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CATASTROPHE ANALYSIS OF PLANAR THREE-SPRING SYSTEMS

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ABSTRACT

A special planar three-spring mechanism is proposed for contact force control. An energy function is defined to describe the behavior of this kind of mechanism. It can be used to perform the catastrophe analysis of this mechanism . The analysis result can be used as a design and control tool. By comparing the three-spring system and a two-spring system, we found the three-spring mechanism has better stability than the two-spring system. A three-spring mechanism which can be used to control a general contact force in a plane is also analyzed .

1 INTRODUCTION

Parallel compliant mechanisms are classically known for their role in mounting and suspension systems. Recently, however , they have been instrumental in the field of robotic force control. A new theory for the simultaneous control of force and displacement for a partially constrained end-effector has been proposed by Griffis and Duffy (1993), Griffis (1991), Pigoski and Duffy (1992).

Figure 1. shows a planar compliant mechanism which is actually a planar two-spring system. It consists of a pair of linear springs connected at one end to a movable base and at the other end to a common pivot which is the axis of the wheel contacting to a surface. The base is connected to a planar two freedom P-P (P

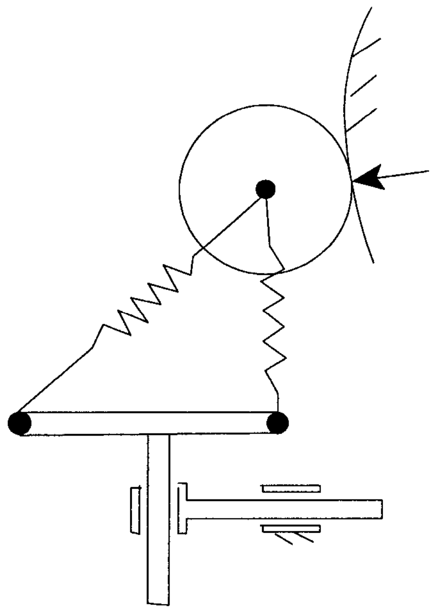

Fig.1 Planar Two-spring System

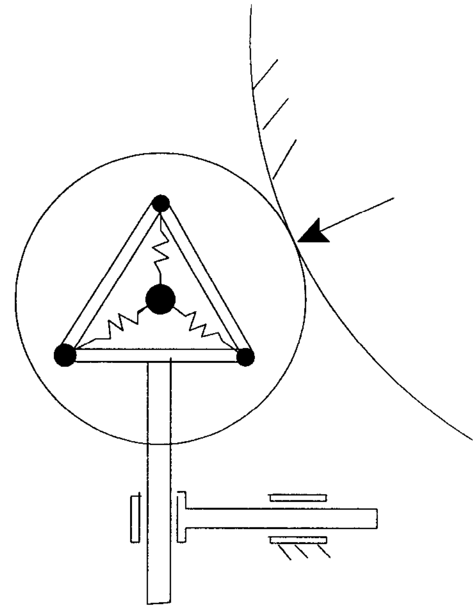

Fig.2 Special Three-spring System

denotes prismatic pair) manipulator . The contact force can be controlled by displacing the two prismatic joints of the manipulator. The required displacements can be calculated from the stiffness mapping. This kind of control was called kinestatic control by Griffis and Duffy (1993).

In order to design the planar two-spring system it is necessary to compute a spring stiffness which will generate a range of displacements of the movable base which can be produced by the prismatic joints over a required range of change in contact force. Clearly if the system is over-designed and the spring stiffness is very high it is always possible to generate any necessary changes in contact force. However such a system will be too sensitive to errors because very small displacements of the platform will generate large changes in contact force. On the other hand if the springs are too soft there can be stability problems (see Hines, Marsh and Duffy (1998) who performed a catastrophe analysis of the planar two spring-system).

In this paper, another compliant mechanism is considered. This is shown in Figure 2. It consists of three linear springs jointed to the triangular frame fixed points and connected at the axis of the wheel. The triangular frame is connected to the planar two freedom P-P manipulator. This mechanism is a special three-spring system. The catastrophe analysis of this system is presented in this paper. Comparing the results demonstrates that three-spring system has better stability characteristics than the two-spring system.

