ABSTRACT

A special planar three-spring mechanism is proposed for contact force control. An energy function is defined to describe the behavior of this kind of mechanism. It can be used to perform the catastrophe analysis of this mechanism. The analysis result can be used as a design and control tool. By comparing the three-spring system and a two-spring system, we found the three-spring mechanism has better stability than the two-spring system. A three-spring mechanism which can be used to control a general contact force in a plane is also analyzed.

1 INTRODUCTION
Fig. 1 Planar Two-spring System
denotes prismatic pair ) manipulator . The contact force can be controlled by displacing the two prismatic joints of the manipulator. The required displacements can be calculated from the stiffness mapping. This kind of control was called kinesthetic control by Griffis and Duffy (1993).

In order to design the planar two-spring system it is necessary to compute a spring stiffness which will generate a range of displacements of the movable base which can be produced by the prismatic joints over a required range of change in contact force. Clearly if the system is over-designed and the spring stiffness is very high it is always possible to generate any necessary changes in contact force. However such a system will be too sensitive to errors because very small displacements of the platform will generate large changes in contact force. On the other hand if the springs are too soft there can be stability problems (see Hines, Marsh and Duffy (1998) who performed a catastrophe analysis of the planar two-spring system).

In this paper, another compliant mechanism is considered. This is shown in Figure 2. It consists of three linear springs jointed to the triangular frame fixed points and connected at the axis of the wheel. The triangular frame is connected to the planar two freedom P-P manipulator. This mechanism is a special three-spring system. The catastrophe analysis of this system is presented in this paper. Comparing the results demonstrates that three-spring system has better stability characteristics than the two-spring system.

A more general planar three-spring compliant mechanism is shown in Fig. 3. This mechanism is connected to the planar three freedom R-R-R (R denotes revolute pair) manipulator. It can be used to control both force and moment. The catastrophe analysis of this mechanism is done in this present paper.

2 THE APPLICATION OF CATASTROPHE THEORY IN SPRING SYSTEM ANALYSIS

Catastrophe theory and singularity theory has been proven beneficial in mechanism analysis of the structural and dynamic
stability of systems with potential functions as well as the analysis of kinematic singularities in robotic systems (Hobbs, 1993, Xiang, 1995). Catastrophe theory will be used to analyze the above three-spring systems. The results may be used as a design and control tool for those spring systems and similar compliant mechanisms.

Intuitively, a catastrophe occurs whenever a smooth change of parameters gives rise to a discontinuous change in behavior. A well-known example that can easily be made to demonstrate a catastrophe in Zeeman's catastrophe machine, see Fig. 4. Zeeman's machine can be constructed by attaching two linear springs to a single point C on a disk that can rotate about O. One of the springs is attached to a fixed pivot at point A and the other spring is attached to point B which can be moved in the x, y plane. The position of point B is called the controlling parameter as it dictates the position of the disk which is defined by the angle θ.

As the point B is moved around in the plane, the disk generally rotates smoothly about point O. However, at certain points in the plane, the disk can jump to some other position and a catastrophe occurs. The set of points in the control parameter space that represent a possible catastrophe is called the bifurcation set or bifurcation curve which is shown in Fig. 4.

The bifurcation curve can be found analytically by using catastrophe theory. The suitable introductions to Catastrophe Theory are (Poston and Stewart, 1978, Gilmore, 1981). Here we shall give a brief description on how to find a bifurcation curve.

Consider a function \( f(x) \). A point \( x_0 \) is said to be a critical point (equilibrium, stationary, or turning point) of a function \( f(x) \) whenever \( f'(x_0) = 0 \). Further, a critical point \( x_0 \) is said to be degenerate whenever \( f''(x_0) = 0 \), otherwise it is non-degenerate. Non-degenerate critical points are either local minimums when \( f''(x_0) > 0 \) or local maximums when \( f''(x_0) < 0 \). The points which satisfy both the critical point condition and the degenerate condition construct the bifurcation curve.

