 2D_1 c_4 + D_2}{2A_1 c_1 c_4 + 2A_2 c_1 + B_1 c_4^{\,2} + B_2 c_4 + B_3} \tag{2-46}$$

Substituting Equations 2-44, 2-45, and 2-46 into 2-43 gives

$$\frac{dU}{dc_4} = k_1(L_1 - L_{01})\left(\frac{a_{12}a_{41}}{L_1}\right)\left(\frac{A_1 c_1^{\,2} + 2B_1 c_1 c_4 + B_2 c_1 + 2D_1 c_4 + D_2}{2A_1 c_1 c_4 + 2A_2 c_1 + B_1 c_4^{\,2} + B_2 c_4 + B_3}\right) + k_2(L_2 - L_{02})\left(\frac{a_{34}a_{41}}{L_2}\right) \tag{2-47}$$

Equating Equation 2-47 to zero, dividing throughout by $a_{41}$ and rearranging gives

$$k_1(L_1 - L_{01})L_2\, a_{12}\, F + k_2(L_2 - L_{02})L_1\, a_{34}\, G = 0 \tag{2-48}$$

where

$$F = A_1 c_1^{\,2} + 2B_1 c_1 c_4 + B_2 c_1 + 2D_1 c_4 + D_2 \tag{2-49}$$

$$G = 2A_1 c_1 c_4 + 2A_2 c_1 + B_1 c_4^{\,2} + B_2 c_4 + B_3 \quad . \tag{2-50}$$

Rearranging this Equation 2-48 gives

$$L_1 L_2 [k_1 a_{12} F + k_2 a_{34} G] = L_1 [k_2 L_{02} a_{34} G] + L_2 [k_1 L_{01} a_{12} F] \quad . \tag{2-51}$$

Squaring both sides and rearranging gives

$$L_1^{\,2} L_2^{\,2} [k_1 a_{12} F + k_2 a_{34} G]^2 - L_1^{\,2} [k_2 L_{02} a_{34} G]^2 - L_2^{\,2} [k_1 L_{01} a_{12} F]^2 = 2L_1 L_2 [k_2 L_{02} a_{34} G][k_1 L_{01} a_{12} F] \tag{2-52}$$

Squaring both sides again gives

$$L_1^{\,4} L_2^{\,4} [k_1 a_{12} F + k_2 a_{34} G]^4 + L_1^{\,4} [k_2 L_{02} a_{34} G]^4 + L_2^{\,4} [k_1 L_{01} a_{12} F]^4$$
$$-2 L_1^{\,4} L_2^{\,2} [k_1 a_{12} F]^2 [k_2 L_{02} a_{34} G]^2 - 2 L_1^{\,2} L_2^{\,4} [k_1 a_{12} F]^2 [k_1 L_{01} a_{12} F]^2$$
$$+2 L_1^{\,2} L_2^{\,2} [k_2 L_{02} a_{34} G]^2 [k_1 L_{01} a_{12} F]^2$$
$$= 4 L_1^{\,2} L_2^{\,2} [k_2 L_{02} a_{34} G]^2 [k_1 L_{01} a_{12} F]^2 \quad . \qquad (2\text{-}53)$$

Rearranging gives,

$$L_1^{\,4} L_2^{\,4} [k_1 a_{12} F + k_2 a_{34} G]^4 + L_1^{\,4} [k_2 L_{02} a_{34} G]^4 + L_2^{\,4} [k_1 L_{01} a_{12} F]^4$$
$$-2 L_1^{\,4} L_2^{\,2} [k_1 a_{12} F]^2 [k_2 L_{02} a_{34} G]^2 - 2 L_1^{\,2} L_2^{\,4} [k_1 a_{12} F]^2 [k_1 L_{01} a_{12} F]^2$$
$$-2 L_1^{\,2} L_2^{\,2} [k_2 L_{02} a_{34} G]^2 [k_1 L_{01} a_{12} F]^2 = 0 \quad . \qquad (2\text{-}54)$$

Substituting for $L_1^2$ and $L_2^2$ using Equations 2-39 and 2-40 and factoring the polynomial in terms of $c_4$ and $c_1$ gives

$$C_{10}\, c_1^{10} + C_9\, c_1^9 + C_8\, c_1^8 + C_7\, c_1^7 + C_6\, c_1^6 + C_5\, c_1^5$$
$$+ C_4\, c_1^4 + C_3\, c_1^3 + C_2\, c_1^2 + C_1 c_1 + C_0 = 0 \qquad (2\text{-}55)$$

where

$$C_{10} = C_{10,2}\, c_4^2 + C_{10,1}\, c_4 + C_{10,0}$$

$$C_9 = C_{9,3}\, c_4^3 + C_{9,2}\, c_4^2 + C_{9,1}\, c_4 + C_{9,0}$$

$$C_8 = C_{8,4}\, c_4^4 + C_{8,3}\, c_4^3 + C_{8,2}\, c_4^2 + C_{8,1}\, c^4 + C_{8,0}$$

$$C_7 = C_{7,5}\, c_4^5 + C_{7,4}\, c_4^4 + C_{7,3}\, c_4^3 + C_{7,2}\, c_4^2 + C_{7,1}\, c^4 + C7_{,0}$$

$$C_6 = C_{6,6}\, c_4^6 + C_{6,5}\, c_4^5 + C_{6,4}\, c_4^4 + C_{6,3}\, c_4^3 + C_{6,2}\, c_4^2 + C_{6,1}\, c4 + C_{6,0}$$

$$C_5 = C_{5,7}\, c_4^7 + C_{5,6}\, c_4^6 + C_{5,5}\, c_4^5 + C_{5,4}\, c_4^4 + C_{5,3}\, c_4^3 + C_{5,2}\, c_4^2 + C_{5,1}\, c^4 + C_{5,0}$$

$$C_4 = C_{4,8}\, c_4^8 + C_{4,7}\, c_4^7 + C_{4,6}\, c_4^6 + C_{4,5}\, c_4^5 + C_{4,4}\, c_4^4 + C_{4,3}\, c_4^3 + C_{4,2}\, c_4^2 + C_{4,1}\, c_4 + C_{4,0}$$

$$C_3 = C_{3,9}\, c_4^9 + C_{3,8}\, c_4^8 + C_{3,7}\, c_4^7 + C_{3,6}\, c_4^6 + C_{3,5}\, c_4^5 + C_{3,4}\, c_4^4 + C_{3,3}\, c_4^3 + C_{3,2}\, c_4^2 + C_{3,1}\, c_4 + C_{3,0}$$

$$C_2 = C_{2,10}\, c_4^{10} + C_{2,9}\, c_4^{9} + C_{2,8}\, c_4^{8} + C_{2,7}\, c_4^{7} + C_{2,6}\, c_4^{6} + C_{2,5}\, c_4^{5} + C_{2,4}\, c_4^{4} + C_{2,3}\, c_4^{3} + C_{2,2}\, c_4^{2} + C_{2,1}\, c_4 + C_{2,0}$$

$$C_1 = C_{1,10}\, c_4^{10} + C_{1,9}\, c_4^{9} + C_{1,8}\, c_4^{8} + C_{1,7}\, c_4^{7} + C_{1,6}\, c_4^{6} + C_{1,5}\, c_4^{5} + C_{1,4}\, c_4^{4} + C_{1,3}\, c_4^{3} + C_{1,2}\, c_4^{2} + C_{1,1}\, c_4 + C_{1,0}$$

$$C_0 = C_{0,10}\, c_4^{10} + C_{0,9}\, c_4^{9} + C_{0,8}\, c_4^{8} + C_{0,7}\, c_4^{7} + C_{0,6}\, c_4^{6} + C_{0,5}\, c_4^{5} + C_{0,4}\, c_4^{4} + C_{0,3}\, c_4^{3} + C_{0,2}\, c_4^{2} + C_{0,1}\, c_4 + C_0 \quad (2\text{-}56)$$

All the terms $C_{i,j}$ have been obtained symbolically and are defined in terms of known mechanism parameters. For example,

$$C_{10,0} = 8\, k_1^4\, a_{12}^6\, a_{41}^4\, a_{34}^2 A_1^4 + 4k_1^4\, a_{12}^6\, a_{41}^6\, A_1^4 + 4k_1^4\, a_{12}^6\, a_{41}^2\, a_{34}^4\, A_1^4$$

$$C_{10,1} = 16k_1^4\, a_{12}^6\, a_{41}^5\, a_{34}\, A_1^4 + 16\, k_1^4\, a_{12}^6\, a_{41}^3\, a_{34}^3\, A_1^4$$

$$C_{10,2} = 16\, k_1^4\, a_{12}^6\, a_{41}^4\, a_{34}^2\, A_1^4\, c_4^2 \quad (2\text{-}57)$$

The remaining terms are quite lengthy and are not listed here.

**2.4.5 Construct Sylvester Matrix for this approach**

Multiplying Equation 2-37 by $c_1, c_1^2, c_1^3, c_1^4, c_1^5, c_1^6, c_1^7, c_1^8, c_1^9$ and 2-55 by $c_1$ gives 12 equations that can be written as follows

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & D \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & D & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & D & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A & B & D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & B & D & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A & B & D & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & A & B & D & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A & B & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A & B & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ A & B & D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & C_{10} & C_9 & C_8 & C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 \\ C_{10} & C_9 & C_8 & C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 & C_0 & 0 \end{bmatrix} \begin{bmatrix} c_1^{11} \\ c_1^{10} \\ c_1^{9} \\ c_1^{8} \\ c_1^{7} \\ c_1^{6} \\ c_1^{5} \\ c_1^{4} \\ c_1^{3} \\ c_1^{2} \\ c_1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} . \quad (2\text{-}58)$$

Expansion of the 12×12 determinant yields a $32^{nd}$ degree polynomial in the parameter $c_4$.

### 2.4.6 Numerical Example for Approach 2

Parameters of Example 1 (section 2.3.4) are used for this numerical example:

| | | |
|---|---|---|
| Strut lengths | $a_{12}$ = 3 in. | $a_{34}$ = 3.5 in. |
| Non-compliant tie lengths | $a_{41}$ = 4 in. | $a_{23}$ = 2 in. |
| Spring 1 free length & spring constant | $L_{01}$ = 0.5 in. | $k_1$ = 4 lbf/in. |
| Spring 2 free length & spring constant | $L_{02}$ = 1 in. | $k_2$ = 2.5 lbf/in. |

For this example, a Maple program was written that would solve the determinant of Sylvester matrix in Equation 2-58. The result was a polynomial of degree 32 with respect to the variable $\cos\theta_4$. The solution to the polynomial for this numerical example resulted in 12 real and 20 complex roots. Eight of the real roots were in acceptable range of -1 to +1. For each value of $\cos\theta_4$ the corresponding value of $\cos\theta_1$ was calculated such that Equations 2-37 and 2-55 are simultaneously satisfied. These values, as well as calculated values for $L_1$ and $L_2$, are presented in Table 2-5.

Table 2-5.Eight feasible real solutions.

| Case | $c_4$ (radian) | $c_1$ (radian) | $L_1$ (inches) | $L_2$ (inches) |
|---|---|---|---|---|
| 1 | -0.8144902900 | 0.2120649698 | -5.4853950884 | 2.3332963547 |
| 2 | -0.7083900124 | 0.1385596475 | -5.3221641782 | -2.9008756697 |
| 3 | -0.9290901467 | -0.9154296793 | -1.7405998094 | -1.4951507917 |
| 4 | -0.8840460933 | -0.9381755562 | -1.5760033787 | 1.8699490332 |
| 5 | -0.9046343262 | -0.9312332602 | 1.6280054527 | 1.7088706403 |
| 6 | -0.9434161072 | -0.8970754537 | 1.8628443599 | -1.3543814070 |
| 7 | -0.6228159543 | 0.0544134519 | 5.1289299905 | -3.2880318246 |
| 8 | -0.8042767224 | 0.2077177765 | 5.4758767915 | 2.3937944296 |

Complex values for $c_1$ that correspond to each complex value of $c_4$ were determined and are presented in Table 2-6. Although these solutions are not realizable, it is important to show that since the $c_4$, $c_1$ pairs satisfy the two equations, no extraneous solutions were introduced in the variable elimination method.

Table 2-6.Twenty complex solutions.

| Case | $c_4$ | $c_1$ |
|---|---|---|
| 1 | -1.2513573 - 0.0853382 I | 1.0200425 + 0.0625503 I |
| 2 | -1.2513573 + 0.0853382 I | 1.0200425 - 0.0625503 I |
| 3 | -1.2371852 - 0.0586866 I | 1.0164651 + 0.0379136 I |
| 4 | -1.2371852 + 0.0586866 I | 1.0164651 - 0.0379136 I |
| 5 | -1.0662774 - 0.0222999 I | -1.0214122 + 0.0110317 I |
| 6 | -1.0662774 + 0.0222999 I | -1.0214122 - 0.0110317 I |
| 7 | -1.0406024 - 0.0082765 I | -1.0447401 + 0.0104582 I |
| 8 | -1.0406024 + 0.0082765 I | -1.0447401 - 0.0104582 I |
| 9 | -1.0322412 - 0.0344383 I | -1.0269253 + 0.0377413 I |
| 10 | -1.0322412 + 0.0344383 I | -1.0269253 - 0.0377413 I |
| 11 | -1.0087191 - 0.0028359 I | -1.1658773 + 0.0338462 I |

Table 2-6. Continued.

| Case | $c_4$ | $c_1$ |
|---|---|---|
| 12 | -1.0087191 + 0.0028359 I | -1.1658773 - 0.0338462 I |
| 13 | 0.3360373 - 0.4127430 I | -0.9379955 + 0.6334257 I |
| 14 | 0.3360373 + 0.4127430 I | -0.9379955 - 0.6334257 I |
| 15 | 0.3571365 - 0.1068179 I | -0.9796431 + 0.3840155 I |
| 16 | 0.3571365 + 0.1068179 I | -0.9796431 - 0.3840155 I |
| 17 | 0.5105685 - 0.0343981 I | -1.0755053 - 0.2990613 I |
| 18 | 0.5105685 + 0.0343981 I | -1.0755053 + 0.2990613 I |
| 19 | 0.5418577 - 0.1704979 I | -1.1122907 + 0.4295136 I |
| 20 | 0.5418577 + 0.1704979 I | -1.1122907 - 0.4295136 I |

Lastly, there were four real values of $\cos\theta_4$ that were not in the range $-1 \leq \cos\theta_4 \leq 1$. These values are listed in Table 2-7. Corresponding values for $\cos\theta_1$ could not be obtained that would satisfy both Equations 2-37 and 2-48. Thus it must be concluded that four extraneous roots were introduced in this second solution approach.

Table 2-7. Four non-feasible real solutions.

| Case | $c_4$ (radian) |
|---|---|
| 1 | $-0.7460150\ 10^{18}$ |
| 2 | -1.0119084 |
| 3 | -1.0061580 |
| 4 | $0.7460150\ 10^{18}$ |

### 2.4.7 Comparison of Results for Approaches 1 and 2

The numerical example from Section 2.3.4 was examined using this second approach. It was not possible to symbolically expand the determinant of the coefficient matrix in Equation 2-58. Rather the coefficients of the polynomials A, B, D, and $C_1$ through $C_{10}$ were obtained numerically and then the determinant was expanded to obtain the single polynomial equation in $c_4$. The numerical example of the second approach resulted in a 32$^{nd}$ degree polynomial in the variable $c_4$. Table 2-5 shows the resulting values of $c_4$ that solved this polynomial as well as the corresponding values of $c_1$ such that Equations 2-37 and 2-55 are simultaneously satisfied. The real solutions are identified in Table 2-5. Values of Table 2-5 were tested in the differential of potential energy to identify equilibrium cases. Analysis showed that cases 3, 4, 5, and 6 of Table 2-5 represent equilibrium configurations. These cases are identically similar to cases presented in Table 2-1.

Comparison of the results of the numerical example using the two approaches outlined in this chapter showed that both approaches agree on the correct numerical answer. Hence, both approaches are valid in finding equilibrium configuration were closed form solution for this structure can not be found.

# CHAPTER 3
# THREE SPRING PLANAR TENSEGRITY

## 3.1 Introduction

This chapter considers the case of a planar tensegrity system with one non-compliant member, $a_{41}$, three compliant members, springs $L_1$, $L_2$, $L_3$, and two struts, $a_{34}$ and $a_{12}$ (see Figure 3-1). In Figure 3-1 the two struts do not intersect but one passes through a slit cut in the other one. The objective of this study is to find all possible equilibrium configurations in stable condition (minimum potential energy) when given the lengths of the struts and the non-compliant tie together with the free length and spring constants for the three compliant members.

The device shown in Figure 3-1 is a two degree of freedom system. Two parameters must be specified, in addition to the constant mechanism parameters, in order to define the configuration of the device. These two parameters will be referred to as the *descriptive parameters* for the system. One obvious set of descriptive parameters are the angles $\theta_4$ and $\theta_1$. Considering the non-compliant member $a_{41}$ as being fixed to ground, specification of $\theta_4$ will define the location of point 3. Similarly, specification of $\theta_1$ will define the location of point 2.

Two approaches to solve this problem are presented in this chapter. Both aim to find a set of descriptive parameters that minimize the potential energy in the system. In the first approach, the lengths of the compliant members $L_1$ and $L_2$ are chosen as the descriptive parameters. Derivatives of the potential energy equation are obtained with respect to $L_1$ and $L_2$ and values for the descriptive parameters are obtained such that these derivatives are zero, corresponding to either a minimum or maximum potential energy state. In the second approach, the cosines of the angles $\theta_4$ and $\theta_1$ were chosen as the descriptive parameters. The cosines of the angles were chosen rather than the angles themselves in the hope that the resulting equations would be simpler in that, for example, a single value of $\cos\theta_4$ accounts for the obvious symmetry in solutions that will occur with respect to the fixed member $a_{41}$.