A more general planar three-spring compliant mechanism is

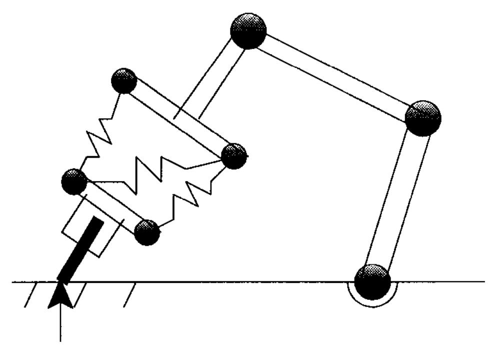

Fig.3 Three-spring System

shown in Fig. 3. This mechanism is connected to the planar three freedom R-R-R (R denotes revolute pair) manipulator. It can be used to control both force and moment. The catastrophe analysis of this mechanism is done in this present paper.

2 THE APPLICATION OF CATASTROPHE THEORY IN SPRING SYSTEM ANALYSIS

Catastrophe theory and singularity theory has been proven beneficial in mechanism analysis of the structural and dynamic stability of systems with potential functions as well as the analysis of kinematic singularities in robotic system(Hobbs,1993, Xiang,1995). Catastrophe theory will be used to analyze the above three-spring systems. The results may be used as a design and control tool for those spring systems and similar compliant mechanisms.

Intuitively, a catastrophe occurs whenever a smooth change of parameters gives rise to a discontinuous change in behavior. A well known example that can easily be made to demonstrate a catastrophe in Zeeman's catastrophe machine , see Fig.4. Zeeman's machine can be constructed by attaching two linear springs to a single point C on a disk that can rotate about O. One of the springs is attached to a fixed pivot at point A and the other spring is attached to point B which can be moved in the x, y plane. The position of point B is called the controlling parameter as it dictates the position of the disk which is defined by the angle θ .

As the point B is moved around in the plane, the disk generally rotates smoothly about point O. However at certain points in the plane, the disk can jump to some other position and a catastrophe occurs. The set of points in the control parameter space that represent a possible catastrophe is called the bifurcation set or bifurcation curve which is shown in Fig. 4.

The bifurcation curve can be found analytically by using catastrophe theory. The suitable introductions to Catastrophe Theory are (Poston and Stewart,1978, Gilmore, 1981). Here we shall give a brief description on how to find a bifurcation curve.

Consider a function f(x). A point x_0 is said to be a critical point (equilibrium, stationary, or turning point) of a function f(x) whenever $f'(x_0)=0$. Further, a critical point x_0 is said to be degenerate whenever $f''(x_0)=0$, otherwise it is non-degenerate. Non-degenerate critical points are either local minimums when $f''(x_0)>0$ or local maximums when $f''(x_0)<0$. The points which satisfy both the critical point condition and the degenerate condition construct the bifurcation curve.

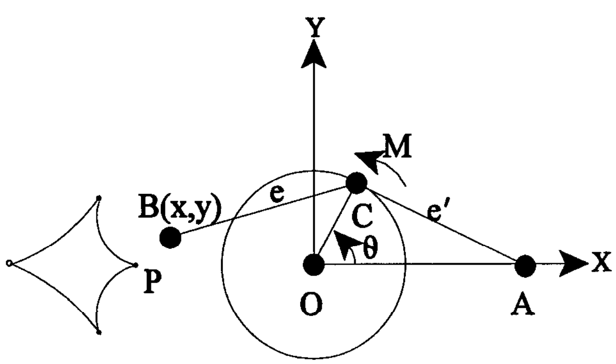

Fig.4 Zeeman's Catastrophe Machine

Catastrophe Theory is a universal mathematic tool. It can be used in many different fields. One important thing is to find a function which describes the behavior of the problem. To find the bifurcation curve for the Zeeman's machine, we have to find a function which describes the system. The potential energy P of this system is defined as the sum of energy stored in the two springs. Here we define the energy function V as the work done by the moment M acting on the disk minus the potential energy.