Catastrophe Theory is a universal mathematical tool. It can be used in many different fields. One important thing is to find a function which describes the behavior of the problem. To find the bifurcation curve for the Zeeman's machine, we have to find a function which describes the system. The potential energy \( P \) of this system is defined as the sum of energy stored in the two springs. Here we define the energy function \( V \) as the work done by the moment \( M \) acting on the disk minus the potential energy.

\[
V = M \theta - \frac{1}{2} k(e - 1)^2 - \frac{1}{2} k(e' - 1)^2
\]  

where \( e \) and \( e' \) are functions of \( x, y \) and \( θ \), \( k \) is spring constants, \( M \) is a constant external moment, natural length of each spring is unit. The critical point condition and degenerate condition are

\[
\frac{∂V}{∂θ} = 0 \quad \frac{∂^2V}{∂θ^2} = 0
\]  

From the equations (2), a function of \( x \) and \( y \) can be found by eliminating the angle \( θ \). The function is a diamond-shaped curve which is actually the bifurcation curve shown in Fig. 4 (here \( M = 0 \)) and hence when \( B \) lies anywhere on this curve a catastrophe may occur.

3 ANALYSIS OF THE THREE-SPRING SYSTEMS

Consider the special three-spring system (Fig. 5) in its unloaded position \( l_i = l_{0i} \) for \( i = 1,2,3 \) and an external force \( F \) is now applied to the point \( P \) at which the three springs are connected.

The stability can be determined by the eigenvalues of the Hessian matrix derived from the second derivative of the energy function \( V \). The energy function is the work done by the force \( F \) minus the sum of the potential energy in each individual spring.

\[
V = (x-x_0)F_x + (y-y_0)F_y - \frac{1}{2} k_1(l_1 - l_{01})^2
- \frac{1}{2} k_2(l_2 - l_{02})^2 - \frac{1}{2} k_3(l_3 - l_{03})^2
\]  

\[
= (x-x_0)F_x + (y-y_0)F_y - \frac{1}{2} k_1(l_1 - l_{01})^2
- \frac{1}{2} k_2(l_2 - l_{02})^2 - \frac{1}{2} k_3(l_3 - l_{03})^2
\]  

Fig.4 Zeeman's Catastrophe Machine
Fig. 5 A Three-spring System

Where $l_{01}$, $l_{02}$ and $l_{03}$ are the free lengths. $l_1$, $l_2$ and $l_3$ are the lengths of these springs when the pivot is at position $(x, y)$ and $k_1$, $k_2$ and $k_3$ are the spring constant. $F_x$ and $F_y$ are components of force $F$ on $x$ and $y$ axes.

The first partial derivatives of the energy function $V$ with respect to independent variables $x$ and $y$ are

$$\frac{\partial V}{\partial x} = F_x - k_1(l_1 - l_{01}) \frac{\partial l_1}{\partial x} - k_2(l_2 - l_{02}) \frac{\partial l_2}{\partial x} - k_3(l_3 - l_{03}) \frac{\partial l_3}{\partial x}$$  \hspace{1cm} (4)

$$\frac{\partial V}{\partial y} = F_y - k_1(l_1 - l_{01}) \frac{\partial l_1}{\partial y} - k_2(l_2 - l_{02}) \frac{\partial l_2}{\partial y} - k_3(l_3 - l_{03}) \frac{\partial l_3}{\partial y}$$  \hspace{1cm} (5)

The second partial derivatives with respect to $x$ and $y$ are

$$\frac{\partial^2 V}{\partial x^2} = k_1 \frac{\partial^2 l_1}{\partial x^2} + k_2 \frac{\partial^2 l_2}{\partial x^2} + k_3 \frac{\partial^2 l_3}{\partial x^2} - k_2(l_2 - l_{02}) \frac{\partial^2 l_2}{\partial x^2}$$  \hspace{1cm} (6)

$$+ k_2(l_2 - l_{02}) \frac{\partial^2 l_2}{\partial y^2} + k_3(l_3 - l_{03}) \frac{\partial^2 l_3}{\partial y^2}$$

$$\frac{\partial^2 V}{\partial y^2} = k_1 \frac{\partial^2 l_1}{\partial y^2} + k_2 \frac{\partial^2 l_2}{\partial y^2} + k_3 \frac{\partial^2 l_3}{\partial y^2}$$

$$- k_2(l_2 - l_{02}) \frac{\partial^2 l_2}{\partial x^2} + k_3(l_3 - l_{03}) \frac{\partial^2 l_3}{\partial x^2}$$

$$+ k_2(l_2 - l_{02}) \frac{\partial^2 l_2}{\partial x^2} + k_3(l_3 - l_{03}) \frac{\partial^2 l_3}{\partial x^2}$$