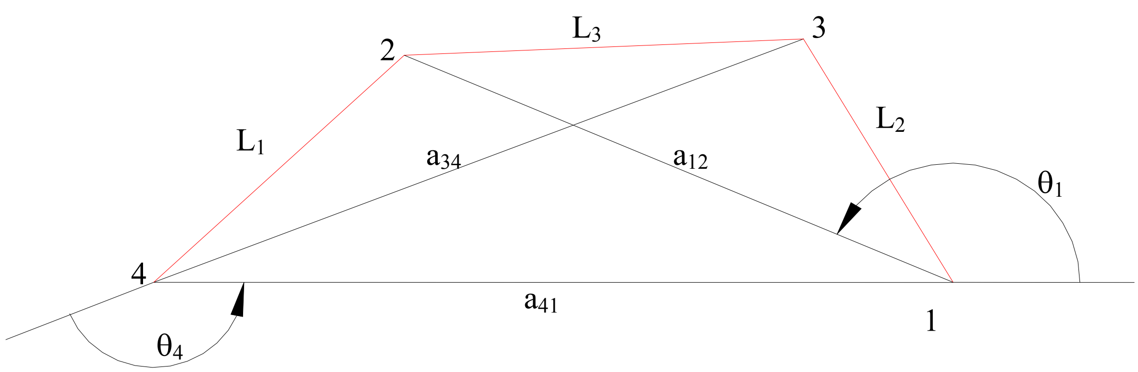


Figure 3-1. Tensegrity structure with two struts, a non-compliant (tie), and three springs.

### 3.2 First Approach Problem Statement (Descriptive Parameters $L_1$, $L_2$)

The problem statement can be explicitly written as:

| | | |
|---|---|---|
| Given: | $a_{12}$, $a_{34}$ | lengths of struts, |
| | $a_{41}$ | length of non-compliant tie |
| | $k_1$, $L_{01}$ | spring constant and free length of compliant tie between points 4 and 2 |
| | $k_2$, $L_{02}$ | spring constant and free length of compliant tie between points 3 and 1 |
| | $k_3$, $L_{03}$ | spring constant and free length of compliant tie between points 2 and 3 |
| Find: | $L_1$ | length of spring 1 at equilibrium position, |
| | $L_2$ | length of spring 2 at equilibrium position, |

$L_3$ corresponding length of spring 3 at equilibrium position

### 3.2.1 Development of Geometric Constraint Equation

Figure 3-2 shows the nomenclature that is used. $L_1$, $L_2$, and $L_3$ are the extended lengths of the compliant ties between points 4 and 2, points 3 and 1, and points 2 and 3.

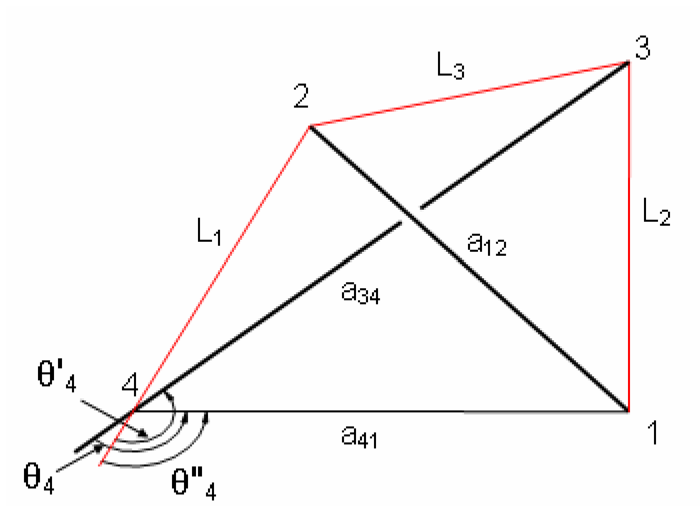


Figure 3-2. Planar tensegrity structure.

The analysis starts by considering different triangles in this structure. Figure 3-3 shows the triangle formed by side $a_{34}$, $L_3$, and $L_1$.

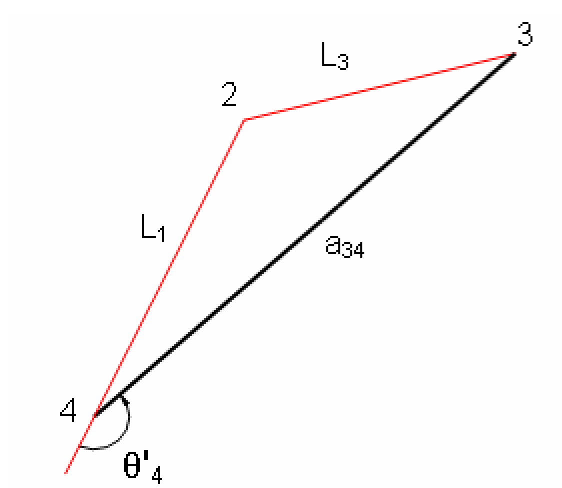


Figure 3-3. Triangle 4-3-2.

A cosine law for this triangle can be written as

$$\frac{L_1^2}{2} + \frac{a_{34}^2}{2} + L_1 a_{34} \cos\theta_4' = \frac{L_3^2}{2} \quad . \qquad (3\text{-}1)$$

Solving for $\cos\theta_4'$ yields

$$\cos\theta_4' = \frac{L_3^2 - L_1^2 - a_{34}^2}{2 L_1 a_{34}} \quad . \qquad (3\text{-}2)$$

Figure 3-4 shows the triangle formed by side $a_{41}$, $a_{12}$, and $L_1$. A cosine law for this triangle can be written as

$$\frac{L_1^2}{2} + \frac{a_{41}^2}{2} + L_1 a_{41} \cos\theta_4'' = \frac{a_{12}^2}{2} \qquad (3\text{-}3)$$

Solving for $\cos\theta_4''$ yields

$$\cos\theta_4'' = \frac{a_{12}^2 - L_1^2 - a_{41}^2}{2 L_1 a_{41}} \quad . \qquad (3\text{-}4)$$

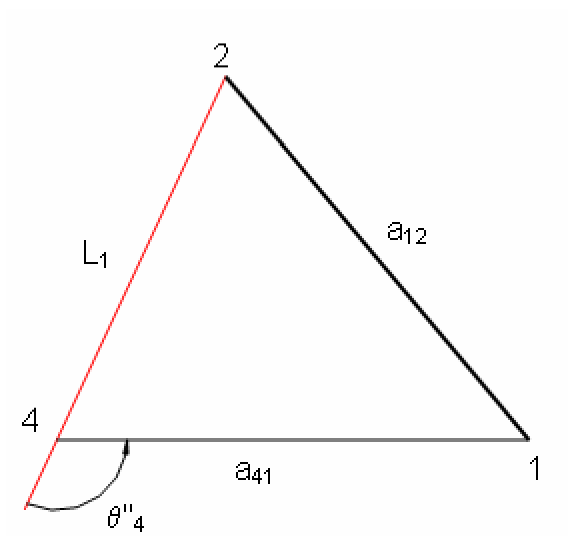


Figure 3-4. Triangle 4-1-2.

Figure 3-5 shows the triangle formed by $a_{41}$, $a_{34}$, and $L_2$. A cosine law for this triangle can be written as

$$\frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34}\, a_{41} \cos\theta_4 = \frac{L_2^2}{2} \quad . \tag{3-5}$$

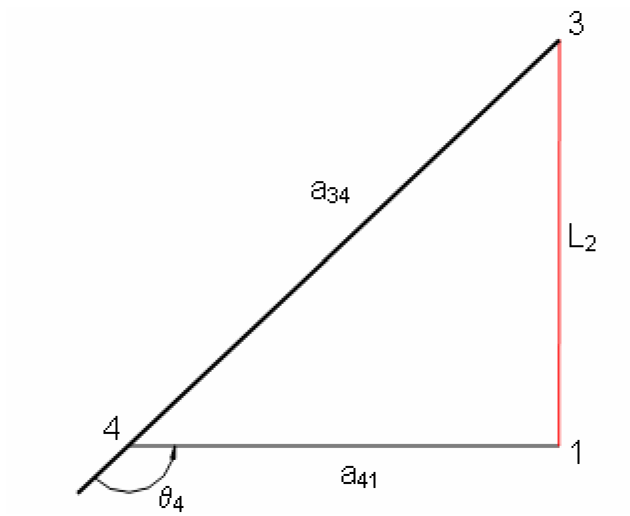


Figure 3-5. Triangle 4-1-3.

Solving for $\cos\theta_4$ yields

$$\cos\theta_4 = \frac{L_2^2 - a_{34}^2 - a_{41}^2}{2 a_{34}\, a_{41}} \quad . \tag{3-6}$$

From Figure 3-2 it is apparent that

$$\theta_4 + \theta_4' = \pi + \theta_4'' \tag{3-7}$$

Equating the cosine of the left and right sides of Equation 3-7 yields

$$\cos(\theta_4 + \theta_4') = \cos(\pi + \theta_4'') \tag{3-8}$$

and expanding this Equation 3-8 yields

$$\cos\theta_4 \cos\theta_4' - \sin\theta_4 \sin\theta_4' = -\cos\theta_4'' \ . \tag{3-9}$$

Rearranging (3-9) gives

$$\cos\theta_4 \cos\theta_4' + \cos\theta_4'' = \sin\theta_4 \sin\theta_4' \ . \tag{3-10}$$

Squaring both sides yields

$$(\cos\theta_4)^2\,(\cos\theta_4')^2 + 2\cos\theta_4\cos\theta_4'\cos\theta_4'' + (\cos\theta_4'')^2 = (\sin\theta_4)^2\,(\sin\theta_4')^2 \qquad (3\text{-}11)$$

Substituting for $(\sin\theta_4)^2$ and $(\sin\theta_4')^2$ in terms of $\cos\theta_4$ and $\cos\theta_4'$ gives

$$(\cos\theta_4)^2\,(\cos\theta_4')^2 + 2\cos\theta_4\cos\theta_4'\cos\theta_4'' + (\cos\theta_4'')^2 = (1-\cos^2\theta_4)\,(1-\cos^2\theta_4') \qquad (3\text{-}12)$$

Equations 3-2, 3-4, and 3-6 are substituted into Equation 3-12 to yield a single equation in the parameters $L_1$, $L_2$, and $L_3$ which can be written as

$$G_1 L_3^4 + (G_2 L_2^2 + G_3)\,L_3^2 + (G_4 L_2^4 + G_5 L_2^2 + G_6) = 0 \qquad (3\text{-}13)$$

where

$$G_1 = a_{41}^2,$$

$$G_2 = G_{2a} L_1^2 + G_{2b},$$

$$G_3 = G_{3a} L_1^2 + G_{3b},$$

$$G_4 = G_{4a} L_1^2,$$

$$G_5 = G_{5a} L_1^4 + G_{5b} L_1^2 + G_{5c},$$

$$G_6 = G_{6a} L_1^2 + G_{6b}\,. \qquad (3\text{-}14)$$

and where

$$G_{2a} = -1, \qquad G_{2b} = a_{12}^2 - a_{41}^2,$$

$$G_{3a} = (a_{34}^2 - a_{41}^2), \qquad G_{3b} = a_{41}^2\,(a_{41}^2 - a_{12}^2 - a_{34}^2) - a_{12}^2\,a_{34}^2,$$

$$G_{4a} = 1,$$

$$G_{5a} = 1, \qquad G_{5b} = (-a_{12}^2 - a_{34}^2 - a_{41}^2), \qquad G_{5c} = a_{34}^2\,(a_{41}^2 - a_{12}^2),$$

$$G_{6a} = a_{12}^2\,(a_{41}^2 - a_{34}^2), \qquad G_{6b} = a_{12}^2\,a_{34}^2\,(a_{12}^2 + a_{34}^2 - a_{41}^2)\,. \qquad (3\text{-}15)$$

### 3.2.2 Development of Potential Energy Equations

The potential energy of the system can be written as

$$U = \frac{1}{2}k_1\,(L_1 - L_{01})^2 + \frac{1}{2}k_2\,(L_2 - L_{02})^2 + \frac{1}{2}k_3\,(L_3 - L_{03})^2 \qquad (3\text{-}16)$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the descriptive parameters $L_1$ and $L_2$ both equal zero. The geometric constraint equation, Equation 3-13, contains three unknown terms, $L_1$, $L_2$, and $L_3$. From this equation, $L_3$ can be considered as a dependent variable of $L_1$ and $L_2$. The following two expressions may be written:

$$\frac{\partial U}{\partial L_1} = k_1 (L_1 - L_{01}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_1} = 0 \ , \tag{3-17}$$

$$\frac{\partial U}{\partial L_2} = k_2 (L_2 - L_{02}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_2} = 0 \ . \tag{3-18}$$

The derivatives $\partial L_3/\partial L_1$ and $\partial L_3/\partial L_2$ can be determined via implicit differentiation from Equation 3-13 as

$$\frac{\partial L_3}{\partial L_1} = \frac{-L_1 [2 L_1^2 L_2^2 + G_{2a} L_2^2 L_3^2 + G_{3a} L_3^2 + L_2^4 + G_{5b} L_2^2 + G_{6a}]}{L_3 [2 G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} , \tag{3-19}$$

$$\frac{\partial L_3}{\partial L_2} = \frac{-L_2 [2 L_1^2 L_2^2 + G_{2a} L_1^2 L_3^2 + G_{2b} L_3^2 + L_1^4 + G_{5b} L_1^2 + G_{5c}]}{L_3 [2 G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} \ . \tag{3-20}$$

Substituting (3-19) into (3-17) and rearranging gives

$$(D_1 L_2^2 + D_2) L_3^3 + (D_3 L_2^2 + D_4) L_3^2 + (D_5 L_2^4 + D_6 L_2^2 + D_7) L_3 + (D_8 L_2^4 + D_9 L_2^2 + D_{10}) = 0 \tag{3-21}$$

where

$D_1 = D_{1a} L_1$ ,

$D_2 = D_{2a} L_1 + D_{2b}$ ,

$D_3 = D_{3a} L_1$ ,

$D_4 = D_{4a} L_1$ ,

$D_5 = D_{5a} L_1$ ,

$D_6 = D_{6a} L_1^3 + D_{6b} L_1^2 + D_{6c} L_1 + D_{6d}$ ,

$D_7 = D_{7a} L_1^3 + D_{7b} L_1^2 + D_{7c} L_1 + D_{7d}$ ,

$D_8 = D_{8a} L_1$ ,

$D_9 = D_{9a} L_1^3 + D_{9b} L_1$

$D_{10} = D_{10a} L_1$ (3.22)

and

$D_{1a} = - G_{2a} k_3$ ,

$D_{2a} = 2 G_1 k_1 – G_{3a} k_3$, $D_{2b} = - 2 G_1 k_1 L_{01}$ ,

$D_{3a} = G_{2a} k_3 L_{03}$ ,

$D_{4a} = G_{3a} k_3 L_{03}$ ,

$D_{5a} = - k_3$ ,

$D_{6a} = G_{2a} k_1 – 2 k_3$ , $D_{6b} = -G_{2a} k_1 L_{01}$ , $D_{6c} = G_{2b} k_1 – G_{5b} k_3$ , $D_{6d} = -G_{2b} k_1 L_{01}$ ,

$D_{7a} = G_{3a} k_1$ , $D_{7b} = -G_{3a} k_1 L_{01}$ , $D_{7c} = G_{3b} k_1 – G_{6a} k_3$ , $D_{7d} = -G_{3b} k_1 L_{01}$ ,

$D_{8a} = k_3 L_{03}$ ,

$D_{9a} = 2 k_3 L_{03}$ , $D_{9b} = G_{5b} k_3 L_{03}$

$D_{10a} = G_{6a} k_3 L_{03}$ (3-23)

Substituting (3-20) into (3-18) and rearranging gives

$$(E_1 L_2 + E_2) L_3^3 + (E_3 L_2) L_3^2 + (E_4 L_2^3 + E_5 L_2^2 + E_6 L_2 + E_7) L_3 + (E_8 L_2^3 + E_9 L_2) = 0$$

(3-24)

where

$E_1 = E_{1a} L_1^2 + E_{1b}$ ,

$E_2 = - 2 G_1 k_2 L_{02}$ ,

$E_3 = E_{3a} L_1^2 + E_{3b}$ ,

$E_4 = E_{4a} L_1^2 + E_{4b}$ ,

$E_5 = E_{5a} L_1^2 + E_{5b}$ ,

$E_6 = E_{6a} L_1^4 + E_{6b} L_1^2 + E_{6c}$ ,

$E_7 = E_{7a} L_1^2 + E_{7b}$ ,

$E_8 = E_{8a} L_1^2$ ,

$E_9 = E_{9a} L_1^4 + E_{9b} L_1^2 + E_{9c}$ (3-25)

and

$E_{1a} = -G_{2a} k_3$ , $E_{1b} = -G_{2b} k_3 + 2 G_1 k_2$ ,

$E_{3a} = G_{2a} k_3 L_{03}$ , $E_{3b} = G_{2b} k_3 L_{03}$ ,

$E_{4a} = G_{2a} k_2 - 2 k_3$ , $E_{4b} = G_{2b} k_2$ ,

$E_{5a} = -G_{2a} k_2 L_{02}$ , $E_{5b} = -G_{2b} k_2 L_{02}$ ,

$E_{6a} = - k_3$ , $E_{6b} = G_{3a} k_2 - G_{5b} k_3$ , $E_{6c} = G_{3b} k_2 - G_{5c} k_3$ ,

$E_{7a} = -G_{3a} k_2 L_{02}$ , $E_{7b} = -G_{3b} k_2 L_{02}$ ,

$E_{8a} = 2 k_3 L_{03}$ ,

$E_{9a} = k_3 L_{03}$ , $E_{9b} = G_{5b} k_3 L_{03}$ , $E_{9c} = G_{5c} k_3 L_{03}$ . (3-26)

### 3.2.3 Create Solution Matrix

Equations 3-13, 3-21, and 3-24 are three equations in the three unknowns $L_1$, $L_2$, and $L_3$. Sylvester's method, reference 11, is applied in order to obtain sets of values for these parameters that simultaneously satisfy all three equations. In this solution, the parameter $L_1$ is embedded in the coefficients of the three equations to yield three equations in the apparent unknowns $L_2$ and $L_3$. Determining the condition that these new coefficients (which contain $L_1$) must satisfy such that the three equations can have common roots for $L_2$ and $L_3$ will yield a single polynomial in $L_1$.