$$V=M\,\theta-\frac{1}{2}k(e-1)^2-\frac{1}{2}k(e'-1)^2 \tag{1}$$

where e and e' are functions of x, y and θ , k is spring constants, M is a constant external moment, natural length of each spring is unit. The critical point condition and degenerate condition are

$$\frac{\partial V}{\partial\theta}=0 \qquad \frac{\partial^2 V}{\partial\theta^2}=0 \tag{2}$$

From the equations (2), a function of x and y can be found by eliminating the angle θ . The function is a diamond shaped curve which is actually the bifurcation curve shown in Fig. 4 (here M=0) and hence when B lies anywhere on this curve a catastrophe may occur.

3 ANALYSIS OF THE THREE-SPRING SYSTEMS

Consider the special three-spring system (Fig.5) in its unloaded position $l_i=l_{0i}$ (i=1,2,3) and an external force F is now applied to the point P at which the three springs are connected.

The stability can be determined by the eigenvalues of the Hessian matrix derived from the second derivative of the energy function V. The energy function is the work done by the force F minus the sum of the potential energy in each individual spring.

$$V=(x-x_0)F_x+(y-y_0)F_y-\frac{1}{2}k_1(l_1-l_{01})^2$$
$$-\frac{1}{2}k_2(l_2-l_{02})^2-\frac{1}{2}k_3(l_3-l_{03})^2 \tag{3}$$

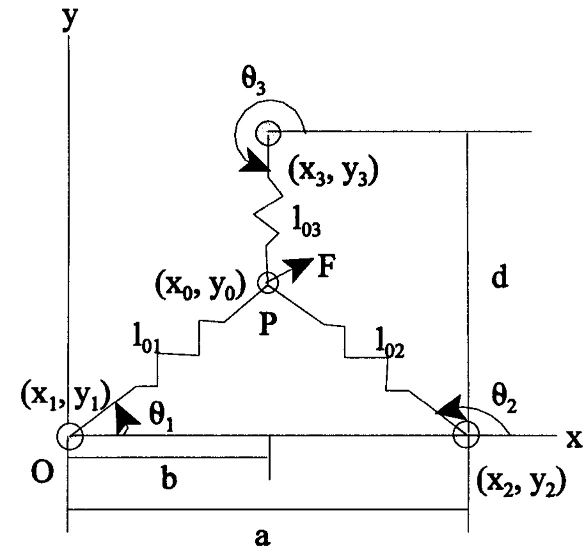

Fig.5 A Three-spring System

Where l_{01}, l_{02} and l_{03} are the free lengths . l_1, l_2 and l_3 are the lengths of these springs when the pivot is at position (x, y) and k_1, k_2 and k_3 are the spring constant . F_x and F_y are components of force F on x and y axes.

The first partial derivatives of the energy function V with respect to independent variables x and y are

$$\frac{\partial V}{\partial x}=F_x-k_1(l_1-l_{01})\frac{\partial l_1}{\partial x}-k_2(l_2-l_{02})\frac{\partial l_2}{\partial x}$$

$$-k_3(l_3-l_{03})\frac{\partial l_3}{\partial x} \tag{4}$$

$$\frac{\partial V}{\partial y}=F_y-k_1(l_1-l_{01})\frac{\partial l_1}{\partial y}-k_2(l_2-l_{02})\frac{\partial l_2}{\partial y}$$

$$-k_3(l_3-l_{03})\frac{\partial l_3}{\partial y} \tag{5}$$

The second partial derivatives with respect to x and y are

$$\frac{\partial^2 V}{\partial x^2}=k_1(\frac{\partial l_1}{\partial x})^2+k_2(\frac{\partial l_2}{\partial x})^2+k_3(\frac{\partial l_3}{\partial x})^2+k_1(l_1-l_{01})\frac{\partial^2 l_1}{\partial x^2}$$

$$+k_2(l_2-l_{02})\frac{\partial^2 l_2}{\partial x^2}+k_3(l_3-l_{03})\frac{\partial^2 l_3}{\partial x^2} \tag{6}$$

