Static Analysis of Tensegrity Structures
Part 2. Numerical Examples

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Abstract

In this paper two examples are provided to demonstrate the use of the mathematical model developed in Part 1, as well as the verification method based on a Newtonian approach. The results are presented both graphically and numerically.

1. Introduction

The methodology to find the equilibrium position for antiprism tensegrity structures is presented. Tensegrity structures with different number of struts and different external loads are analyzed. Each example is developed in detail and numerical solutions are obtained. In addition to the numerical results, figures showing the structures in their equilibrium positions are also provided.

The software Matlab was used to generate and solve numerically the equations necessary to model the structure. The software also graphs the results.

The static analysis is performed in two steps: initially the equilibrium position of the structure in its unloaded position is evaluated, then the external loads are considered and the new equilibrium position is found.

Due to the complexity of the equations it was advisable to verify the numerical results independently. This was accomplished using an algorithm based on the Newton's Third Law.

Some of the equations that are derived in detail in Part 1 have been repeated for convenience in order to minimize repeated reference to Part 1.

2. Analysis of Tensegrity Structures in their Unloaded Positions.

When there are no external loads applied, the equilibrium position can be determined using Yin's results [1]. In order to determine the unloaded equilibrium position the lengths of the struts are specified, L_S, which are assumed to be all the same, together with the stiffness of the top ties k_T (assumed equal), bottom ties k_B (assumed equal), connecting ties k_L (assumed equal) and the free lengths of the top ties T_0 (assumed equal), bottom ties B_0 (assumed equal) and connecting ties L_0 (assumed equal).

The generalized coordinates of each strut of the structure in its unloaded position must be evaluated. This is accomplished first by computing the three unknowns R_B, R_T and the length of the connecting ties L in the following equations given in Section 10 of Part 1

$$k_L \left(1-\frac{L_o}{L}\right) R_B - 2k_T \left(R_T - R_{To}\right) \sin\frac{\gamma}{2} = 0 \qquad (32)$$

$$k_L \left(1-\frac{L_o}{L}\right) R_T - 2k_B \left(R_B - R_{Bo}\right) \sin\frac{\gamma}{2} = 0 \qquad (33)$$

$$L - \sqrt{L_S^2 + 2R_B R_T \left[\cos(\alpha+\gamma) - \cos\alpha\right]} = 0 \quad (34)$$

where

$$R_{T0} = \frac{T_o}{2\sin\frac{\gamma}{2}} \text{ and } R_{B0} = \frac{B_o}{2\sin\frac{\gamma}{2}} \quad (35)$$

And the angles γ and α are given by

$$\gamma = \frac{2\pi}{n} \text{ and } \alpha = \frac{\pi}{2} - \frac{\pi}{n} \quad (36)$$

where n is the number of struts.

The values of R_B and R_T are then substituted into the equations (37) through (41) which yield the generalized coordinates for the unloaded position.

$$a_{j,0} = R_B \cos\left((j-1)\ \gamma \right) , \quad j = 1, 2, \ldots , n \quad (37)$$

$$b_{j,0} = R_B \sin\left((j-1)\ \gamma \right) , \quad j = 1, 2, \ldots , n \quad (38)$$

$$\tan\varepsilon_{j,0} = \frac{b_{j,0} - R_T \sin(\ (j-1)\ \gamma + \alpha)}{H} \quad (39)$$

$$\tan\beta_{j,0} = \frac{R_T \cos(\ (j-1)\ \gamma + \alpha) - a_{j,0}}{\left(\frac{b_{j,0} - R_T \sin(\ (j-1)\ \gamma + \alpha)}{\sin\varepsilon_{j,0}}\right)} \quad (40)$$

$$\text{where } H = \sqrt{L_S^2 - R_B^2 - R_T^2 - 2R_B R_T \sin\frac{\gamma}{2}} \quad (41)$$

and if $j = 1$ then $j - 1 = n$

3. Analysis of Loaded Tensegrity Structures

The external loads acting on a tensegrity structure may be external forces and external moments. According to the restrictions of this study, only one external force and two external moments may be applied per strut. In addition the directions of the external moments are along the axes of the universal joint used to model the strut, see Figure 1.

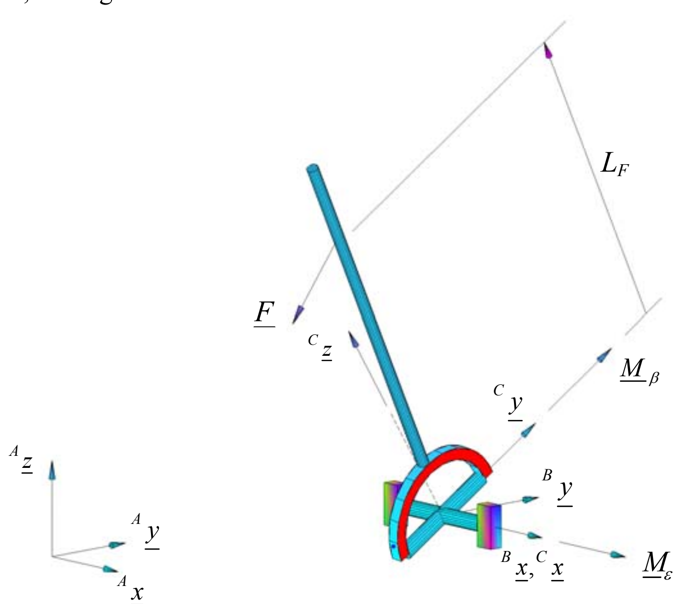

Figure 1. External loads applied to one of the struts of a tensegrity structure.