Equation 3-13 was multiplied by $L_2$, $L_3$, $L_2L_3$, $L_3^2$, $L_2^2$, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, $L_3^3$, $L_2^3$, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2^3L_3^3$. Equation 3-21 was multiplied by $L_3$, $L_2$, $L_3^2$, $L_2L_3^2$, $L_3^3$, $L_2^2$, $L_2L_3^3$, $L_2^2L_3$, $L_2^2L_3^2$, $L_2^3$, $L_3^4$, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2L_3^4$. Equation 3-24 was multiplied by $L_2$, $L_3$, $L_2L_3$, $L_3^2$, $L_2^2$, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, $L_3^3$, $L_2^3$, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, $L_2^4$, $L_3^4$, $L_2^4L_3$, $L_2^4L_3^2$, $L_2L_3^4$, and $L_2^2L_3^4$. This resulted in a set of 52 equations that can be written in matrix for as

$$\mathbf{M}\,\boldsymbol{\lambda} = \mathbf{0}\,. \qquad (3\text{-}27)$$

The vector $\boldsymbol{\lambda}$ is written as

$$\boldsymbol{\lambda} = [L_2^7L_3^3, L_2^5L_3^5, L_2^3L_3^7, L_2^7L_3^2, L_2^6L_3^3, L_2^5L_3^4, L_2^4L_3^5, L_2^3L_3^6, L_2^2L_3^7, L_2^7L_3, L_2^6L_3^2, L_2^5L_3^3, L_2^4L_3^4, L_2^3L_3^5, L_2^2L_3^6, L_2L_3^7, L_2^7, L_2^6L_3, L_2^5L_3^2, L_2^4L_3^3, L_2^3L_3^4, L_2^2L_3^5, L_2L_3^6, L_3^7, L_2^6, L_2^5L_3, L_2^4L_3^2, L_2^3L_3^3, L_2^2L_3^4, L_2L_3^5, L_3^6, L_2^5, L_2^4L_3, L_2^3L_3^2, L_2^2L_3^3, L_2L_3^4, L_3^5, L_2^4, L_2^3L_3, L_2^2L_3^2, L_2L_3^3, L_3^4, L_2^3, L_2^2L_3, L_2L_3^2, L_3^3, L_2^2, L_2L_3, L_3^2, L_2, L_3, 1]^T\,. \qquad (3\text{-}28)$$

The coefficient matrix **M** is a 52×52 matrix. To show this 52×52 coefficient matrix, it is subdivided into the four 26×26 submatrices $\mathbf{M}_{11}$, $\mathbf{M}_{12}$, $\mathbf{M}_{21}$, $\mathbf{M}_{22}$ as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix}. \qquad (3\text{-}29)$$

The four sub matrices are:

$$
\mathbf{M}_{11} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&G_4 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&D_5 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&D_5&0&D_1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&D_5&0&D_1&0&0&0&0&D_8&0&D_3&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&E_4 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0&E_8 \\
0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0&E_8&E_5&E_3&E_2&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&G_4&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&D_5&0&D_1&0&0&0&0&D_8&0&D_3&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&D_5&0&D_1&0&0&0&0&D_8&0 \\
0&0&0&0&0&D_5&0&D_1&0&0&0&D_8&0&D_3&0&0&0&0&0&0&D_6&0&D_2&0&0&0
\end{bmatrix}
$$

$$
\mathbf{M}_{21} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&D_5&0&D_1&0&0&0&0&D_8&0&D_3&0&0&0&0&0&0 \\
0&0&0&0&D_5&0&D_1&0&0&0&D_8&0&D_3&0&0&0&0&0&0&D_6&0&D_2&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0&E_8&E_5 \\
0&0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0&E_8&E_5&E_3&E_2&0&0&0 \\
0&0&0&0&0&E_4&0&E_1&0&0&0&E_8&E_5&E_3&E_2&0&0&0&0&0&E_6&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0&E_8&E_5&E_3&E_2&0&0&0&0&0 \\
0&0&0&0&E_4&0&E_1&0&0&0&E_8&E_5&E_3&E_2&0&0&0&0&0&E_6&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0&0&0&0&0&0&0&G_5 \\
0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0 \\
G_4&G_2&G_1&0&0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&D_5&0&D_1&0&0&0&0&D_8&0&D_3&0&0&0&0&0&0&D_6 \\
0&0&0&0&0&0&D_5&0&D_1&0&0&0&D_8&0&D_3&0&0&0&0&0&0&D_6&0&D_2&0&0 \\
0&0&0&D_5&0&D_1&0&0&0&D_8&0&D_3&0&0&0&0&0&0&D_6&0&D_2&0&0&0&0&D_9 \\
D_5&D_1&0&D_8&0&D_3&0&0&0&0&0&D_6&0&D_2&0&0&0&0&D_9&0&D_4&0&0&0&0&0 \\
0&D_5&D_1&0&0&D_8&0&D_3&0&0&0&0&0&D_6&0&D_2&0&0&0&0&D_9&0&D_4&0&0&0 \\
0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0&E_8&E_5&E_3&E_2&0&0&0&0&0&E_6 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&0&0&0&E_8&E_5&E_3&E_2&0&0 \\
0&0&0&E_4&0&E_1&0&0&0&E_8&E_5&E_3&E_2&0&0&0&0&0&E_6&0&0&0&0&0&0&E_9 \\
E_4&E_1&0&E_8&E_5&E_3&E_3&0&0&0&0&E_6&0&0&0&0&0&0&E_9&E_7&0&0&0&0&0&0 \\
0&0&0&0&0&0&E_4&0&E_1&0&0&0&E_8&E_5&E_3&E_2&0&0&0&0&E_6&0&0&0&0&0 \\
0&E_4&E_1&0&0&E_8&E_5&E_3&E_2&0&0&0&0&E_6&0&0&0&0&0&0&E_9&E_7&0&0&0&0
\end{bmatrix}
$$

$$
\mathbf{M}_{12} = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&G_5&0&G_3&0&0&G_6 \\
0&0&0&0&0&0&D_5&0&D_1&0&0&D_8&0&D_3&0&0&0&D_6&0&D_2&D_9&0&D_4&0&D_7&D_{10} \\
0&0&0&0&0&0&0&0&0&0&0&0&E_4&0&E_1&0&E_8&E_5&E_3&E_2&0&E_6&0&E_9&E_7&0 \\
D_5&0&D_1&0&0&0&D_8&0&D_3&0&0&0&0&D_6&0&D_2&0&D_9&0&D_4&0&0&D_7&0&D_{10}&0 \\
0&0&0&0&0&0&E_4&0&E_1&0&0&E_8&E_5&E_3&E_2&0&0&E_6&0&0&E_9&E_7&0&0&0&0 \\
0&0&0&0&0&0&0&E_4&0&E_1&0&0&E_8&E_5&E_3&E_2&0&0&E_6&0&0&E_9&E_7&0&0&0 \\
E_4&0&E_1&0&0&0&E_8&E_5&E_3&E_2&0&0&0&E_6&0&0&0&E_9&E_7&0&0&0&0&0&0&0 \\
0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&G_6&0&0 \\
0&0&0&0&0&0&G_4&0&G_2&0&G_1&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&G_6&0 \\
0&G_2&0&G_1&0&0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&G_6&0&0&0&0 \\
0&D_1&0&0&0&D_8&0&D_3&0&0&0&0&D_6&0&D_2&0&D_9&0&D_4&0&0&D_7&0&D_{10}&0&0 \\
D_8&0&D_3&0&0&0&0&0&D_6&0&D_2&0&0&D_9&0&D_4&0&0&0&D_7&0&0&D_{10}&0&0&0 \\
0&D_6&0&D_2&0&0&0&D_9&0&D_4&0&0&0&0&D_7&0&0&0&D_{10}&0&0&0&0&0&0&0 \\
0&E_4&0&E_1&0&0&0&E_8&E_5&E_3&E_2&0&0&0&E_6&0&0&0&E_9&E_7&0&0&0&0&0&0 \\
0&E_1&0&0&0&E_8&E_5&E_3&E_2&0&0&0&E_6&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0 \\
E_8&E_5&E_3&E_2&0&0&0&0&E_6&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0 \\
E_5&E_3&E_2&0&0&0&0&E_6&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&E_6&0&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
G_4&0&G_2&0&G_1&0&0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&G_6&0&0&0 \\
G_2&0&G_1&0&0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&G_6&0&0&0&0&0 \\
0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0&0 \\
G_5&0&G_3&0&0&0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&D_6&0&D_2&0&0&0&D_9&0&D_4&0&0&0&0&D_7&0&0&0&D_{10}&0&0&0&0&0&0 \\
D_3&0&0&0&0&0&D_6&0&D_2&0&0&D_9&0&D_4&0&0&0&D_7&0&0&D_{10}&0&0&0&0&0 \\
0&D_9&0&D_4&0&0&0&0&0&D_7&0&0&0&0&D_{10}&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
$$

$$
\mathbf{M}_{22} = \begin{bmatrix}
D_6&0&D_2&0&0&0&D_9&0&D_4&0&0&0&0&D_7&0&0&0&D_{10}&0&0&0&0&0&0&0&0 \\
D_9&0&D_4&0&0&0&0&0&D_7&0&0&0&0&D_{10}&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&E_8&E_5&E_3&E_2&0&0&0&0&E_6&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0 \\
E_3&E_2&0&0&0&0&E_6&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&E_6&0&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
E_6&0&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0 \\
0&0&0&0&0&G_5&0&G_3&0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0&0&0 \\
0&G_5&0&G_3&0&0&0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&G_3&0&0&0&0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&G_6&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&G_6&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&D_2&0&0&0&D_9&0&D_4&0&0&0&0&D_7&0&0&0&D_{10}&0&0&0&0&0&0&0&0&0 \\
0&0&D_9&0&D_4&0&0&0&0&0&D_7&0&0&0&0&D_{10}&0&0&0&0&0&0&0&0&0&0 \\
0&D_4&0&0&0&0&0&D_7&0&0&0&0&D_{10}&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&D_7&0&0&0&0&0&D_{10}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&D_7&0&0&0&0&0&D_{10}&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&E_6&0&0&0&0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&E_9&E_7&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}
\tag{3-30}
$$

In order for a solution for the vector **λ** to exist, it is necessary for the 52 resulting equations to be linearly dependent. This will occur if the determinant of **M** equals zero.

It was not possible to symbolically expand the determinant of matrix **M**. A numerical case was analyzed and a polynomial of degree 158 in the variable $L_1$ was obtained. It was not possible to solve this high degree polynomial for the values of $L_1$, although several commercial and in-house written algorithms were attempted. Because of this, a different method was attempted to solve the set of equations 3-13, 3-21, and 3-24.

### 3.2.4 Solution of Three Simultaneous Equations in Three Unknowns – Continuation Method

The continuation method (Morgan [18], Mora [19], Cullen [20], Strang [21], Garcia and Li [22], Morgan [23-24], Wampler et al., [25]) which is a numerical technique to solve a set of equations in multiple variables are used. This is as opposed to Sylvester's method which would lead to a symbolic solution of the problem.

A concise description of the continuity method is presented by Tsai, 1999. Suppose one wishes to solve the set of equations F(**x**) which are defined by

$$F(\mathbf{x}):\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases} \tag{3-31}$$

F(**x**) is called the target system.

The continuation method begins by first estimating the total number of possible solution sets (sets of values for $L_1$, $L_2$, and $L_3$ for our case) that satisfy the given equations. For example, Bezout's theorem states that a polynomial of total degree *n* has at most *n* isolated solutions in the complex Euclidean space. Including solutions at infinity, the Bezout number of a polynomial system is equal to the total degree of the system.

Next, an initial system, G(**x**) =0, is obtained, whose solution will be of the same degree as that of F(**x**), but whose solution set is known in closed form. In other words, G(**x**) maintains the same polynomial structure as F(**x**).

Finally, a homotopy function H(**x**, t) is prepared such as

$$H(\mathbf{x}, t) = \gamma\,(1-t)\,G(\mathbf{x}) + t\,F(\mathbf{x}) \tag{3-32}$$

where $\gamma$ is a random complex constant. When t=0, the homotopy function equals the initial system, G(**x**). When t=1, the homotopy function equals the target system, F(**x**). Recall that the solutions to G(**x**) are known. As the parameter *t* is increased in small steps from 0 to 1, the solutions of H(**x**, t) can be tracked (referred to as path tracking) and when t =1, these solutions will be the solutions to the original target system. If the degree of the solution set was overestimated, some of the solutions will track to infinity and these can easily be discarded.

### 3.2.5 Numerical Example

The following information is given:

strut lengths:

$a_{12}$ = 14 in.  $a_{34}$ = 12 in.

non-compliant tie lengths:

$a_{41}$ = 10 in.

spring 1 free length & spring constant:

$L_{01}$ = 8 in.  $k_1$ = 1 lbf/in.

spring 2 free length & spring constant:

$L_{02}$ = 2 in.  $k_2$ = 2.687 lbf/in.

spring 3 free length & spring constant:

$L_{03}$ = 2.5 in.  $k_3$ = 3.465 lbf/in.

Based on these values, the coefficients in equations 3-13, 3-21, and 3-24 were evaluated to yield the three equations

$$100\, L_3^4 + [(-L_1^2 + 96)\, L_2^2 + 44\, L_1^2 - 52224]\, L_3^2 + L_1^2\, L_2^4 + (L_1^4 - 440\, L_1^2 - 13824)\, L_2^2 - 8624\, L_1^2 + 6773760 = 0 \qquad (3\text{-}33)$$

$$(2.5\, L_1\, L_2^2 + 90\, L_1 - 1600)\, L_3^3 + (-8.663\, L_1\, L_2^2 + 381.165\, L_1)\, L_3^2 + [-2.5\, L_1\, L_2^4 + (-6\, L_1^3 + 8\, L_1^2 + 1196\, L_1 - 768)\, L_2^2 + 44\, L_1^3 - 352\, L_1^2 - 30664\, L_1 + 417792]\, L_3 + 8.663\, L_1\, L_2^4 + (17.326\, L_1^3 - 3811.651\, L_1)\, L_2^2 - 74708.354\, L_1 = 0 \qquad (3\text{-}34)$$

$$[(2.5\, L_1^2 + 160)\, L_2 - 1074.637]\, L_3^3 + (831.633 - 8.663\, L_1^2)\, L_2\, L_3^2 + [(-7\, L_1^2 + 192)\, L_2^3 + (-515.826 + 5.373\, L_1^2)\, L_2^2 + (-2.5\, L_1^4 + 1188\, L_1^2 - 69888)\, L_2 - 236.420\, L_1^2 + 280609.161]\, L_3 + 17.326\, L_1^2\, L_2^3 + (-119755.135 + 8.663\, L_1^4 - 3811.651\, L_1^2)\, L_2 = 0\,. \qquad (3\text{-}35)$$

The continuation method was run on this set of three equations in three unknowns to obtain all solution sets for the three spring lengths, $L_1$, $L_2$, and $L_3$, for the particular numerical example. The software PHCpack (Verschelde [26]) was used to implement the method.

The PHCpack software estimated the number of possible solutions to be 136. Seven real solutions were obtained and these are listed in Table 3-1.

Table 3-1: Seven real solutions for three-spring planar tensegrity system(units are inches)

| Case | $L_1$ | $L_2$ | $L_3$ |
|---|---|---|---|
| 1 | 13.000 | 8.000 | 7.017 |
| 2 | -11.376 | -10.371 | -5.333 |
| 3 | -7.585 | 9.097 | 10.106 |
| 4 | -11.029 | 12.557 | -3.044 |
| 5 | 13.969 | -5.800 | 9.164 |
| 6 | 14.248 | -9.373 | -4.774 |
| 7 | 13.181 | 11.599 | -2.488 |

An equilibrium analysis was conducted for the seven cases and only the first case was indeed in equilibrium. Case 1 is shown in Figure 3-6.

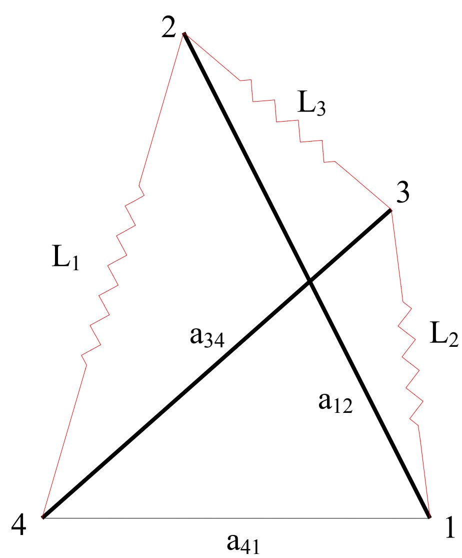


Figure 3-6. Case 1, equilibrium solution.

### 3.3 Second Approach Problem Statement (calculate $\cos\theta_4$, $\cos\theta_1$)

In the second approach, the cosines of the angles $\theta_4$ and $\theta_1$ were chosen as the descriptive parameters. The cosines of the angles were chosen rather than the angles themselves in the hope that the resulting equations would be simpler in that, for example, a single value of $\cos\theta_4$ accounts for the obvious symmetry in solutions that will occur with respect to the fixed member $a_{41}$.