$$\frac{\partial^2 V}{\partial y^2}=k_1(\frac{\partial l_1}{\partial y})^2+k_2(\frac{\partial l_2}{\partial y})^2+k_3(\frac{\partial l_3}{\partial y})^2+k_1(l_1-l_{01})\frac{\partial^2 l_1}{\partial y^2}$$

$$+k_2(l_2-l_{02})\frac{\partial^2 l_2}{\partial y^2}+k_3(l_3-l_{03})\frac{\partial^2 l_3}{\partial y^2} \tag{7}$$

$$\frac{\partial^2 V}{\partial x \partial y}=k_1\frac{\partial l_1}{\partial x}\frac{\partial l_1}{\partial y}+k_2\frac{\partial l_2}{\partial x}\frac{\partial l_2}{\partial y}+k_3\frac{\partial l_3}{\partial x}\frac{\partial l_3}{\partial y}+$$

$$+k_1(l_1-l_{01})\frac{\partial^2 l_1}{\partial x \partial y}+k_2(l_2-l_{02})\frac{\partial^2 l_2}{\partial x \partial y}+k_3(l_3-l_{03})\frac{\partial^2 l_3}{\partial x \partial y} \tag{8}$$

$$\frac{\partial^2 V}{\partial y \partial x}=\frac{\partial^2 V}{\partial x \partial y} \tag{9}$$

The spring lengths l_i and the position coordinates (x,y) of the connector are given

$$l_i^2=(x-x_i)^2+(y-y_i)^2 \qquad i = 1,2,3 \tag{10}$$

and their partial derivatives with respect to x and y for i=1,2,3 are

$$\frac{\partial l_i}{\partial x}=\frac{x-x_i}{l_i} \qquad \frac{\partial l_i}{\partial y}=\frac{y-y_i}{l_i} \tag{11}$$

$$\frac{\partial^2 l_i}{\partial x^2}=\frac{1}{l_i}(1-\frac{x-x_i}{l_i}\frac{\partial l_i}{\partial x})=\frac{1}{l_i}[1-(\frac{\partial l_i}{\partial x})^2] \tag{12}$$

$$\frac{\partial^2 l_i}{\partial y^2}=\frac{1}{l_i}(1-\frac{y-y_i}{l_i}\frac{\partial l_i}{\partial y})=\frac{1}{l_i}[1-(\frac{\partial l_i}{\partial y})^2] \tag{13}$$

$$\frac{\partial^2 l_i}{\partial x \partial y}=\frac{\partial^2 l_i}{\partial y \partial x}=-\frac{1}{l_i}\frac{x-x_i}{l_i}\frac{\partial l_i}{\partial y}=-\frac{1}{l_i}\frac{\partial l_i}{\partial x}\frac{\partial l_i}{\partial y} \tag{14}$$

The Hessian takes the form

$$H=\begin{pmatrix} \frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\ \frac{\partial^2 V}{\partial y \partial x} & \frac{\partial^2 V}{\partial y^2} \end{pmatrix} \tag{15}$$

The vanishing of equations (4) and (5) yields the critical point condition of this system while equating the determinant of the Hessian to zero yields the degenerate condition. If we only consider the case :l_i >=0 i=1,2,3, l_i can be calculated from (10) .

$$l_i=\sqrt{(x-x_i)^2+(y-y_i)^2} \tag{16}$$

Substituting l_i into the H=0 , ∂V/∂x =0 and ∂V/∂y=0. Now, the Hessian is a function of x and y, the equation ∂V/∂x =0 is a function of x,y and F_x while the equation ∂V/∂y=0 is a function of x, y and F_y. For a set of parameters chosen as $l_{01}=l_{02}=l_{03}$=10 cm, $k_1=k_2=k_3$=0.5 kg/cm, a=20, b=10 and d=17.32 cm. The degenerate points of the Hessian are shown in Fig. 6 which are the three curves designated with Π_1, Π_2 and Π_3. Using values of (x,y) on these curves , F_x and F_y can be calculated from the critical conditions ∂V/∂x =∂V/∂y=0. The result is shown in Fig. 7. All these results are numerically calculated out by software Maple.