The spring lengths $l_i$ and the position coordinates $(x, y)$ of the connector are given

$$l_i^2 = (x - x_i)^2 + (y - y_i)^2 \hspace{1cm} i = 1, 2, 3$$  \hspace{1cm} (10)

and their partial derivatives with respect to $x$ and $y$ for $i=1,2,3$ are
The Hessian takes the form

\[
H = \begin{pmatrix}
\frac{\partial^2 V}{\partial x^2} & \frac{\partial^2 V}{\partial x \partial y} \\ \\
\frac{\partial^2 V}{\partial y \partial x} & \frac{\partial^2 V}{\partial y^2}
\end{pmatrix}
\]  

The vanishing of equations (4) and (5) yields the critical point condition of this system while equating the determinant of the Hessian to zero yields the degenerate condition. If we only consider the case \( \lambda_i \geq 0 \) \( i=1,2,3 \), \( \lambda_i \) can be calculated from (10).

\[
l_i = \sqrt{(x-x_i)^2 + (y-y_i)^2}
\]  

Substituting \( \lambda_i \) into the \( H=0 \), \( \partial V/\partial x =0 \) and \( \partial V/\partial y =0 \). Now, the Hessian is a function of \( x \) and \( y \), the equation \( \partial V/\partial x =0 \) is a function of \( x, y \) and \( F_x \) while the equation \( \partial V/\partial y =0 \) is a function of \( x, y \) and \( F_y \). For a set of parameters chosen as \( l_{01}=l_{02}=l_{03}=10 \text{ cm} \), \( k_1=k_2=k_3=0.5 \text{ kg/cm} \), \( a=20 \), \( b=10 \) and \( d=17.32 \text{ cm} \). The degenerate points of the Hessian are shown in Fig. 6 which are the three curves designated with \( \Pi_1, \Pi_2 \) and \( \Pi_3 \). Using values of \( (x,y) \) on these curves, \( F_x \) and \( F_y \) can be calculated from the critical conditions \( \partial V/\partial x =\partial V/\partial y =0 \). The result is shown in Fig. 7. All these results are numerically calculated out by software Maple.
For the two spring system shown in Fig. 8. A set of parameters chosen as \( l_01 = l_02 = 10 \text{ cm}, k_1 = k_2 = 0.5 \text{ kg/cm} \) and \( a = 10 \text{ cm} \). \( F_x \) and \( F_y \) are control parameters. The catastrophe curves and bifurcation curves are shown in Fig. 9 and Fig. 10.

It is most important from a practical viewpoint to identify the stable unloaded position \((F_x = F_y = 0)\) labeled by P in Fig. 5 and 8. There are a number of other unloaded configurations for both the two-spring and the three-spring systems. For the special three-spring system, the catastrophe curves divide the \( x, y \) plane into four parts, \( \Pi_1, \Pi_2, \Pi_3, \) and \( \Pi_4 \). The bifurcation curves divided the \( F_x, F_y \) plane to four corresponding parts. Catastrophe theory shows when the mechanism equilibrium positions cross any catastrophe curve from one area into another area, there may be one (or more) configurations lost or there may be an increase in the numbers of configurations. At the same time, the control parameters \( F_x \) and \( F_y \) must cross a bifurcation curve, from one to another area (Fig. 7 and Fig. 10). It is necessary to perform a detailed investigation to show if configurations are lost or increased. In the \( x, y \) plane, an area without any catastrophe curves is said an absolute safe area for a certain configuration whenever its unloaded state is in that area. For instance, \( \Pi_4 \) is an absolute safe area since it contains the stable unloaded configuration at point P of the three-spring system (in Fig. 6). \( \Pi_2 \) is an absolute safe area and it contains the stable unloaded configuration at point P of the two-spring system. When the stable configurations for these two systems are considered and their absolute safe areas are compared, it is clear that the three-spring system has better stability. First, the absolute safe area of the three-spring system has better symmetry than two-spring system (comparing the catastrophe curves in Fig. 6 and Fig. 9), and hence better symmetry of sensitivity; Second, the load on three-spring system can be reached to unlimited magnitude theoretically except three diamond areas by comparing the bifurcation curves in Fig. 7 and Fig. 9. On the other hand, the force sensitivity of the system can be found by comparing the stable absolute safe area on catastrophe curve plane and bifurcation curve plane of the same spring system. The suitable stable work area and sensitivity can be designed by...
adjudging the spring constants and mechanism sizes and comparing the catastrophe curve and bifurcation curve repeatedly.