The problem statement is presented as follows:

Given: Strut lengths $a_{12}$, $a_{34}$, tie length $a_{41}$, and the spring constants and free lengths $k_1$, $k_2$, $k_3$, $L_{01}$, $L_{02}$, $L_{03}$.

Find: $\cos\theta_4$ and $\cos\theta_1$ when the system is in equilibrium.

The solution approach is as follows:

1. Obtain expressions for $L_1$, $L_2$, and $L_3$ in terms of $\cos\theta_4$ and $\cos\theta_1$

2. Write the potential energy equation

3. Determine values of $\cos\theta_4$ and $\cos\theta_1$ such that $dU/d\cos\theta_4 = dU/d\cos\theta_1 = 0$

**3.3.1 Development of Geometric Equations**

Figure 3-1 shows the nomenclature used. Springs $L_1$, $L_2$, and $L_3$ are the extended lengths of the compliant ties between points 4 and 2, 2 and 3, and 1 and 3 respectively. The unloaded lengths of springs are given by $L_{01}$, $L_{02}$, and $L_{03}$. A cosine law for the quadrilateral 1-2-3-4 can be written as:

$$Z_{41} = \frac{L_3^2}{2} \tag{3-36}$$

where,

$$Z_{41} = -a_{12}(X_4 s_1 + Y_4 c_1) + Z_4 + \frac{a_{12}^2}{2} \tag{3-37}$$

and where

$$X_4 = a_{34}\, s_4 \tag{3-38}$$

$$Y_4 = -(a_{41} + a_{34}\, c_4) \tag{3-39}$$

$$Z_4 = \frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} c_4\ . \tag{3-40}$$

Substituting (3-37) into (3-36) and rearranging gives

$$a_{12}X_4s_1 = -a_{12}Y_4c_1 + Z_4 + \frac{a_{12}^{\ 2}}{2} - \frac{L_3^{\ 2}}{2} \ . \qquad (3\text{-}41)$$

Substituting (3-38), (3-39), and (3-40) into (3-41), then squaring it, and substituting for $s_4^2 = 1 - c_4^2$ and $s_1^2 = 1 - c_1^2$ and multiplying the entire equation by 4 yields

$$L_3^4 + A\,L_3^2 + B = 0 \qquad (3\text{-}42)$$

where

$$A = A_1\,c_4 + A_2\,c_1 + A_3\,c_1\,c_4 + A_4\,, \qquad (3\text{-}43)$$

$$B = B_1\,c_1^2 + B_2\,c_1^2\,c_4 + B_3\,c_1 + B_4\,c_1\,c_4 + B_5\,c_1\,c_4^2 + B_6\,c_4^2 + B_7\,c_4 + B_8 \qquad (3\text{-}44)$$

and where

$A_1 = -\,4\,a_{34}\,a_{41}$

$A_2 = -\,4\,a_{12}\,a_{41}$

$A_3 = -\,4\,a_{12}\,a_{34}$

$A_4 = -\,2\,(a_{12}^2 + a_{34}^2 + a_{41}^2)$

$B_1 = 4\,a_{12}^2\,(a_{34}^2 + a_{41}^2)$

$B_2 = 8\,a_{12}^2\,a_{34}\,a_{41}$

$B_3 = 4\,a_{12}\,a_{41}\,(a_{41}^2 + a_{12}^2 + a_{34}^2)$

$B_4 = 4\,a_{12}\,a_{34}\,(a_{34}^2 + a_{12}^2 + 3\,a_{41}^2)$

$B_5 = 8\,a_{12}\,a_{41}\,a_{34}^2$

$B_6 = 4\,a_{34}^2\,(a_{12}^2 + a_{41}^2)$

$B_7 = 4\,a_{34}\,a_{41}\,(a_{41}^2 + a_{34}^2 + a_{12}^2)$

$$B_8 = (a_{41}^2 + a_{34}^2 - 2\,a_{12}\,a_{34} + a_{12}^2)\,(a_{41}^2 + a_{34}^2 + 2\,a_{12}\,a_{34} + a_{12}^2)\ . \qquad (3\text{-}45)$$

Equation 3-42 expresses $L_3$ as a function of $c_4$ and $c_1$. A cosine law for triangle 4-1-2 may be written as

$$\frac{a_{12}^2}{2}+\frac{a_{41}^2}{2}+a_{12}a_{41}c_1=\frac{L_1^2}{2}\ . \qquad (3\text{-}46)$$

A cosine law for the triangle 3-4-1 may be written as

$$\frac{a_{34}^2}{2}+\frac{a_{41}^2}{2}+a_{34}a_{41}c_4=\frac{L_2^2}{2}\ . \qquad (3\text{-}47)$$

Equations 3-42, 3-46, and 3-47 define the three spring lengths in terms of the variables $\cos\theta_4$ and $\cos\theta_1$.

### 3.3.2 Development of Potential Energy Equations

The total potential energy stored in all three springs is given by,

$$U = ½\, k_1\, (L_1\text{-}L_{01})^2 + ½\, k_2\, (L_2\text{-}L_{02})^2 + ½\, k_3\, (L_3\text{-}L_{03})^2\ . \qquad (3\text{-}48)$$

Differentiating the potential energy with respect to $c_4$ and $c_1$ and then evaluating values for $c_4$ and $c_1$ that cause the derivative of the potential energy to equal zero, will identify configurations of either minimum or maximum potential energy. These derivatives may be written as,

$$dU/dc_4 = k_1\, (L_1 - L_{01})\, dL_1/dc_4 + k_2\, (L_2 - L_{02})\, dL_2/dc_4 + k_3\, (L_3 - L_{03})\, dL_3/dc_4 \qquad (3\text{-}49)$$

$$dU/dc_1 = k_1\, (L_1 - L_{01})\, dL_1/dc_1 + k_2\, (L_2 - L_{02})\, dL_2/dc_1 + k_3\, (L_3 - L_{03})\, dL_3/dc_1 \qquad (3\text{-}50)$$

Since from 3-46, $L_1$ is not a function of $c_4$, $dL_1/dc_4 = 0$. Similarly, from 3-47, $L_2$ is not a function of $c_1$ and thus $dL_2/dc_1 = 0$. Equations 3-49 and 3-50 reduce to

$$dU/dc_4 = k_2\, (L_2 - L_{02})\, dL_2/dc_4 + k_3\, (L_3 - L_{03})\, dL_3/dc_4\ , \qquad (3\text{-}51)$$

$$dU/dc_1 = k_1\, (L_1 - L_{01})\, dL_1/dc_1 + k_3\, (L_3 - L_{03})\, dL_3/dc_1\ . \qquad (3\text{-}52)$$

The term $dL_2/dc_4$ is evaluated from 3-47 as

$$\frac{dL_2}{dc_4} = \frac{a_{34}a_{41}}{L_2} \tag{3-53}$$

and the term $dL_1/dc_1$ is evaluated from 3-46 as

$$\frac{dL_1}{dc_1} = \frac{a_{12}a_{41}}{L_1} \ . \tag{3-54}$$

Implicit differentiation of 3-42 for $L_3$ with respect to $c_4$ and $c_1$ yields

$$\frac{dL_3}{dc_4} = -\frac{(A_1 + A_3c_1)L_3^{\ 2} + B_2c_1^{\ 2} + B_4c_1 + 2B_5c_1c_4 + 2B_6c_4 + B_7}{2L_3(2L_3^{\ 2} + A_1c_4 + A_2c_1 + A_3c_1c_4 + A_4)} \tag{3-55}$$

$$\frac{dL_3}{dc_1} = -\frac{(A_2 + A_3c_4)L_3^{\ 2} + 2B_1c_1 + 2B_2c_1c_4 + B_3 + B_4c_4 + B_5c_4^{\ 2}}{2L_3(2L_3^{\ 2} + A_1c_4 + A_2c_1 + A_3c_1c_4 + A_4)} \ . \tag{3-56}$$

Substituting 3-53 and 3-55 into 3-51 and 3-54 and 3-56 into 3-52 and equating to zero yields

$$(M_1 L_3^3 + M_2 L_3^2 + M_3 L_3 + M_4) L_2 + M_5 L_3^3 + M_6 L_3 = 0 \tag{3-57}$$

$$(N_1 L_3^3 + N_2 L_3^2 + N_3 L_3 + N_4) L_1 + N_5 L_3^3 + N_6 L_3 = 0 \tag{3-58}$$

Where the coefficients $M_i$ and $N_i$, i=1..6, are functions of $c_4$ and $c_1$ as

$M_1 = -k_3 (A_3 c_1 + A_1) + 4 k_2 a_{34} a_{41}$

$M_2 = k_3 L_{03} (A_3 c_1 + A_1)$

$M_3 = -k_3 B_2 c_1^2 + [2 (k_2 a_{34} a_{41}A_3 – k_3 B_5) c_4 + 2 k_2 a_{34} a_{41} A_2 – k_3 B_4) c_1$

$+ 2 (k_2 a_{34} a_{41} A_1 – k_3 B_6) c_4 – k_3 B_7 + 2 k_2 a_{34} a_{41} A_4$

$M_4 = k_3 L_{03} B_2 c_1^2 + k_3 L_{03} B_4 c_1 + 2 k_3 L_{03} B_5 c_1 c_4 + 2 k_3 L_{03} B_6 c_4 + k_3 L_{03} B_7$

$M_5 = -4 k_2 a_{34} a_{41} L_{02}$

$M_6 = -2 k_2 (a_{34} a_{41}L_{02} A_2 c_1 + a_{34} a_{41} L_{02} A_3 c_1 c_4 + a_{34} a_{41} L_{02} A_1 c_4$

$+ a_{34} a_{41} L_{02} A_4)$ (3-59)

$N_1 = - k_3 (A_3 c_4 + A_2) + 4 k_1 a_{12} a_{41}$

$N_2 = k_3 L_{03} (A_3 c_4 + A_2)$

$N_3 = 2\ k_1\ a_{12}\ a_{41}\ (A_1\ c_4 + A_2\ c_1 + A_3\ c_1\ c_4 + A_4) - k_3\ (B_4\ c_4 + B_5\ c_4^2 + 2\ B_1\ c_1$

$+ 2\ B_2\ c_1\ c_4 + B_3)$

$N_4 = k_3\ L_{03}\ (B_4\ c_4 + B_5\ c_4^2 + 2\ B_1\ c_1 + B_2\ c_1\ c_4 + B_3)$

$N_5 = -\ 4\ k_1\ a_{12}\ a_{41}\ L_{01}$

$N_6 = -2\ k_1\ a_{12}\ a_{41}\ L_{01}\ (A_4 + A_1\ c_4 + A_2\ c_1 + A_3\ c_1\ c_4)\ .$ (3-60)

Equations 3-57 and 3-58 may be written as

$$(M_1\ L_3^3 + M_2\ L_3^2 + M_3\ L_3 + M_4)\ L_2 = -\ M_5\ L_3^3 - M_6\ L_3\ , \quad (3\text{-}61)$$

$$(N_1\ L_3^3 + N_2\ L_3^2 + N_3\ L_3 + N_4)\ L_1 = -\ N_5\ L_3^3 - N_6\ L_3\ . \quad (3\text{-}62)$$

Squaring both sides of both equations gives

$$(M_1\ L_3^3 + M_2\ L_3^2 + M_3\ L_3 + M_4)^2\ L_2^2 = (M_5\ L_3^3 + M_6\ L_3)^2\ , \quad (3\text{-}63)$$

$$(N_1\ L_3^3 + N_2\ L_3^2 + N_3\ L_3 + N_4)^2\ L_1^2 = (N_5\ L_3^3 + N_6\ L_3)^2\ . \quad (3\text{-}64)$$

Using 3-46 and 3-47 to substitute for $L_1^2$ and $L_2^2$ will yield two equations in the parameters $c_1$, $c_4$, and $L_3$. These two equations can be arranged as

$$\begin{aligned}(p_1\ c_1^2 + p_2\ c_1 + p_3)\ c_4^3 + (p_4\ c_1^3 + p_5\ c_1^2 + p_6\ c_1 + p_7)\ c_4^2 \\ + (p_8\ c_1^4 + p_9\ c_1^3 + p_{10}\ c_1^2 + p_{11}\ c_1 + p_{12})\ c_4 \\ + (p_{13}\ c_1^4 + p_{14}\ c_1^3 + p_{15}\ c_1^2 + p_{16}\ c_1 + p_{17}) = 0\end{aligned} \quad (3\text{-}65)$$

$$\begin{aligned}(q_1\ c_1 + q_2)\ c_4^4 + (q_3\ c_1^2 + q_4\ c_1 + q_5)\ c_4^3 \\ + (q_6\ c_1^3 + q_7\ c_1^2 + q_8\ c_1 + q_9)\ c_4^2 + (q_{10}\ c_1^3 + q_{11}\ c_1^2 + q_{12}\ c_1 + q_{13})\ c_4 \\ + (q_{14}\ c_1^3 + q_{15}\ c_1^2 + q_{16}\ c_1 + q_{17}) = 0\end{aligned} \quad (3\text{-}66)$$

where

$p_1 = p_{1a}\ L_3^2 + p_{1b}\ L_3 + p_{1c}$

$p_2 = p_{2a}\ L_3^2 + p_{2b}\ L_3 + p_{2c}$

$p_3 = p_{3a}\ L_3^2 + p_{3b}\ L_3 + p_{3c}$

$$p_4 = p_{4a} L_3^2 + p_{4b} L_3 + p_{4c}$$

$$p_5 = p_{5a} L_3^4 + p_{5b} L_3^3 + p_{5c} L_3^2 + p_{5d} L_3 + p_{5e}$$

$$p_6 = p_{6a} L_3^4 + p_{6b} L_3^3 + p_{6c} L_3^2 + p_{6d} L_3 + p_{6e}$$

$$p_7 = p_{7a} L_3^4 + p_{7b} L_3^3 + p_{7c} L_3^2 + p_{7d} L_3 + p_{7e}$$

$$p_8 = p_{8a} L_3^2 + p_{8b} L_3 + p_{8c}$$

$$p_9 = p_{9a} L_3^4 + p_{9b} L_3^3 + p_{9c} L_3^2 + p_{9d} L_3 + p_{9e}$$

$$p_{10} = p_{10a} L_3^6 + p_{10b} L_3^5 + p_{10c} L_3^4 + p_{10d} L_3^3 + p_{10e} L_3^2 + p_{10f} L_3 + p_{10g}$$

$$p_{11} = p_{11a} L_3^6 + p_{11b} L_3^5 + p_{11c} L_3^4 + p_{11d} L_3^3 + p_{11e} L_3^2 + p_{11f} L_3 + p_{11g}$$

$$p_{12} = p_{12a} L_3^6 + p_{12b} L_3^5 + p_{12c} L_3^4 + p_{12d} L_3^3 + p_{12e} L_3^2 + p_{12f} L_3 + p_{12g}$$

$$p_{13} = p_{13a} L_3^2 + p_{13b} L_3 + p_{13c}$$

$$p_{14} = p_{14a} L_3^4 + p_{14b} L_3^3 + p_{14c} L_3^2 + p_{14d} L_3 + p_{14e}$$

$$p_{15} = p_{15a} L_3^6 + p_{15b} L_3^5 + p_{15c} L_3^4 + p_{15d} L_3^3 + p_{15e} L_3^2 + p_{15f} L_3 + p_{15g}$$

$$p_{16} = p_{16a} L_3^6 + p_{16b} L_3^5 + p_{16c} L_3^4 + p_{16d} L_3^3 + p_{16e} L_3^2 + p_{16f} L_3 + p_{16g}$$

$$p_{17} = p_{17a} L_3^6 + p_{17b} L_3^5 + p_{17c} L_3^4 + p_{17d} L_3^3 + p_{17e} L_3^2 + p_{17f} L_3 + p_{17g} \qquad (3\text{-}67)$$

$$q_1 = q_{1a} L_3^2 + q_{1b} L_3 + q_{1c}$$

$$q_2 = q_{2a} L_3^2 + q_{2b} L_3 + q_{2c}$$

$$q_3 = q_{3a} L_3^2 + q_{3b} L_3 + q_{3c}$$

$$q_4 = q_{4a} L_3^4 + q_{4b} L_3^3 + q_{4c} L_3^2 + q_{4d} L_3 + q_{4e}$$

$$q_5 = q_{5a} L_3^4 + q_{5b} L_3^3 + q_{5c} L_3^2 + q_{5d} L_3 + q_{5e}$$

$$q_6 = q_{6a} L_3^2 + q_{6b} L_3 + q_{6c}$$

$$q_7 = q_{7a} L_3^4 + q_{7b} L_3^3 + q_{7c} L_3^2 + q_{7d} L_3 + q_{7e}$$

$$q_8 = q_{8a} L_3^6 + q_{8b} L_3^5 + q_{8c} L_3^4 + q_{8d} L_3^3 + q_{8e} L_3^2 + q_{8f} L_3 + q_{8g}$$

$$q_9 = q_{9a} L_3^6 + q_{9b} L_3^5 + q_{9c} L_3^4 + q_{9d} L_3^3 + q_{9e} L_3^2 + q_{9f} L_3 + q_{9g}$$

$$q_{10} = q_{10a} L_3^2 + q_{10b} L_3 + q_{10c}$$

$$q_{11} = q_{11a} L_3^4 + q_{11b} L_3^3 + q_{11c} L_3^2 + q_{11d} L_3 + q_{11e}$$

$$q_{12} = q_{12a} L_3^6 + q_{12b} L_3^5 + q_{12c} L_3^4 + q_{12d} L_3^3 + q_{12e} L_3^2 + q_{12f} L_3 + q_{12g}$$

$$q_{13} = q_{13a} L_3^6 + q_{13b} L_3^5 + q_{13c} L_3^4 + q_{13d} L_3^3 + q_{13e} L_3^2 + q_{13f} L_3 + q_{13g}$$

$$q_{14} = q_{14a} L_3^2 + q_{14b} L_3 + q_{14c}$$

$$q_{15} = q_{15a} L_3^4 + q_{15b} L_3^3 + q_{15c} L_3^2 + q_{15d} L_3 + q_{15e}$$

$$q_{16} = q_{16a} L_3^6 + q_{16b} L_3^5 + q_{16c} L_3^4 + q_{16d} L_3^3 + q_{16e} L_3^2 + q_{16f} L_3 + q_{16g}$$

$$q_{17} = q_{17a} L_3^6 + q_{17b} L_3^5 + q_{17c} L_3^4 + q_{17d} L_3^3 + q_{17e} L_3^2 + q_{17f} L_3 + q_{17g} \,. \qquad (3\text{-}68)$$

The coefficients $p_{1a}$ through $q_{17g}$ are functions of the given constant parameters. These coefficients are defined in the Appendix.