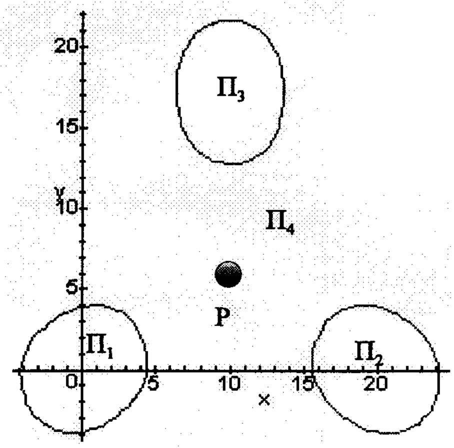

Fig.6 Special Three Spring System Catastrophe Curves

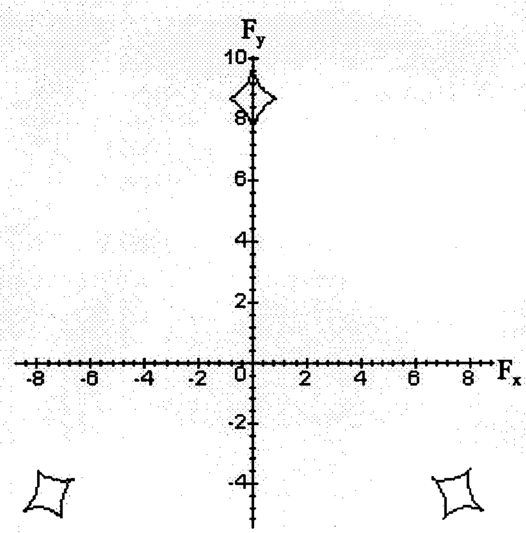

Fig.7 Special Three Spring Bifurcation Curves

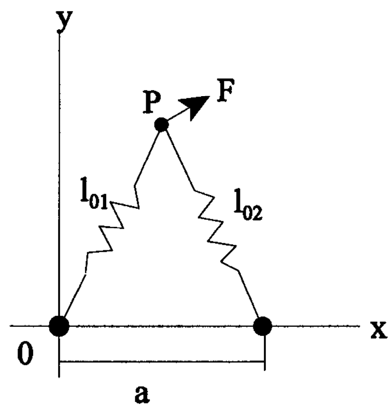

Fig. 8 Two Spring System

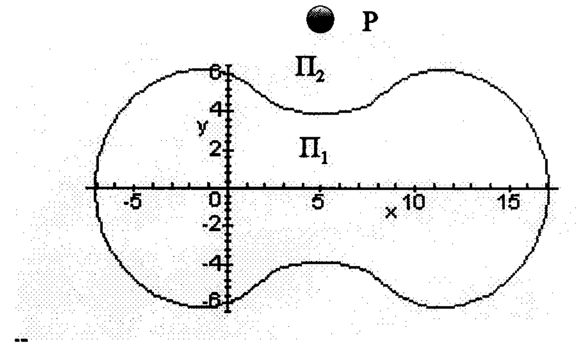

Fig.9 Two Spring System Catastrophe Vurve

For the two spring system shown in Fig.8. A set of parameters chosen as $l_{01}=l_{02}=10$ cm, $k_1=k_2=0.5$ kg/cm and a=10 cm. F_x and F_y are control parameters. The catastrophe curves and bifurcation curves are shown in Fig. 9 and Fig.10.