For the three-spring system shown in Fig. 11. Similarly, the energy function of the system is the work done by external force and moment minus the potential energy stored in the three springs.

\[
V = (x-x_0)F_x + (y-y_0)F_y + (\alpha - \alpha_0)M \\
- \frac{1}{2} k_1 (l_1 - l_{01})^2 - \frac{1}{2} k_2 (l_2 - l_{02})^2 \\
- \frac{1}{2} k_3 (l_3 - l_{03})^2
\]

(17)

Where \((x_0, y_0)\) and \(\alpha_0\) are the values of the coordinates of the point \(P(x, y)\) and the angle \(\alpha\) when the system is at its unloaded position. \(l_{01}, l_{02}\) and \(l_{03}\) are free lengths. \(k_1, k_2\) and \(k_3\) are spring constants. \(F_x\) and \(F_y\) are components of force \(F\) on axes \(x\) and \(y\). \(l_1, l_2\) and \(l_3\) are spring lengths when both force \(F\) and moment \(M\) are applied at point \(P\). The relations between \(l_i\) (i=1,2,3) and \(x, y\) and \(\alpha\) are expressed as follows.

\[
l_1^2 = x_1^2 + y_1^2 \\
l_2^2 = x_2^2 + y_2^2 \\
l_3^2 = (x_2 - b)^2 + y_2^2
\]

(18)

Where

\[
x_1 = x - a_1 \cos(\alpha) \\
y_1 = y - a_1 \sin(\alpha) \\
x_2 = x + a_2 \cos(\alpha) \\
y_2 = y + a_2 \sin(\alpha)
\]
Obviously, V is a function of variables x, y and a. It is not difficult to calculate the first partial derivatives, \( \partial V / \partial x \), \( \partial V / \partial y \) and \( \partial V / \partial a \), and the second partial derivatives, \( \partial^2 V / \partial x^2 \), \( \partial^2 V / \partial y^2 \), \( \partial^2 V / \partial x \partial y \), \( \partial^2 V / \partial y \partial a \) and \( \partial^2 V / \partial x \partial a \).

Using the degenerate condition, the catastrophe surface of the system is drawn (shown in Fig. 12). From the critical conditions, the corresponding bifurcation surface of this system can be calculated (shown in Fig. 13).

The catastrophe surface in Fig. 12 separates the x, y and a space into two parts. The space above the surface is an absolute safe space for the configuration shown in Fig. 11. While the bifurcation surface in Fig. 13 separates the \( F_x \), \( F_y \) and M space into corresponding two parts too. The left part is the absolute safe load space for the configuration.

In order to analyze the surfaces accurately, we can cut the surface by planes and draw the corresponding curves on the planes. For instance, using the plane \( a=0.0 \) to cut the catastrophe surface, a catastrophe curve C can be found (shown in Fig. 14). For the catastrophe curve C, the corresponding critical loads (\( M, F_x, F_y \)) can be calculated from critical conditions (shown in Fig. 15). It must be a curve on the bifurcation surface in Fig. 13. The absolute safe space and load space for certain a can be read from the figures accurately. This will be helpful for critical design. To show this, assuming the curve S in Fig. 14 is an arbitrary safe trajectory (notice, any point of curve S is above the curve C) of the three-spring system, the corresponding loads (\( SM, SF_x, SF_y \)) are calculated from the critical conditions (shown in Fig. 15).

To design a proper three-spring system for certain application, one can choose different amount for parameters \( b, l_{01}, l_{02}, l_{03}, a_1, a_2, k_1, k_2, k_3 \) and analyze the catastrophe surface and bifurcation surface. Generally, the catastrophe surface is mainly determined by geometric parameters, while the bifurcation surface mainly dependents on spring constants.

4 CONCLUSION

This paper has presented a general energy function to describe planar spring systems. The function can be used to analyze the characteristics of stability and sensitivity of these spring systems by using catastrophe theory. Two planar three-spring systems are analyzed as examples. The conclusion that the special three-spring system presented in this paper has better stability than that of corresponding two-spring system was draft.

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REFERENCE