The coefficients $p_{1a}$ through $q_{17g}$ have been obtained symbolically. For example, the terms $p_{1a}$ through $p_{1c}$ are written as

$$p_{1a} = 8\, k_2^2\, a_{34}^3\, a_{41}^3\, A_3^2 - 16\, k_2\, a_{34}^2\, a_{41}^2\, A_3\, k_3\, B_5 + 8\, k_3^2\, B_5^2\, a_{34}\, a_{41}$$

$$p_{1b} = -16\, k_3^2\, B_5^2\, L_{03}\, a_{34}\, a_{41} + 16\, k_2\, a_{34}^2\, a_{41}^2\, A_3\, k_3\, L_{03}\, B_5$$

$$p_{1c} = 8\, k_3^2\, L_{03}^2\, B_5^2\, a_{34}\, a_{41}$$

The remaining terms are not listed here due to their complexity.

Similarly, equation 3-42 is an equation in the same parameters, $c_1$, $c_4$, and $L_3$. This equation can be factored as

$$(r_1\, c_1 + r_2)\, c_4^2 + (r_3\, c_1^2 + r_4\, c_1 + r_5)\, c_4 + (r_6\, c_1^2 + r_7\, c_1 + r_8) = 0 \qquad (3\text{-}69)$$

where

$$r_1 = 8\, a_{12}\, a_{41}\, a_{34}^2$$

$$r_2 = 4\, a_{34}^2\, (a_{12}^2 + a_{41}^2)$$

$$r_3 = 8\, a_{34}\, a_{41}\, a_{12}^2$$

$r_4 = (-4\, a_{12}\, a_{34})\, L_3^2 + 4\, a_{12}\, a_{34}\, (a_{34}^2 + a_{12}^2 + 3\, a_{41}^2)$

$r_5 = (-4\, a_{34}\, a_{41})\, L_3^2 + 4\, a_{34}\, a_{41}\, (a_{34}^2 + a_{12}^2 + a_{41}^2)$

$r_6 = 4\, a_{12}^2\, (a_{34}^2 + a_{41}^2)$

$r_7 = (-4\, a_{12}\, a_{41})\, L_3^2 + 4\, a_{12}\, a_{41}\, (a_{41}^2 + a_{34}^2 + a_{12}^2)$

$r_8 = L_3^4 - 2\, (a_{34}^2 + a_{12}^2 + a_{41}^2)\, L_3^2$ (3-70)

**3.3.3 Solution of Equation Set for $c_4$, $c_1$, and $L_3$**

Equations 3-65, 3-66, and 3-69 are factored such that the parameter $L_3$ is embedded in the coefficients $p_1$ through $r_8$. Sylvester's method was used in an attempt to solve these equations for all possible sets of values of $L_3$, $c_1$, and $c_4$. Equation 3-65 was multiplied by $c_1$, $c_1^2$, $c_4$, $c_4^2$, $c_4^3$, $c_1^4$, $c_{12}c_4$, $c_1c_4^2$, and $c_1^2c_4^2$ to obtain 10 equations (including itself). Equation 3-66 was multiplied by $c_1$, $c_1^2$, $c_1^3$, $c_4$, $c_4^2$, $c_1c_4$, $c_1^2c_4$, $c_1^3c_4$, $c_1c_4^2$ and $c_1^2c_4^2$ to obtain 11 equations (including itself). Equation 3-69 was multiplied by $c_1$, $c_1^2$, $c_1^3$, $c_1^4$, $c_4$, $c_4^2$, $c_4^3$, $c_4^4$, $c_1c_4$, $c_1^2c_4$, $c_1^3c_4$, $c_1^4c_4$, $c_1c_4^2$, $c_1^2c_4^2$, $c_1^3c_4^2$, $c_1^4c_4^2$, $c_1c_4^3$, $c_1^2c_4^3$, $c_1^3c_1^4$, $c_1c_4^4$ and $c_1^2c_4^4$ to obtain 22 equations including itself. This resulted in a set of 43 "homogeneous" equations in 43 unknowns.

The condition for a solution to this set of equations was that they are linearly dependent, i.e. the determinant of the 43×43 coefficient matrix must equal zero. This would yield a polynomial in the single variable $L_3$. Due to the complexity of the problem, it was not possible to expand this determinant symbolically. Similarly, it was not possible to root the polynomial that resulted from a particular numerical example. Because of this, the Continuation Method was used again to determine solutions for the same numerical example presented in Section 3.2.5.

Substituting the given numerical values into equations 3-65, 3-66, and 3-69 are yield three equations for this numerical example. For this case, the coefficients $p_{1a}$ through $q_{17g}$ and $r_1$ through $r_8$ are evaluated and presented in Appendix b.

The continuation method was run on this set of three equations in three unknowns to obtain all solution sets for the three variables, $L_3$, $c_1$, and $c_4$, for the particular numerical example. The software PHCpack (Verschelde [26]) was again used to implement the method.

The PHCpack software estimated the number of possible solutions to be 136. Seven real solutions were obtained and these are listed in Table 3-2.

Table 3-2. Real value answers using cosine of $\theta_1$, $\theta_4$ in radians and $L_3$ in inches.

| Case | Cosine $\theta_1$ | Cosine $\theta_4$ | $L_3$ |
|---|---|---|---|
| 1 | -0.453571523 | -0.749999925 | 7.01658755 |
| 2 | -0.594965126 | -0.568509675 | -5.33336481 |
| 3 | -0.851695156 | -0.671871253 | 10.1059910 |
| 4 | -0.622722694 | -0.359683532 | -3.04401046 |
| 5 | -0.360201938 | -0.876513972 | 9.16391281 |
| 6 | -0.332122619 | -0.650641695 | -4.77378239 |
| 7 | -.0436671807 | -0.456122910 | -2.48758125 |

The lengths of the springs $L_1$ and $L_2$ were calculated for each of the seven cases and it was determined that the seven real solutions here correspond directly with the seven real solutions that were determined from the first problem formulation and listed in Table 3-1. As before, case 1 was the only case that was indeed in equilibrium and the mechanism is shown in this configuration in Figure 3-6.

### 3.4 Conclusion

Two approaches were presented to solve the three-spring planar tensegrity equilibrium problem. Both approaches resulted in high degree polynomials. These polynomials could not be derived symbolically and were in fact difficult to derive numerically.

A continuation approach was presented to solve the set of three equations for each of the two solution approaches. A force balance analysis was conducted to identify all realizable solutions; in this case one.

It is apparent that further analysis of Sylvester's variable elimination procedure may be necessary to reduce the number of simultaneous equations, and thereby the size of the univariate polynomial that results. As a result of the work in this chapter it can be said that the degree of the solution has been bounded, although further work may yield a simpler result.

CHAPTER 4
FOUR SPRING PLANAR TENSEGRITY

### 4.1 Introduction

In this chapter planar tensegrity mechanisms with two struts and four elastic ties (see Figure 4-1) are analyzed to determine all possible equilibrium configurations for the device when no external forces or moments are applied. The stable equilibrium position is determined by identifying the configurations at which the potential energy stored in the four springs is a minimum.

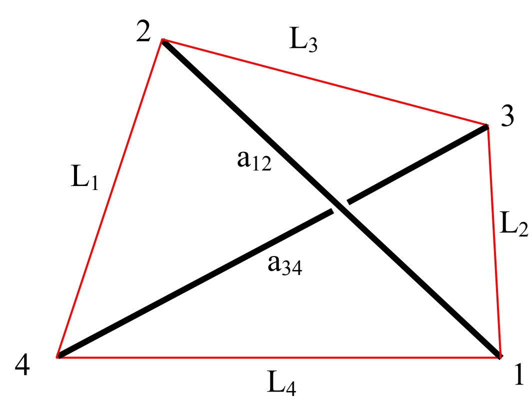


Figure 4-1. Two strut, four spring planar tensegrity device

Four non-linear equations are obtained in the variables $L_1$, $L_2$, $L_3$, and $L_4$ which correspond to the equilibrium lengths of the four compliant ties. One equation is associated with the geometry constraint that must be satisfied in order for the mechanism to be assembled. The other three equations correspond to the derivative of the potential energy equation taken with respect to three of the compliant tie lengths.

Solution of these four equations in order to find sets of values for the parameters $L_1$ through $L_4$ that simultaneously satisfy all four equations proved to be a very difficult process. For this case, the continuity method was used in an attempt to find all equilibrium solutions for two numerical cases. The results of the continuity method are then checked to first verify that the set of solutions simultaneously satisfy all four equations and then to determine whether each solution set represents a minimum potential energy and an equilibrium state.

### 4.2 Problem Statement

The problem statement can be explicitly written as:

Given

| | |
|---|---|
| $a_{12}$, $a_{34}$ | lengths of struts, |
| $k_1$, $L_{01}$ | spring constant and its free length between points 4 and 2, |
| $k_2$, $L_{02}$ | spring constant and its free length between points 3 and 1, |
| $k_3$, $L_{03}$ | spring constant and its free length between points 3 and 2, |
| $k_4$, $L_{04}$ | spring constant and its free length between points 4 and 1. |

Find at equilibrium position

| | |
|---|---|
| $L_1$ | length of spring 1, |
| $L_2$ | length of spring 2, |
| $L_3$ | length of spring 3, |
| $L_4$ | length of spring 4, |

It should be noted that the problem statement could be formulated in a variety of ways. The solution presented here uses $L_1$, $L_2$, and $L_3$ as the three generalized parameters, knowledge of which completely describes the system (i.e. the value for $L_4$, the last remaining parameter, can be deduced). The geometry equation that relates $L_4$ in terms of the three generalized parameters is derived first. Following this, a derivative of the potential energy equation is taken with respect to each of the three generalized parameters to yield an additional three equations.

### 4.3 Development of Geometric and Potential Energy Constraint Equations

The geometry equation is derived in a manner similar to that presented in Section 3.2.1. The parameter $a_{41}$ in equations (3.15) can be replaced by $L_4$ to yield an equation in the unknown terms $L_1$, $L_2$, $L_3$, and $L_4$. This equation can be rearranged as

$$G_1 (L_1^4 L_2^4) + G_2 (L_1^4 L_2^2) + G_3 (L_3^2 L_1^2 L_2^2) + G_4 (L_1^2 L_2^2) + G_5 (L_3^2 L_2^2) + G_6 (L_2^2) + G_7( L_3^2 L_1^2 ) + G_8( L_1^2 ) + G_9( L_3^4 ) + G_{10}( L_3^2 ) + G_{11} = 0 \tag{4-1}$$

where

$$G_1 = 1$$

$$G_2 = 1$$

$$G_3 = -1$$

$$G_4 = -L_4^2 - a_{12}^2 - a_{34}^2$$

$$G_5 = a_{12}^2 - L_4^2$$

$$G_6 = L_4^2 a_{34}^2 - a_{34}^2 a_{12}^2$$

$$G_7 = -L_4^2 L_1^2$$

$$G_8 = - a_{34}^2 a_{12}^2 - a_{12}^2 L_4^2$$

$$G_9 = L_4^2$$

$$G_{10} = L_4^4 - a_{12}^2 L_4^2 - a_{34}^2 a_{12}^2 - a_{34}^2 L_4^2$$

$$G_{11} = -L_4^2 a_{34}^2 a_{12}^2 + a_{34}^4 a_{12}^2 - a_{12}^4 a_{34}^2 \quad . \tag{4-2}$$

The potential energy of the system can be evaluated as

$$U = \frac{1}{2}k_1 (L_1 - L_{01})^2 + \frac{1}{2}k_2 (L_2 - L_{02})^2 + \frac{1}{2}k_3 (L_3 - L_{03})^2 + \frac{1}{2}k_4 (L_4 - L_{04})^2 \tag{4-3}$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the lengths $L_1$, $L_2$, and $L_3$ all equal zero, i.e.

$$\frac{dU}{dL_1} = k_1 (L_1 - L_{01}) + k_4 (L_4 - L_{04}) \frac{dL_4}{dL_1} = 0 \ , \tag{4-4}$$

$$\frac{dU}{dL_2} = k_2 (L_2 - L_{02}) + k_4 (L_4 - L_{04}) \frac{dL_4}{dL_2} = 0 \ , \tag{4-5}$$

$$\frac{dU}{dL_3} = k_3 (L_3 - L_{03}) + k_4 (L_4 - L_{04}) \frac{dL_4}{dL_3} = 0 \ . \tag{4-6}$$

The derivatives $dL_4/dL_1$, $dL_4/dL_2$, and $dL_4/dL_3$ are determined by implicit differentiation of Equation 4-1 as

$$\frac{dL_4}{dL_1} = \frac{-L_1 [L_2^4 - L_2^2 a_{34}^2 + 2L_2^2 L_1^2 - L_2^2 L_3^2 - L_2^2 a_{12}^2 - L_2^2 L_4^2 + a_{12}^2 L_4^2 - L_4^2 L_3^2 - a_{34}^2 a_{12}^2 + L_3^2 a_{34}^2)]}{L_4 (L_2^2 a_{34}^2 - L_2^2 L_1^2 - L_2^2 L_3^2 + a_{12}^2 L_1^2 - L_3^2 L_1^2 - a_{34}^2 a_{12}^2 + 2L_4^2 L_3^2 + L_3^4 - L_3^2 a_{34}^2 - L_3^2 a_{12}^2} \tag{4-7}$$

$$\frac{dL_4}{dL_2} = \frac{-L_2 [-2L_2^2 L_1^2 + a_{34}^2 L_1^2 - L_1^4 + L_3^2 L_1^2 + a_{12}^2 L_1^2 - L_4^2 a_{34}^2 + L_4^2 L_1^2 + a_{34}^2 a_{12}^2 + L_4^2 L_3^2 - L_3^2 a_{12}^2)]}{L_4 (L_2^2 a_{34}^2 - L_2^2 L_1^2 - L_2^2 L_3^2 + a_{12}^2 L_1^2 - L_3^2 L_1^2 - a_{34}^2 a_{12}^2 + 2L_4^2 L_3^2 + L_3^4 - L_3^2 a_{34}^2 - L_3^2 a_{12}^2)} \tag{4-8}$$

$$\frac{dL_4}{dL_3} = \frac{-L_3 [-L_2^2 L_1^2 - L_2^2 L_4^2 - L_2^2 a_{12}^2 - L_4^2 L_1^2 + L_4^4 + 2L_4^2 L_3^2 - L_4^2 a_{34}^2 - a_{12}^2 L_4^2 - a_{34}^2 a_{12}^2 - a_{34}^2 L_1^2)]}{L_4 (L_2^2 a_{34}^2 - L_2^2 L_1^2 - L_2^2 L_3^2 + a_{12}^2 L_1^2 - L_3^2 L_1^2 - a_{34}^2 a_{12}^2 + 2L_4^2 L_3^2 + L_3^4 - L_3^2 a_{34}^2 - L_3^2 a_{12}^2)} \tag{4-9}$$

Substituting Equation 4-7 into Equation 4-4 and rearranging gives

$$\begin{aligned} &D_1(L_1 L_2^4) + D_2(L_1^3 L_2^2) + D_3(L_1^2 L_2^2) + D_4(L_3^2 L_1 L_2^2) + D_5(L_1 L_2^2) \\ &+ D_6(L_3^2 L_2^2) + D_7(L_2^2) + D_8(L_3^4) + D_9(L_3^2) + D_{10} + D_{11}(L_3^2 L_1^3) + D_{12}(L_1^3) \\ &+ D_{13}(L_3^2 L_1^2) + D_{14}(L_1^2) + D_{15}(L_3^4 L_1) + D_{16}(L_3^2 L_1) + D_{17} (L_1) = 0 \end{aligned} \tag{4-10}$$

where

$D_1 = -k_4L_4 + k_4L_{04}$ ,

$D_2 = -2k_4L_4 + 2k_4L_{04} - k_1L_4$ ,

$D_3 = k_1L_4L_{01}$ ,

$D_4 = -k_1L_4 - k_4L_{04} + k_4L_4$ ,

$D_5 = -k_4L_{04}L_4^2 + k_4L_4^3 + k_1L_4a_{34}^2 + k_4L_4a_{12}^2 - k_4L_{04}a_{12}^2 + k_4a_{34}^2L_4 - k_4L_{04}a_{34}^2$ ,