It is most important from a practical view point to identify the stable unloaded position ($F_x=F_y=0$) labeled by P in Fig. 5 and 8 . There are a number of other unloaded configurations for both the two-spring and the three-spring systems. For the special three-spring system, the catastrophe curves divide the x, y plane into four parts ,Π_1, Π_2, Π_3 and Π_4 . The bifurcation curves divided the F_x , F_y plane to four corresponding parts. Catastrophe theory shows when the mechanism equilibrium positions cross any catastrophe curve from one area into another area, there may be one (or more) configurations lost or there may be an increase in the numbers of configurations. At the same time , the control parameters F_x and F_y must cross a bifurcation curve , from one to another area(Fig. 7 and Fig. 10). It is necessary to perform a detailed investigation to show if configurations are lost or increased. In the x,y plane, an area without any catastrophe curves is said an absolute safe area for a certain configuration whenever its unloaded state is in that area. For instance, Π_4 is an absolute safe area since it contains the stable unloaded configuration at point P of the three-spring system (in Fig. 6), $\pi 2$ is an absolute safe area and it contains the stable unloaded configuration at point P of the two-spring system . When the stable configurations for these two systems are considered and their absolute safe areas are compared, it is clear that the three-spring system has better stability. First, the absolute safe area of the three-spring system has better symmetry than two-spring system (comparing the catastrophe curves in Fig.6 and Fig. 9), and hence better symmetry of sensitivity; Second, the load on three-spring system can be reached to unlimited magnitude theoretically except three diamond areas by comparing the bifurcation curves in Fig. 7 and Fig. 9. On the other hand, the force sensitivity of the system can be found by comparing the stable absolute safe area on catastrophe curve plane and bifurcation curve plane of the same spring system. The suitable stable work area and sensitivity can be designed by

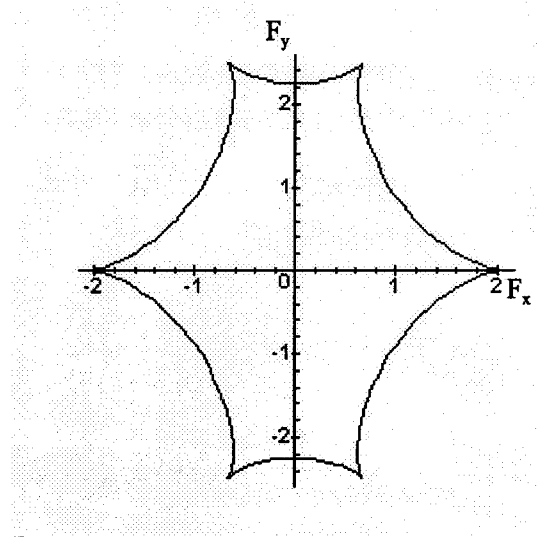

Fig.10 Two Spring System Bifurcation Curve

adjudging the spring constants and mechanism sizes and comparing the catastrophe curve and bifurcation curve repeatedly.

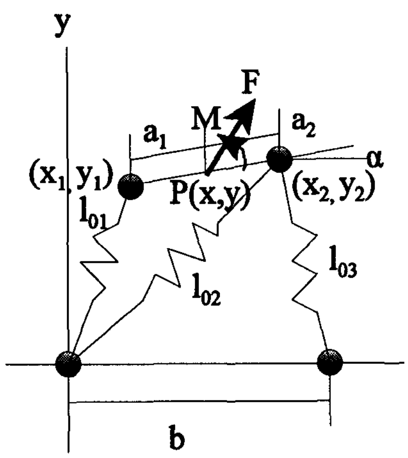

Fig.11 Three-spring system

For the three-spring system shown in Fig.11. Similarly, the energy function of the system is the work done by external force and moment minus the potential energy stored in the three springs.

$$V = (x-x_0)F_x + (y-y_0)F_y + (\alpha-\alpha_0)M$$
$$-\tfrac{1}{2}k_1(l_1-l_{01})^2 - \tfrac{1}{2}k_2(l_2-l_{02})^2$$
$$-\tfrac{1}{2}k_3(l_3-l_{03})^2 \qquad (17)$$

Where (x_0 , y_0) and α_0 are the values of the coordinates of the point P(x,y) and the angle α when the system is at its unloaded position. l_{01} , l_{02} and l_{03} are free lengths. k_1 ,k_2 and k_3 are spring constants. F_x and F_y are components of force F on axes x and y. l_1 ,l_2 and l_3 are spring lengths when both force F and moment M are applied at point P. The relations between l_i (i=1,2,3) and x,y and α are expressed as follows.