$D_6 = k_1L_4L_{01}$ ,

$D_7 = -k_1L_4L_{01}a_{34}^2$ ,

$D_8 = k_1L_4L_{01}$ ,

$D_9 = k_1L_4L_{01}a_{12}^2 - 2k_1L_4^3L_{01} + k_1L_4L_{01}a_{34}^2$ ,

$D_{10} = k_1L_4L_{01}a_{34}^2a_{12}^2$ ,

$D_{11} = k_1L_4$ ,

$D_{12} = k_1L_4a_{12}^2$ ,

$D_{13} = k_1L_4L_{01}$ ,

$D_{14} = -k_1L_4L_{01}a_{12}^2$ ,

$D_{15} = k_1L_4$ ,

$D_{16} = 2k_1L_4^3 + k_4L_4^3 + k_4L_{04}a_{34}^2 - k_1L_4a_{12}^2 - k_4a_{34}^2L_4 + k_1L_4a_{34}^2 - k_4L_{04}L_4^2$ ,

$D_{17} = -k_4a_{12}^2L_4^3 - k_1L_4a_{34}^2a_{12}^2 + k_4L_4a_{34}^2a_{12}^2 + k_4L_{04}a_{12}^2L_4^2 - k_4L_{04}a_{34}^2a_{12}$. (4-11)

Substituting Equation 4-8 into Equation 4-5 and rearranging gives

$$E_1(L_1^2L_2^3) + E_2(L_3^2L_2^3) + E_3(L_2^3) + E_4(L_1^2L_2^2) + E_5(L_3^2L_2^2) + E_6(L_2^2) + E_7(L_1^4L_2) + E_8(L_3^2L_2) + E_9(L_1^2L_2) + E_{10}(L_3^4L_2) + E_{11}(L_3^2L_1^2L_2) + E_{12}(L_2) + E_{13}(L_3^2) + E_{14}(L_1^2) + E_{15}(L_3^4) + E_{16} = 0 \quad (4\text{-}12)$$

where

$E_1 = -k_2L_4 - 2k_4L_4 + 2k_4L_{04}$ ,

$E_2 = -k_2L_4$ ,

$E_3 = k_2L_4a_{34}^2$ ,

$E_4 = k_2L_4L_{02}$ ,

$E_5 = k_2L_4L_{02}$ ,

$E_6 = -k_2L_4L_{02}a_{34}^2$ ,

$E_7 = k_4L_{04} - k_4L_4$ ,

$E_8 = k_4L_4^3 - k_4L_{04}L_4^2 + 2k_2L_4^3 - k_2L_4a_{34}^2 - k_2L_4a_{12}^2 - k_4L_4a_{12}^2 + k_4L_{04}a_{12}^2$ ,

$E_9 = k_4L_4^3 + k_2L_4a_{12}^2 + k_4L_4a_{34}^2 + k_4L_4a_{12}^2 – k_4L_{04}a_{34}^2 – k_4L_{04}a_{12}^2 – k_4L_{04}L_4^2$ ,

$E_{10} = k_2L_4$ ,

$E_{11} = -k_2L_4 + k_4L_4 – k_4L_{04}$ ,

$E_{12} = -k_4L_4^3a_{34}^2 + k_4L_4a_{34}^2a_{12}^2 - k_4L_{04}a_{34}^2a_{12}^2 - k_2L_4a_{34}^2a_{12}^2 + k_4L_{04}L_4^2a_{34}^2$ ,

$E_{13} = k_2L_4L_{02}a_{12}^2 – 2k_2L_4^3L_{02} + k_2L_4L_{02}a_{34}^2 + k_2L_4L_{02}L_1^2$

$E_{14} = -k_2L_4L_{02}a_{12}^2$ ,

$E_{15} = -k_2L_4L_{02}$ ,

$E_{16} = k_2L_4L_{02}a_{34}^2a_{12}^2$ . (4-13)

Lastly, substituting Equation 4-9 into Equation 4-6 and rearranging gives

$$F_1(L_1^2L_2^2) + F_2(L_3L_1^2L_2^2) + F_3(L_3^3L_2^2) + F_4(L_3^2L_2^2) + F_5(L_3L_2^2) + F_6(L_2^2)$$
$$+ F_7(L_3^3L_1^2) + F_8(L_3 L_1^2) + F_9(L_1^2) + F_{10}(L_3^5) + F_{11}(L_3^4) + F_{12}(L_3^3) + F_{13}(L_3^2)$$
$$+ F_{14}(L_3) + F_{15} = 0 \qquad (4\text{-}14)$$

where

$F_1 = k_3L_4L_{03}$ ,

$F_2 = k_4L_4 - k_4L_{04} - k_3L_4$ ,

$F_3 = -k_3L_4$ ,

$F_4 = k_3L_4L_{03}$ ,

$F_5 = k_3L_4a_{34}^2 – k_4L_4a_{12}^2 + k_4L_4^3 – k_4L_{04}L_4^2 + k_4L_{04}a_{12}^2$ ,

$F_6 = -k_3L_4L_{03}a_{34}^2$ ,

$F_7 = -k_3L_4 + k_3L_4L_{03}$ ,

$F_8 = k_4L_4^3 + k_3L_4a_{12}^2 - k_4L_4a_{34}^2 – k_4L_{04}L_4^2 + k_4L_{04}a_{34}^2$ ,

$F_9 = -k_3L_4L_{03}a_{12}^2$ ,

$F_{10} = k_3L_4$ ,

$F_{11} = -k_3k_4L_{03}$ ,

$F_{12} = 2k_3L_4^3 – k_3L_4a_{34}^2 – k_3L_4a_{12}^2 + 2k_4L_{04}L_4^2 – 2k_4L_4^3$ ,

$F_{13} = -2k_3L_4^3L_{03} + k_3L_4L_{03}a_{34}^2 + k_3L_4L_{03}a_{12}^2$ ,

$$F_{14} = k_4L_4^3a_{34}^2 + k_4a_{12}^2L_4^3 + L_4L_{04}L_4^4 + k_3L_4a_{34}^2a_{12}^2 + k_4L_4a_{34}^2a_{12}^2$$
$$- k_4L_{04}L_4^2a_{34}^2 - k_4L_{04}a_{12}^2L_4^2 - k_4L_{04}a_{34}^2a_{12}^2 - k_4L_4^5 ,$$
$$F_{15} = k_3L_4L_{03}a_{34}^2a_{12}^2 . \qquad (4\text{-}15)$$

## 4.4 Solution of Geometry and Energy Equations

The problem at hand is to determine sets of values for the parameters $L_1$, $L_2$, $L_3$, and $L_4$ that will simultaneously satisfy Equations (4-1), (4-10), (4-12), and (4-14). Sylvester's method is one approach for solving this set of equations for all possible solutions. The parameter $L_4$ has been embedded in the coefficients of Equations (4-1), (4-10), (4-12), and (4-14) and Table 4-1 shows the 37 products of $L_1$, $L_2$, and $L_3$ that exist in these four equations. The Table indicates which products that are multiplied by which coefficients for the four equations.

Table 4-1. Coefficient G, D, E, and F.

| | $L_1^x L_2^y L_3^z$ | G's | D's | E's | F's |
|---|---|---|---|---|---|
| 1 | $L_1^4 L_2^4$ | $G_1$ | | | |
| 2 | $L_1^4 L_2^2$ | $G_2$ | | | |
| 3 | $L_3^2 L_1^2 L_2^2$ | $G_3$ | | | |
| 4 | $L_1^2 L_2^2$ | $G_4$ | $D_3$ | $E_4$ | $F_1$ |
| 5 | $L_3^2 L_2^2$ | $G_5$ | $D_6$ | $E_5$ | $F_4$ |
| 6 | $L_2^2$ | $G_6$ | $D_7$ | $E_6$ | $F_6$ |
| 7 | $L_3^2 L_1^2$ | $G_7$ | $D_{13}$ | | |
| 8 | $L_1^2$ | $G_8$ | $D_{14}$ | $E_{14}$ | $F_9$ |
| 9 | $L_3^4$ | $G_9$ | $D_8$ | $E_{15}$ | $F_{11}$ |
| 10 | $L_3^2$ | $G_{10}$ | $D_9$ | $E_{13}$ | $F_{13}$ |
| 11 | 1 | $G_{11}$ | $D_{10}$ | $E_{16}$ | $F_{15}$ |
| 12 | $L_1 L_2^4$ | | $D_1$ | | |
| 13 | $L_1^3 L_2^2$ | | $D_2$ | | |
| 14 | $L_3^2 L_1 L_2^2$ | | $D_4$ | | |
| 15 | $L_1 L_2^2$ | | $D_5$ | | |
| 16 | $L_3^2 L_1^3$ | | $D_{11}$ | | |
| 17 | $L_1^3$ | | $D_{12}$ | | |
| 18 | $L_3^4 L_1$ | | $D_{15}$ | | |
| 19 | $L_3^2 L_1$ | | $D_{16}$ | | |
| 20 | $L_1$ | | $D_{17}$ | | |
| 21 | $L_1^2 L_2^3$ | | | $E_1$ | |
| 22 | $L_3^2 L_2^3$ | | | $E_2$ | |
| 23 | $L_2^3$ | | | $E_3$ | |
| 24 | $L_1^4 L_2$ | | | $E_7$ | |
| 25 | $L_3^2 L_2$ | | | $E_8$ | |
| 26 | $L_1^2 L_2$ | | | $E_9$ | |

Table 4-1. Continued.

| | $L_1^x L_2^y L_3^z$ | G's | D's | E's | F's |
|---|---|---|---|---|---|
| 27 | $L_3^4 L_2$ | | | $E_{10}$ | |
| 28 | $L_3^2 L_1^2 L_2$ | | | $E_{11}$ | |
| 29 | $L_2$ | | | $E_{12}$ | |
| 30 | $L_3 L_1^2 L_2^2$ | | | | $F_2$ |
| 31 | $L_3^3 L_2^2$ | | | | $F_3$ |
| 32 | $L_3 L_2^2$ | | | | $F_5$ |
| 33 | $L_3^3 L_1^2$ | | | | $F_7$ |
| 34 | $L_3 L_1^2$ | | | | $F_8$ |
| 35 | $L_3^5$ | | | | $F_{10}$ |
| 36 | $L_3^3$ | | | | $F_{12}$ |
| 37 | $L_3$ | | | | $F_{14}$ |

It is apparent from the complexity of the four equations that attempting to multiply the four equations by different products of $L_1$, $L_2$, and $L_3$ in order to obtain a set of "homogeneous" equations where the number of equations equals the number of unknowns is impractical. For this reason, a numerical case will be considered and this case will be solved using the continuation approach.

### 4.5 Numerical Case Study

The following parameters were selected as a numerical example:

Strut lengths: $a_{12} = 14$ in., $a_{34} = 12$ in.

Spring free lengths and spring constants:

$L_{01} = 8$ in. $\quad k_1 = 1$ lbf/in.

$L_{02} = 2.68659245$ in. $\quad k_2 = 2.0$ lbf/in.

$L_{03} = 3.46513678$ in. $\quad k_3 = 2.5$ lbf/in.

$L_{04} = 7.3082878$ in. $\quad k_4 = 1.5$ lbf/in.

Equations (4-1), (4-10), (4-12), and (4-14) can now be expressed respectively as

$$L_2^4L_1^2 + (L_1^4 - L_3^2L_1^2 - 440.0\,L_1^2 + 96.0\,L_3^2 - 13824.0)\,L_2^2 - 8424.0\,L_1^2 - 8624.0\,L_1^2 + 44.0\,L_3^2\,L_1^2 + 6773760.0 - 52224.0\,L_3^2 + L_3^4 = 0 \qquad (4\text{-}16)$$

$- 4.03756830\, L_2^4 L_1 + (-11520.0 - 5.96243170\, L_3^2 L_1 + 80.0\, L_1^2 - 18.07513660.0\, L_1^3 + 3216.53005200\, L_1 + 80.0\, L_3^2) L_2^2 + 11200.0\, L_3^2 + 1960.0\, L_1^3 - 10\, L_3^2 L_1^3 - 1577.65300520\, L_3^2 L_1 + 10.0\, L_1 L_3^4 - 80.0\, L_3^4 + 0.2257920\, 10^7 - 15680.0\, L_1^2 - 247420.01098080\, L_1 + 80.\, L_3^2 L_1^2 = 0$

(4-17)

$(-28.07513660\, L_1^2 + 2880.0 - 20.0\, L_3^2) L_2^3 + (-7737.38625600 + 53.73184900\, L_3^2 + 53.73184900\, L_1^2) L_2^2 + (-3187.60655680\, L_3^2 + 5696.53005200\, L_1^2 - 15.96243170\, L_3^2 L_1^2 + 20.0\, L_3^4 - 4.03756830\, L_1^4 - 508664.65582080) L_2 + 7522.45886000\, L_3^2 + 0.151652770617600\, 10^7 - 53.73184900\, L_3^4 - 10531.44240400\, L_1^2 + 53.73184900\, L_3^2 L_1^2 = 0$

(4-18)

$- 494742.03310080\, L_3 + 12127.978730000\, L_3^2 - 16979.170222000\, L_1^2 + 86.628419500\, L_3^2 L_1^2 - 86.628419500\, L_3^4 - 4307.51366000\, L_3^3 + 4722.34699480\, L_3 L_1^2 + 25.0\, L_3^5 - 25.0\, L_3^3 L_1^2 + (-20.96243170\, L_3 L_1^2 + 86.628419500\, L_1^2 + 3212.39344320\, L_3 - 25.0\, L_3^3 - 12474.492408000 + 86.628419500\, L_3^2) L_2^2 + 0.2445000511968000\, 10^7 = 0$

(4-19)

The Polynomial Continuation Method was used to find simultaneous roots for the four equations. Eighteen real solutions and 490 complex solutions were found. All the real and complex solutions were shown to satisfy Equations (4-16) through (4-19) with neglible residuals. The real solutions are listed in Table 4-2.

Table 4-2. Eighteen real solutions. (Solutions that are geometrically impossible are shaded.)

| | $L_1$, in. | $L_2$, in. | $L_3$, in. | $L_4$, in. |
|---|---|---|---|---|
| 1 | 19.028528 | 3.624423 | 3.108440 | 4.306682 |
| 2 | 13.000001 | 7.999996 | 7.016593 | 9.999998 |
| 3 | 11.632986 | -4.613617 | 9.950038 | 12.269992 |

Table 4-2. Continued.

| | $L_1$, in. | $L_2$, in. | $L_3$, in. | $L_4$, in. |
|---|---|---|---|---|
| 4 | 3.141567 | 2.156404 | 5.241983 | 18192258 |
| 5 | -7.985150 | 10.049547 | -6.186358 | 14.286022 |
| 6 | -0.398136 | 4.497517 | 10.533965 | 15.669134 |
| 7 | -6.868035 | 1.105366 | 3.486019 | 20.971818 |
| 8 | 4.200027 | -3.538652 | 10.462012 | 15.411513 |
| 9 | 19.507910 | 4.034579 | 3.169571 | 5.347832 |
| 10 | -16.748403 | 8.590494 | 2.764444 | 2.329743 |
| 11 | 4.520731 | 0.603385 | -7.518882 | 18.852623 |
| 12 | 4.077402 | 4.708715 | 9.316941 | 16.646605 |
| 13 | 13.473294 | 11.823371 | -2.235364 | 7.821817 |
| 14 | 22.800126 | 6.225147 | 13.541713 | -5.861200 |
| 15 | 14.616270 | -9.580282 | -4.547034 | 8.863331 |
| 16 | 20.778750 | -0.084697 | 4.369638 | 6.733159 |
| 17 | -12.774217 | 12.496006 | -1.650677 | 8.309365 |
| 18 | 3.806954 | 2.744884 | 8.098941 | 17.877213 |

Cases 1, 4, 7, 9, 10, 11, 14, 16, and 18 are not physically realizable. For example, $L_1$ in case 1 is greater than sum of $a_{12}$ and $L_4$, and hence point 2 cannot be constructed.

The solutions in Table 4-2 can correspond to either maximum or minimum potential energy configurations. The second derivative of the potential energy with respect to the variables $L_1$, $L_2$, and $L_3$ are now evaluated to identity the minimal potential energy cases.

Derivatives of Equations (4-4) through (4-6) with respect to $L_1$, $L_2$, and $L_3$, respectively can be written as

$$\frac{d^2U}{dL_1^{\ 2}} = k_1 + k_4(\frac{dL_4}{dL_1})^2 + k_4(L_4 - L_{04})\frac{d^2L_4}{dL^2_{\ 1}} \ ,$$

$$\frac{d^2U}{dL_2^{\ 2}} = k_2 + k_4(\frac{dL_4}{dL_2})^2 + k_4(L_4 - L_{04})\frac{d^2L_4}{dL^2_{\ 2}} \ ,$$

$$\frac{d^2U}{dL_3^{\ 2}} = k_3 + k_4(\frac{dL_4}{dL_3})^2 + k_4(L_4 - L_{04})\frac{d^2L_4}{dL^2_{\ 3}} \ .$$

(4-20)

Table 4-3 shows the value of the potential energy and the second derivative for each of the remaining nine cases. It is seen that cases 2, 3, 13, 15, and 17 have all positive second derivative values and represent possible equilibrium solutions. Each of these solutions was analyzed to determine if it truly corresponds to an equilibrium configuration and only cases 2, 3, 13, and 15 were indeed in equilibrium.