$$l_1^2 = x_1^2 + y_1^2 \qquad l_2^2 = x_2^2 + y_2^2$$
$$l_3^2 = (x_2-b)^2 + y_2^2 \qquad (18)$$

Where

$$x_1 = x - a_1\cos(\alpha) \qquad y_1 = y - a_1\sin(\alpha)$$
$$x_2 = x + a_2\cos(\alpha) \qquad y_2 = y + a_2\sin(\alpha)$$

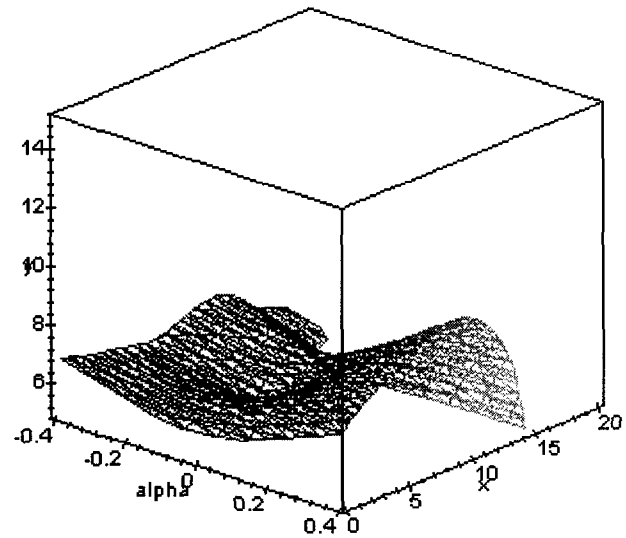

Fig.12 Three-spring System Catastrophe Surface

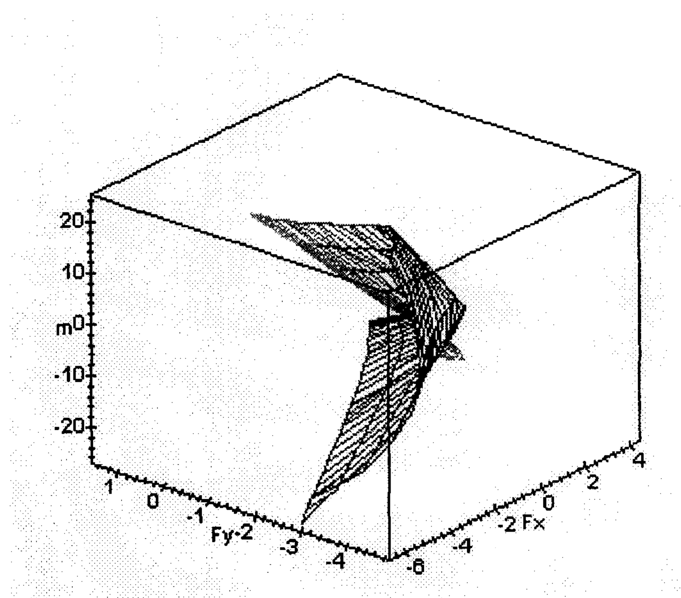

Fig.13 Three-spring System Bifurcation Surface

Obviously, V is a function of variables x, y and α. It is not difficult to calculate the first partial derivatives, $\partial V/\partial x$, $\partial V/\partial y$ and $\partial V/\partial \alpha$, and the second partial derivatives, $\partial V^2/\partial x^2$, $\partial V^2/\partial y^2$, $\partial V^2/\partial \alpha^2$, $\partial V^2/(\partial x\, \partial y)$, $\partial V^2/(\partial y\, \partial \alpha)$ and $\partial V^2/(\partial x \partial \alpha)$.

Using the degenerate condition, the catastrophe surface of the system is drawn (shown in Fig.12). From the critical conditions, the corresponding bifurcation surface of this system can be calculated (shown in Fig.13).

The catastrophe surface in Fig. 12 separates the x, y and α space into two parts. The space above the surface is an absolute safe space for the configuration shown in Fig. 11. While the bifurcation surface in Fig. 13 separates the F_x, F_y and M space into corresponding two parts too. The left part is the absolute safe load space for the configuration.