Table 4-3. Identification of minimum potential energy state. (Solutions that do not have all positive second derivatives are shaded.)

| Case | $L_1$, in | $L_2$, in | $L_3$, in | $L_4$, in | U, in-lbf | $\frac{\partial^2 U}{\partial L_1^2}$ lbf/in | $\frac{\partial^2 U}{\partial L_2^2}$ lbf/in | $\frac{\partial^2 U}{\partial L_3^2}$ lbf/in |
|---|---|---|---|---|---|---|---|---|
| 2 | 13.000 | 7.999 | 7.016 | 9.999 | 61.932 | 1.146 | 3.052 | 4.569 |
| 3 | 11.632 | -4.613 | 9.950 | 12.269 | 130.923 | 0.740 | 0.566 | 4.065 |
| 5 | -7.985 | 10.049 | -6.186 | 14.286 | 334.931 | -0.470 | 1.241 | 1.594 |
| 6 | -0.398 | 4.497 | 10.533 | 15.669 | 153.432 | 3.597 | 1.889 | -0.750 |
| 8 | 4.200 | -3.538 | 10.462 | 15.411 | 156.415 | 0.659 | -16.134 | -2494.334 |
| 12 | 4.077 | 4.708 | 9.316 | 16.646 | 119.990 | -0.221 | -15.521 | -379.009 |
| 13 | 13.473 | 11.823 | -2.235 | 7.821 | 139.276 | 4.967 | 51.239 | 56.665 |
| 15 | 14.616 | -9.580 | -4.547 | 8.863 | 25.420 | 2.405 | 25.608 | 32.573 |
| 17 | -12.774 | 12.496 | -1.650 | 8.309 | 345.474 | 10.529 | 50.466 | 39.862 |

Figure 4-2 shows the four feasible solutions that are in stable equilibrium drawn to scale. Dashed lines are used to identify the springs that have negative spring lengths. It must be noted that when a spring has a negative spring length, the two points at the end of the spring want to move towards one another as the spring attempts to return to its free length.

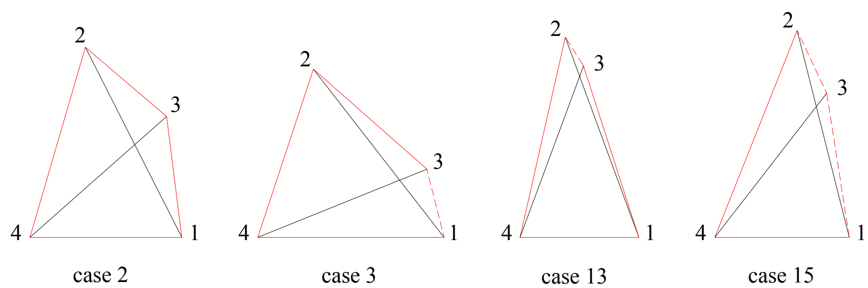


Figure 4-2. Four equilibrium solutions

A force analysis was conducted for each of the four cases and these cases were found to be in static equilibrium. Table 4-4 shows the calculated force in each of the four springs and the two struts for the equilibrium cases. Negative values in the table correspond to the springs which have a negative spring length.

Table 4-4. Calculated forces in struts, and elastic ties.

| Case | Force in Spring $L_1$, lbf | Force in Spring $L_2$, lbf | Force in Spring $L_3$, lbf | Force in Spring $L_4$, lbf | Force in Strut $a_{12}$, lbf | Force in Strut $a_{34}$, lbf |
|---|---|---|---|---|---|---|
| 2 | 5.000 | 10.625 | 8.877 | 4.036 | 11.828 | 7.220 |
| 3 | 3.632 | - 14.599 | 16.212 | 7.441 | 17.942 | 9.253 |
| 13 | 5.473 | 18.273 | - 14.250 | 0.769 | 18.5294 | 5.8228 |
| 15 | 6.616 | - 24.533 | - 20.030 | 2.332 | 25.005 | 7.6394 |

## 4.6 Conclusions

The problem of determining the equilibrium configurations of a four-spring planar tensegrity system was formulated by developing four equations in terms of the four spring lengths. It was not practical to use Sylvester's method to solve this set of equations. Rather, the continuation method was used for a particular numerical example and four equilibrium configurations were obtained. Cases may exist that would yield more than four realizable configurations, and this could be investigated in future work. The objective here, however, was to determine if it was tractable at all to solve the equilibrium problem for multiple (if not all) solutions, either symbolically or numerically, and this goal was met.

# CHAPTER 5
# FUTURE WORK, SUMMARY, AND CONCLUSION

## 5.1 Future Work

### 5.1.1 The 3D Tensegrity Platform

As part of future suggested research work, an introduction to the position analysis of a general three dimensional parallel platform device consisting of 3 struts, 3 springs, and 6 ties (Figure 5-1) is presented here. The problem statement is given as follows:

Given: length of non-compliant ties, struts, and free lengths and spring constants of the compliant ties.

Find: final length of compliant ties at all equilibrium configurations.

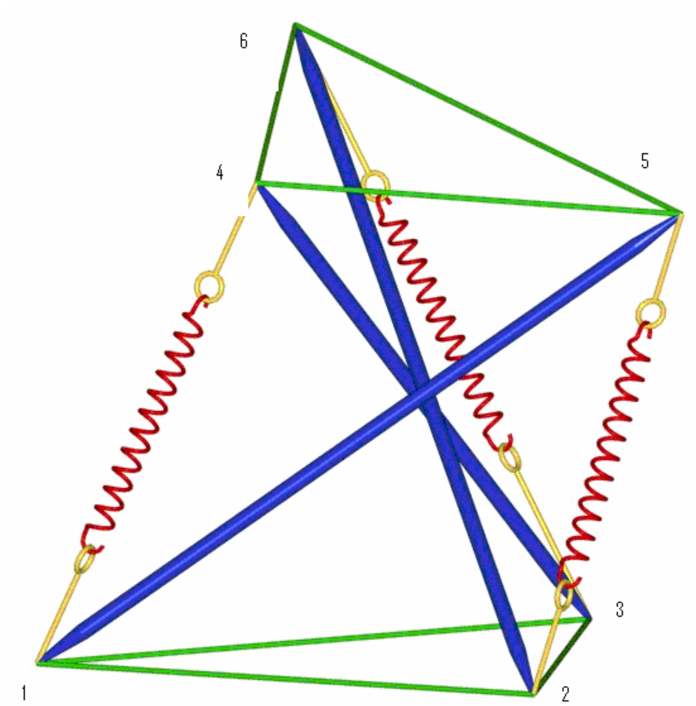


Figure 5-1. Three-dimensional parallel platform of 3 struts, 3 springs, and 6 ties.

Three approaches to this problem are presented here:

1. Minimum potential energy analysis.
2. Force balance analysis.
3. Linear dependence of 6 connector lines.

The unexpected complexity, i.e. high degree polynomials, encountered in the planar analysis may indicate that the analysis of the general three strut tensegrity system will be very complex.

This is a three degree of freedom system. One approach is to define the system by six angles, subject to three constraints. That is, the coordinates of the tip of each strut (points 4, 5, and 6) can be defined by angles $\alpha_i$ and $\beta_i$, i = 1..3, which define the orientation of each of the three struts relative to the base triangle. The three constraints are that the distances between points 4, 5, and 6 are given. The potential energy can be written in terms of these six angles and three equations can be generated by equating the partial derivative of the potential energy with respect to three of the angles to zero. These three equations, together with the three distance constraint equations would give a set of six equations in the six angles.

In the second approach, force balance equations at points 4, 5, and 6 could be written as functions of the lengths of the three springs. This is complicated in that the positions of points 4, 5, and 6 would have to be determined from a forward analysis of the device, i.e. the device is equivalent to a 3-3 parallel mechanism, whose forward analysis was solved by Griffis and Duffy [27]. The solution of the forward analysis of the 3-3 parallel mechanism requires rooting a fourth degree polynomial in the square of one parameter, resulting in eight possible configurations for the device. The nature of the forward analysis makes it infeasible to obtain a symbolic solution to the equilibrium configuration problem. A numerical solution may be possible.

The third approach utilizes the fact that if the device is in equilibrium with no external forces applied, then the Plücker coordinates of the lines along the three struts and the three springs must be linearly dependent. For this solution, it is then necessary to obtain the Plücker coordinates of these lines as functions of the three spring lengths. This again requires a forward analysis of the 3-3 parallel mechanism. Further, only a single equation results from equating the determinant of the 6×6 matrix of line coordinates to zero. This leads one to believe that the fact that the lines must be linearly dependent is a necessary, but not sufficient condition for equilibrium. Further analysis is necessary.

### 5.1.2 The 2D Tensegrity structures

Cases of two dimensional tensegrity structures proved to be mathematically challenging in spite of their simple configurations. Different methods were tried to solve these problems. It was expected the final closed form solution be a simple mathematical expression. In each case the intermediate mathematical expressions become complicated to solve and the relevant matrices become large to manipulate. Other mathematical procedures might exist to solve these problems with less complexity.

## 5.2: Summary and Conclusion

In this research, all possible combinations of two dimensional tensegrity structures consisting of two struts, springs, and non-compliant members were analyzed. A closed form solution could only be obtained for case of 2 spring, 2 struts, and 2 non-compliant members. Finding a simple solution approach for a closed form solution to the other two dimensional tensegrity structures consisting of either 3 or 4 springs was not possible due to the complexity of mathematical solutions. A numerical solution using the Continuity Method was used to solve individual numerical examples.

The contribution of this research was to set a basic mathematical foundation for the closed form solutions of tensegrity structures and also to present numerical approaches where a closed form solution could not be obtained. The contribution of this research leads to a better understanding of the complexity of these devices and hopes to lay a ground work for future analyses of more complex systems.

# APPENDIX
# SAMPLE OUTPUT

The following coefficients belong to the numerical example of a Maple sessions programming.

$$(p_1 c_1^2 + p_2 c_1 + p_3) c_4^3 + (p_4 c_1^3 + p_5 c_1^2 + p_6 c_1 + p_7) c_4^2 + (p_8 c_1^4 + p_9 c_1^3 + p_{10} c_1^2 + p_{11} c_1 + p_{12}) c_4 + (p_{13} c_1^4 + p_{14} c_1^3 + p_{15} c_1^2 + p_{16} c_1 + p_{17}) = 0 \quad (3\text{-}65)$$

$$(q_1 c_1 + q_2) c_4^4 + (q_3 c_1^2 + q_4 c_1 + q_5) c_4^3 + (q_6 c_1^3 + q_7 c_1^2 + q_8 c_1 + q_9) c_4^2 + (q_{10} c_1^3 + q_{11} c_1^2 + q_{12} c_1 + q_{13}) c_4 + (q_{14} c_1^3 + q_{15} c_1^2 + q_{16} c_1 + q_{17}) = 0 \quad (3\text{-}66)$$

$$(r_1 c_1 + r_2) c_4^2 + (r_3 c_1^2 + r_4 c_1 + r_5) c_4 + (r_6 c_1^2 + r_7 c_1 + r_8) = 0 \quad (3\text{-}69)$$

```
> #-------------------------------------------------------------
> # Numerical example
```

$a12 := 14 \quad a34 := 12 \quad a41 := 10$

$L01 := 8 \quad k1 := 1$

$L02 := 2 \quad k2 := 2.687$

$L03 := 2.5 \quad k3 := 3.465$

$p1 := 0.5773663961\ 10^{15}\ L3^2 - 0.2080248063\ 10^{16}\ L3 + 0.1873780685\ 10^{16}$

$p2 := 0.1110100300\ 10^{16}\ L3^2 - 0.4198962128\ 10^{16}\ L3 + 0.3961707733\ 10^{16}$

$p3 := 0.5335964670\ 10^{15}\ L3^2 - 0.2114119475\ 10^{16}\ L3 + 0.2094045517\ 10^{16}$

$p4 := 0.4853912146\ 10^{15}\ L3^2 - 0.2087909024\ 10^{16}\ L3 + 0.2186077465\ 10^{16}$

$$p5 := -0.1733540051\ 10^{13}\ L3^4 + 0.7456817937\ 10^{13}\ L3^3 + 0.2423387462\ 10^{16}\ L3^2 - 0.9462608682\ 10^{16}\ L3 + 0.9212755033\ 10^{16}$$

$$p6 := -0.3864997467\ 10^{13}\ L3^4 + 0.1452389258\ 10^{14}\ L3^3 + 0.3406073884\ 10^{16}\ L3^2 - 0.1266504024\ 10^{17}\ L3 + 0.1176377359\ 10^{17}$$

$$p7 := -0.2113487353\ 10^{13}\ L3^4 + 0.7162795261\ 10^{13}\ L3^3 + 0.1466145513\ 10^{16}\ L3^2 - 0.5300984714\ 10^{16}\ L3 + 0.4722921577\ 10^{16}$$

$$p8 := 0.1020169484\ 10^{15}\ L3^2 - 0.5100847420\ 10^{15}\ L3 + 0.6376059274\ 10^{15}$$

$$p9 := -0.7286924884\ 10^{12}\ L3^4 + 0.3643462442\ 10^{13}\ L3^3 + 0.1068305616\ 10^{16}\ L3^2 - 0.4737062676\ 10^{16}\ L3 + 0.5137282043\ 10^{16}$$

$$p10 := 0.1301236586\ 10^{10}\ L3^6 - 0.6506182936\ 10^{10}\ L3^5 - 0.4747632005\ 10^{13}\ L3^4 + 0.2052962276\ 10^{14}\ L3^3 + 0.3082075504\ 10^{16}\ L3^2 - 0.1236996613\ 10^{17}\ L3 + 0.1219206631\ 10^{17}$$

$$p11 := 0.3300435984\ 10^{10}\ L3^6 - 0.1289836349\ 10^{11}\ L3^5 - 0.7989672476\ 10^{13}\ L3^4 + 0.2941695331\ 10^{14}\ L3^3 + 0.3455740168\ 10^{16}\ L3^2 - 0.1248534533\ 10^{17}\ L3 + 0.1113665191\ 10^{17}$$

$$p12 := 0.2092793462\ 10^{10}\ L3^6 - 0.5893635686\ 10^{10}\ L3^5 - 0.3983027155\ 10^{13}\ L3^4 + 0.1246857458\ 10^{14}\ L3^3 + 0.1339547765\ 10^{16}\ L3^2 - 0.4345164946\ 10^{16}\ L3 + 0.3440522646\ 10^{16}$$

$$p13 := 0.1037172309\ 10^{15}\ L3^2 - 0.5185861544\ 10^{15}\ L3 + 0.6482326928\ 10^{15}$$

$$p14 := -0.7408373632\ 10^{12}\ L3^4 + 0.3704186816\ 10^{13}\ L3^3 + 0.5844049572\ 10^{15}\ L3^2 - 0.2657927757\ 10^{16}\ L3 + 0.2963349453\ 10^{16}$$

$$p15 := 0.1322923863\ 10^{10}\ L3^6 - 0.6614619316\ 10^{10}\ L3^5 - 0.3034952946\ 10^{13}\ L3^4 + 0.1316433327\ 10^{14}\ L3^3 + 0.1235296500\ 10^{16}\ L3^2 - 0.4981478398\ 10^{16}\ L3 + 0.4841901337\ 10^{16}$$

$$p16 := 0.3355443251\ 10^{10}\ L3^6 - 0.1311333621\ 10^{11}\ L3^5 - 0.4128749122\ 10^{13}\ L3^4 + 0.1489517912\ 10^{14}\ L3^3 + 0.1159476235\ 10^{16}\ L3^2 - 0.4015136920\ 10^{16}\ L3 + 0.3326208569\ 10^{16}$$

$$p17 := 0.2121019430\ 10^{10}\ L3^6 - 0.5991862948\ 10^{10}\ L3^5 - 0.1862278592\ 10^{13}\ L3^4 + 0.5272839394\ 10^{13}\ L3^3 + 0.4069170754\ 10^{15}\ L3^2 - 0.1160024667\ 10^{16}\ L3 + 0.8167029969\ 10^{15}$$

$$q1 := 0.8744309862\ 10^{14}\ L3^2 - 0.4372154932\ 10^{15}\ L3 + 0.5465193664\ 10^{15}$$

$$q2 := 0.9243984712\ 10^{14}\ L3^2 - 0.4621992356\ 10^{15}\ L3 + 0.5777490445\ 10^{15}$$

$$q3 := 0.4669520350\ 10^{15}\ L3^2 - 0.2187549572\ 10^{16}\ L3 + 0.2550423709\ 10^{16}$$

$$q4 := -0.7286924880\ 10^{12}\ L3^4 + 0.3643462442\ 10^{13}\ L3^3 + 0.9975040455\ 10^{15}\ L3^2 - 0.4749518798\ 10^{16}\ L3 + 0.5610932160\ 10^{16}$$

$$q5 := -0.7703320590\ 10^{12}\ L3^4 + 0.3851660295\ 10^{13}\ L3^3 + 0.5326615535\ 10^{15}\ L3^2 - 0.2576221616\ 10^{16}\ L3 + 0.3081328237\ 10^{16}$$

$$q6 := 0.6233888280\ 10^{15}\ L3^2 - 0.2723886871\ 10^{16}\ L3 + 0.2975494328\ 10^{16}$$

$$q7 := -0.1945633479\ 10^{13}\ L3^4 + 0.9114789880\ 10^{13}\ L3^3 + 0.2467565040\ 10^{16}\ L3^2 - 0.1115537420\ 10^{17}\ L3 + 0.1253958324\ 10^{17}$$

$$q8 := 0.1518109351\ 10^{10}\ L3^6 - 0.7590546756\ 10^{10}\ L3^5 - 0.4948249499\ 10^{13}\ L3^4 + 0.2326400716\ 10^{14}\ L3^3 + 0.2978451536\ 10^{16}\ L3^2 - 0.1366694272\ 10^{17}\ L3 + 0.1548714436\ 10^{17}$$

$$q9 := 0.1604858457\ 10^{10}\ L3^6 - 0.8024292286\ 10^{10}\ L3^5 - 0.3056661932\ 10^{13}\ L3^4 + 0.1440713627\ 10^{14}\ L3^3 + 0.1127231047\ 10^{16}\ L3^2 - 0.5199241185\ 10^{16}\ L3 + 0.5873781950\ 10^{16}$$

$$q10 := 0.1238733084\ 10^{16}\ L3^2 - 0.5475596549\ 10^{16}\ L3 + 0.6050171798\ 10^{16}$$