In order to analyze the surfaces accurately, we can cut the surface by planes and draw the corresponding curves on the planes. For instance, using the plane α=0.0 to cut the catastrophe surface, a catastrophe curve C can be found (shown in Fig.14). For the catastrophe curve C, the corresponding critical loads (M, F_x, F_y) can be calculated from critical conditions (shown in Fig. 15). It must be a curve on the bifurcation surface in Fig.13. The absolute safe space and load space for certain α can be read from the figures accurately. This will be helpful for critical design. To show this, assuming the curve S in Fig. 14 is an arbitrary safe

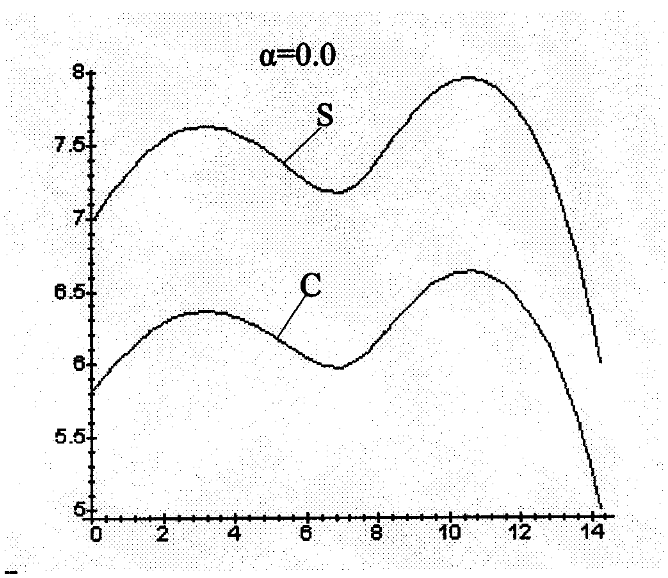

Fig.14 Catastrophe Curve C and an Arbitrary Safe Trajectory S on plane α=0.0

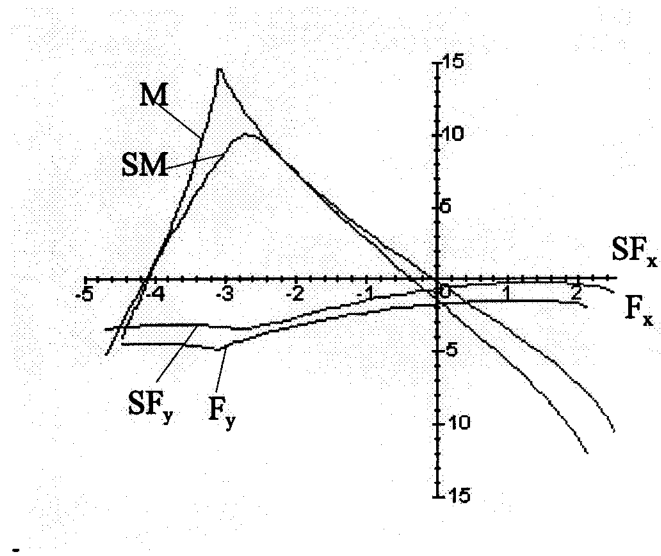

Fig.15 Critical Load M,F_x, F_y and Safe Load SM, SF_x, SF_y

trajectory(notice, any point of curve S is above the curve C) of the three-spring system, the corresponding loads(SM,SF_x,SF_y) are calculated from the critical conditions (shown in Fig. 15).

To design a proper three-spring system for certain application, one can choose different amount for parameters b, l_{01}, l_{02}, l_{03}, a_1, a_2 and k_1, k_2, k_3 and analyze the catastrophe surface and bifurcation surface. Generally, the catastrophe surface is mainly determined by geometric parameters, while the bifurcation surface mainly dependents on spring constants.

4 CONCLUSION

This paper has presented a general energy function to describe planar spring systems. The function can be used to analyze the characteristics of stability and sensitivity of these spring systems by using catastrophe theory. Two planar three-spring systems are analyzed as examples. The conclusion that the special three-spring system presented in this paper has better stability than that of corresponding two-spring system was draft.

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