$$q11 := -0.4022366239\ 10^{13}\ L3^4 + 0.1777220459\ 10^{14}\ L3^3 + 0.3554116422\ 10^{16}\ L3^2 - 0.1568854809\ 10^{17}\ L3 + 0.1720685863\ 10^{17}$$

$$q12 := 0.3260393580\ 10^{10}\ L3^6 - 0.1447643958\ 10^{11}\ L3^5 - 0.7932329962\ 10^{13}\ L3^4 + 0.3478741954\ 10^{14}\ L3^3 + 0.3352073482\ 10^{16}\ L3^2 - 0.1470298991\ 10^{17}\ L3 + 0.1588185279\ 10^{17}$$

$$q13 := 0.3446701784\ 10^{10}\ L3^6 - 0.1530366470\ 10^{11}\ L3^5 - 0.3895030685\ 10^{13}\ L3^4 + 0.1691392672\ 10^{14}\ L3^3 + 0.1035832980\ 10^{16}\ L3^2 - 0.4479338270\ 10^{16}\ L3 + 0.4707584806\ 10^{16}$$

$$q14 := 0.6153702088\ 10^{15}\ L3^2 - 0.2751416753\ 10^{16}\ L3 + 0.3075503998\ 10^{16}$$

$$q15 := -0.2075805867\ 10^{13}\ L3^4 + 0.8667876209\ 10^{13}\ L3^3 + 0.1553312052\ 10^{16}\ L3^2 - 0.6722506098\ 10^{16}\ L3 + 0.7212669096\ 10^{16}$$

$$q16 := 0.1750560045\ 10^{10}\ L3^6 - 0.6792486624\ 10^{10}\ L3^5 - 0.3717087746\ 10^{13}\ L3^4 + 0.1514057165\ 10^{14}\ L3^3 + 0.1284193314\ 10^{16}\ L3^2 - 0.5346826116\ 10^{16}\ L3 + 0.5463422536\ 10^{16}$$

$$q17 := -0.7180628716\ 10^{10}\ L3^5 - 0.1603893518\ 10^{13}\ L3^4 + 0.6318953271\ 10^{13}\ L3^3 + 0.1830521647\ 10^{10}\ L3^6 - 0.1390169719\ 10^{16}\ L3 + 0.1348526898\ 10^{16} + 0.3482593233\ 10^{15}\ L3^2$$

$$r1 := 8\ a12\ a41\ a34^2$$

$$r2 := 4\ a41^2\ a34^2 + 4\ a34^2\ a12^2$$

$$r3 := 8\ a34\ a41\ a12^2$$

$$r4 := 12\ a12\ a41^2\ a34 - 4\ a12\ a34\ L3^2 + 4\ a12\ a34^3 + 4\ a12^3\ a34$$

$$r5 := -4\ a34\ a41\ L3^2 + 4\ a41^3\ a34 + 4\ a34\ a41\ a12^2 + 4\ a34^3\ a41$$

$$r6 := 4\ a41^2\ a12^2 + 4\ a34^2\ a12^2$$

$$r7 := 4\ a12^3\ a41 - 4\ a12\ a41\ L3^2 + 4\ a12\ a41\ a34^2 + 4\ a12\ a41^3$$

$$r8 := L3^4 + (-2\ a34^2 - 2\ a41^2 - 2\ a12^2)\ L3^2 + a41^4 + a34^4 + a12^4 + 2\ a41^2\ a34^2 + 2\ a41^2\ a12^2 - 2\ a34^2\ a12^2$$

The following portion of Maple program presents the three equations used in continuation method.

(3-71) = **eqn1_for_cont**

(3-72) = **eqn3_16_sq**

(3-73) = **eqn3_17_sq**

```
> # 3 springs cosine of theta1, cosine of theta2 and L23, Continuation method
#+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
> a41:= 10 ;  a12:= 14.0 ;  a34:= 12 ;
```

$$a41 := 10$$

$$a12 := 14.0$$

$$a34 := 12$$

```
> k24:= 1 ;  k31:= 2.0 ;  k23:= 2.5 ;
```

$$k24 := 1$$

$$k31 := 2.0$$

$$k23 := 2.5$$

>

```
> L024:= 8.0;   L031:= 2.686592430;   L023:= 3.465136806;
```

$$L024 := 8.0$$

$$L031 := 2.686592430$$

$$L023 := 3.465136806$$

```
>  eqn1_for_cont:= eqn1_for_cont ;
```

$$eqn1\_for\_cont := 61600.000\ c1 + 42624.00\ c4^2 - 168.0\ c1\ c4\ L23^2 + \frac{1}{4} L23^4 - 220.0000000\ L23^2 + 47824.00\ c1^2 + 52800.00\ c4 - 140.0\ c1\ L23^2 + 47040.00\ c1^2\ c4 + 40320.0\ c1\ c4^2 + 107520.000\ c1\ c4 + 20176.00000 - 120\ c4\ L23^2$$

>

```
> eqn3_16_sq:= eqn3_16_sq ;
```

$$eqn3\_16\_sq := 0.2094210873\ 10^{16}\ c4^3 + 0.1119744000\ 10^{10}\ L23^6\ c4 + 0.1131754495\ 10^{10}\ L23^6 - 0.4383037685\ 10^{10}\ L23^5 + 0.3326471226\ 10^{16}\ c1 + 0.4723294525\ 10^{16}\ c4^2 + 0.5399138304\ 10^{14}\ L23^2\ c1^4 + 0.1817487566\ 10^{16}\ c1\ c4\ L23^2 - 0.3619570961\ 10^{16}\ L23\ c1^2 + 0.3038915898\ 10^{15}\ L23^2\ c1^3 - 0.4221575075\ 10^{13}\ L23^4\ c1\ c4 - 0.9917251153\ 10^{12}\ L23^4 + 0.2153950907\ 10^{15}\ L23^2 + 0.4842283683\ 10^{16}\ c1^2 - 0.8485560960\ 10^{15}\ L23 + 0.3440794329\ 10^{16}\ c4 + 0.1488688451\ 10^{14}\ L23^3\ c1^2\ c4 - 0.2488945515\ 10^{13}\ L23^4\ c1^2\ c4 + 0.6376562766\ 10^{15}\ c1^4\ c4 + 0.1275251673\ 10^{16}\ L23^2\ c1^2\ c4^2 + 0.1614293490\ 10^{16}\ L23^2\ c1^2\ c4 + 0.1795299633\ 10^{16}\ L23^2\ c4^2\ c1 - 0.3793305600\ 10^{12}\ L23^4\ c1^3\ c4 - 0.4694401018\ 10^{10}\ L23^5\ c1^2\ c4 + 0.6773760000\ 10^{9}\ L23^6\ c1^2\ c4 - 0.2043740160\ 10^{13}\ L23^4\ c1\ c4^2 - 0.9388802036\ 10^{10}\ L23^5\ c1\ c4 - 0.1535416328\ 10^{16}\ L23\ c4^3 + 0.1741824000\ 10^{10}\ L23^6\ c1\ c4 + 0.5580683989\ 10^{15}\ L23^2\ c1^3\ c4 + 0.2141076067\ 10^{14}\ L23^3\ c1\ c4 + 0.2628864571\ 10^{13}\ L23^3\ c1^3\ c4 + 0.9213482524\ 10^{16}\ c1^2\ c4^2 + 0.2186250090\ 10^{16}\ c1^3\ c4^2 + 0.6089439462\ 10^{15}\ c1\ L23^2 - 0.9103933440\ 10^{12}\ L23^4\ c1^2\ c4^2 + 0.5137687713\ 10^{16}\ c1^3\ c4 + 0.1055837852\ 10^{14}\ L23^3\ c1\ c4^2 + 0.6445019392\ 10^{15}\ L23^2\ c1^2 + 0.5407949973\ 10^{13}\ L23^3\ c1^2\ c4^2 + 0.3857073162\ 10^{13}\ L23^3 + 0.1873928649\ 10^{16}\ c1^2\ c4^3$$

$+ 0.3962020571\ 10^{16}\ c1\ c4^3 - 0.3741750572\ 10^{15}\ L23\ c1^4$
$+ 0.5310627840\ 10^{14}\ L23^2\ c1^4\ c4 - 0.4311184608\ 10^{10}\ L23^5\ c4$
$+ 0.2963583455\ 10^{16}\ c1^3 + 0.2814309827\ 10^{15}\ L23^2\ c4^3$
$+ 0.2549101363\ 10^{15}\ L23^2\ c1^3\ c4^2 + 0.1219302908\ 10^{17}\ c1^2\ c4$
$+ 0.5868135383\ 10^{15}\ L23^2\ c4^3\ c1 - 0.2119474277\ 10^{13}\ L23^4\ c4$
$- 0.3053174043\ 10^{16}\ L23\ c4^3\ c1 - 0.3856527360\ 10^{12}\ L23^4\ c1^3$
$- 0.9545282070\ 10^{10}\ L23^5\ c1 - 0.1122729984\ 10^{13}\ L23^4\ c4^2$
$+ 0.6482838809\ 10^{15}\ c1^4 + 0.9545282070\ 10^{13}\ L23^3\ c1^2$
$+ 0.7738436783\ 10^{15}\ L23^2\ c4^2 + 0.1176470253\ 10^{17}\ c1\ c4^2$
$- 0.4772641035\ 10^{10}\ L23^5\ c1^2 + 0.1113753133\ 10^{17}\ c1\ c4$
$+ 0.6886656000\ 10^{9}\ L23^6\ c1^2 + 0.1085434933\ 10^{14}\ L23^3\ c1$
$- 0.1924328866\ 10^{16}\ L23\ c1^3 - 0.2180321636\ 10^{13}\ L23^4\ c1$
$+ 0.5224006096\ 10^{13}\ L23^3\ c4^2 - 0.2927947095\ 10^{16}\ L23\ c1$
$- 0.1514225992\ 10^{16}\ L23\ c1^2\ c4^3 + 0.1770854400\ 10^{10}\ L23^6\ c1$
$- 0.6875307066\ 10^{16}\ L23\ c1^2\ c4^2 - 0.9212468531\ 10^{16}\ L23\ c1\ c4^2$
$+ 0.3058921636\ 10^{15}\ L23^2\ c1^2\ c4^3 - 0.3171517401\ 10^{16}\ L23\ c4$
$- 0.3859569282\ 10^{16}\ L23\ c4^2 + 0.9104915321\ 10^{13}\ L23^3\ c4$
$- 0.9090163022\ 10^{16}\ L23\ c1\ c4 + 0.8167674885\ 10^{15}$
$- 0.1589435265\ 10^{13}\ L23^4\ c1^2 + 0.7079662564\ 10^{15}\ c4\ L23^2$
$- 0.3432245583\ 10^{16}\ L23\ c1^3\ c4 - 0.8985008439\ 10^{16}\ L23\ c1^2\ c4$
$+ 0.2672678980\ 10^{13}\ L23^3\ c1^3 - 0.3680410399\ 10^{15}\ L23\ c1^4\ c4$
$- 0.1514225992\ 10^{16}\ L23\ c1^3\ c4^2$

```
> eqn3_17_sq:= eqn3_17_sq ;
```

$eqn3\_17\_sq := 0.3081571556\ 10^{16}\ c4^3 + 0.1949337600\ 10^{10}\ L23^6\ c4$
$+ 0.1117043200\ 10^{10}\ L23^6 - 0.5628934554\ 10^{10}\ L23^5 + 0.5463853959\ 10^{16}\ c1$
$+ 0.5874245780\ 10^{16}\ c4^2 + 0.1911253794\ 10^{16}\ c1\ c4\ L23^2$
$- 0.5115729560\ 10^{16}\ L23\ c1^2 + 0.3470298604\ 10^{15}\ L23^2\ c1^3$
$- 0.4526879307\ 10^{13}\ L23^4\ c1\ c4 - 0.9760319345\ 10^{12}\ L23^4$
$+ 0.2101294121\ 10^{15}\ L23^2 + 0.7213238649\ 10^{16}\ c1^2 - 0.1089761729\ 10^{16}\ L23$
$+ 0.4707956544\ 10^{16}\ c4 + 0.1339260450\ 10^{14}\ L23^3\ c1^2\ c4$
$- 0.2286520320\ 10^{13}\ L23^4\ c1^2\ c4 + 0.1380473656\ 10^{16}\ L23^2\ c1^2\ c4^2$
$+ 0.2019950461\ 10^{16}\ L23^2\ c1^2\ c4 + 0.1648149759\ 10^{16}\ L23^2\ c4^2\ c1$
$- 0.2693873525\ 10^{13}\ L23^4\ c1\ c4^2 - 0.1095360238\ 10^{11}\ L23^5\ c1\ c4$
$- 0.1889778074\ 10^{16}\ L23\ c4^3 + 0.1843968000\ 10^{10}\ L23^6\ c1\ c4$
$+ 0.7038352097\ 10^{15}\ L23^2\ c1^3\ c4 + 0.2635290684\ 10^{14}\ L23^3\ c1\ c4$

$+ 0.1254057344\ 10^{17}\ c1^2\ c4^2 + 0.2975729290\ 10^{16}\ c1^3\ c4^2$
$+ 0.7563577221\ 10^{15}\ c1\ L23^2 - 0.1062125568\ 10^{13}\ L23^4\ c1^2\ c4^2$
$+ 0.6050649557\ 10^{16}\ c1^3\ c4 + 0.1721030006\ 10^{14}\ L23^3\ c1\ c4^2$
$+ 0.8939345721\ 10^{15}\ L23^2\ c1^2 + 0.6747419064\ 10^{13}\ L23^3\ c1^2\ c4^2$
$+ 0.4953462407\ 10^{13}\ L23^3 + 0.2550625106\ 10^{16}\ c1^2\ c4^3$
$+ 0.5611375232\ 10^{16}\ c1\ c4^3 - 0.1157952251\ 10^{11}\ L23^5\ c4 + 0.3075746857\ 10^{16}\ c1^3$
$- 0.3334902483\ 10^{15}\ c4^4\ L23 + 0.5465625225\ 10^{15}\ c4^4\ c1$
$+ 0.2779085403\ 10^{13}\ c4^3\ L23^3 + 0.2839097906\ 10^{15}\ L23^2\ c4^3$
$+ 0.3568741909\ 10^{15}\ L23^2\ c1^3\ c4^2 + 0.1720821738\ 10^{17}\ c1^2\ c4$
$+ 0.5380397453\ 10^{15}\ L23^2\ c4^3\ c1 + 0.1348633385\ 10^{16}$
$- 0.2234879229\ 10^{13}\ L23^4\ c4 - 0.3499544516\ 10^{16}\ L23\ c4^3\ c1$
$+ 0.5777946668\ 10^{15}\ c4^4 - 0.5324667821\ 10^{10}\ L23^5\ c1$
$- 0.4010065920\ 10^{12}\ L23^4\ c4^3 + 0.2628864571\ 10^{13}\ c4^3\ L23^3\ c1$
$- 0.1660829642\ 10^{13}\ L23^4\ c4^2 - 0.3154637484\ 10^{15}\ c4^4\ L23\ c1$
$+ 0.6662224379\ 10^{13}\ L23^3\ c1^2 + 0.6209287703\ 10^{15}\ L23^2\ c4^2$
$+ 0.1548836731\ 10^{17}\ c1\ c4^2 + 0.1588310692\ 10^{17}\ c1\ c4$
$+ 0.8354304000\ 10^{9}\ L23^6\ c4^2 - 0.5789761256\ 10^{10}\ L23^5\ c4^2$
$+ 0.1172863060\ 10^{14}\ L23^3\ c1 - 0.2066277815\ 10^{16}\ L23\ c1^3$
$- 0.2220502329\ 10^{13}\ L23^4\ c1 + 0.1065316071\ 10^{14}\ L23^3\ c4^2$
$- 0.4129741774\ 10^{16}\ L23\ c1 - 0.1619380575\ 10^{16}\ L23\ c1^2\ c4^3$
$+ 0.1075648000\ 10^{10}\ L23^6\ c1 - 0.8361892426\ 10^{16}\ L23\ c1^2\ c4^2$
$- 0.1021681926\ 10^{17}\ L23\ c1\ c4^2 + 0.2549101363\ 10^{15}\ L23^2\ c1^2\ c4^3$
$- 0.3430626536\ 10^{16}\ L23\ c4 - 0.3890719565\ 10^{16}\ L23\ c4^2$
$+ 0.7902720000\ 10^{9}\ L23^6\ c4^2\ c1 - 0.5476801188\ 10^{10}\ L23^5\ c4^2\ c1$
$+ 0.1289186839\ 10^{14}\ L23^3\ c4 + 0.4812079104\ 10^{14}\ L23^2\ c4^4$
$- 0.3793305600\ 10^{12}\ L23^4\ c4^3\ c1 + 0.4551966720\ 10^{14}\ L23^2\ c4^4\ c1$
$- 0.1116496308\ 10^{17}\ L23\ c1\ c4 - 0.1221936128\ 10^{13}\ L23^4\ c1^2$
$+ 0.5952551997\ 10^{15}\ c4\ L23^2 - 0.4127784728\ 10^{16}\ L23\ c1^3\ c4$
$- 0.1185533798\ 10^{17}\ L23\ c1^2\ c4 - 0.2061029824\ 10^{16}\ L23\ c1^3\ c4^2$

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#+++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++++
> # end
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# BIOGRAPHICAL SKETCH

I have extensive educational background in both engineering and science, and also years of work experience with nuclear reactors and its components for the United States Naval Nuclear Reactors.  My prior education and experience were in the fields of mechanical engineering, nuclear engineering, mathematics and natural science. In addition, I also had experience as a design and structural analyst of mechanical and nuclear reactor equipment and related components. Furthermore, I worked in maintenance dealing with overhauling the nuclear reactors and their power plant equipment for the United States Naval Nuclear Reactors. I plan to work in academic filed related to my past education and work experience.