To be able to perform the static analysis the components (*Fx, Fy, Fz*) and the point of application L_F measured along the strut for each force must be known, together with the magnitudes and senses of the external moments M_ε and M_β.

Any strut of a tensegrity structure constrained to remain on the horizontal plane has four degrees of freedom, two associated with its longitudinal displacements *a* and *b*, and two associated with its rotations *ε* and *β*, see Figure 2. Therefore the whole structure posses *4*n* degrees of freedom where *n* is the number of struts. However if some of the freedoms of the system are constrained the degrees of freedom decrease. Hence in addition to the knowledge of the external loads it is necessary to know the number of freedoms of the structure.

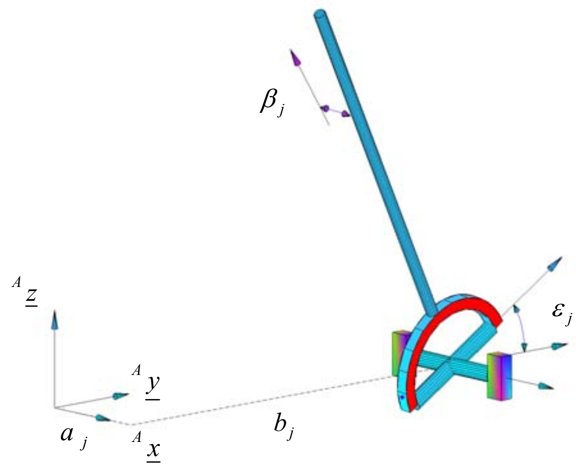

Figure 2. Degrees of freedom associated with one of the struts of a tensegrity structure.

The equilibrium position of the structure is determined for a system of g equations where g is the number of freedoms of the system. These equations are obtained by expanding and equating to zero equations (27) through (30) from part I of this paper:

$$f_i = {}^AF_{i\,x} - \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial a_i} - \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial a_i} - \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial a_i} \quad (27)$$

$$f_{n+i} = {}^AF_{i\,y} - \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial b_i} - \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial b_i} - \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial b_i} \quad (28)$$

$$f_{2n+i} = L_{Fi} \left[-{}^AF_{i\,y} \cos\varepsilon_i \cos\beta_i - {}^AF_{i\,z} \sin\varepsilon_i \cos\beta_i\right] + M_{\varepsilon\,i} - \sum_{j=1}^{n} k_T \left(T_j - T_o\right)\frac{\partial T_j}{\partial \varepsilon_i}$$
$$- \sum_{j=1}^{n} k_B \left(B_j - B_o\right)\frac{\partial B_j}{\partial \varepsilon_i} - \sum_{j=1}^{n} k_L \left(L_j - L_o\right)\frac{\partial L_j}{\partial \varepsilon_i} \quad (29)$$

$$f_{3n+i} = L_{Fi} \left[{}^AF_{i\,x} \cos\beta_i + {}^AF_{i\,y} \sin\varepsilon_i \sin\beta_i - {}^AF_{i\,z} \cos\varepsilon_i \sin\beta_i\right] + M_{\beta\,i}$$
$$- \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial \beta_i} - \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial \beta_i} - \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial \beta_i} \quad (30)$$

$i = 1, \; 2, \; . \; . \; . \; . \; , \; n$

In order to obtain the values of the generalized coordinates the resultant system must be solved numerically. To enhance the performance of the numerical method it is advisable to increase the external loads gradually in a step-by-step procedure. In this way the generalized coordinates evaluated at each step are the initial values for the next step.

The following equations determine the coordinates of the lower and upper ends of the struts, ${}^{A}\underline{E}_j$ and ${}^{A}\underline{A}_j$ respectively, in the global reference system *A*, in terms of the generalized coordinates yielded by the solution of equations (27) through (30),

$$ {}^{A}\underline{E}_j = \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} \tag{7} $$

$$ {}^{A}\underline{A}_j = \begin{bmatrix} L_s \sin\beta_j + a_j \\ -L_s \sin\varepsilon_j \cos\beta_j + b_j \\ L_s \cos\varepsilon_j \cos\beta_j \end{bmatrix} \tag{8} $$

4. Example 1: Analysis of a Tensegrity Structure with 3 Struts

4.1 ANALYSIS FOR THE UNLOADED POSITION

It is required to evaluate the unloaded equilibrium position of a tensegrity structure with 3 struts and with the stiffness and free lengths shown in Table 1. Each of the struts has a length $L_s = 100mm$.

Table 1. Stiffness and free lengths for the structure of the example 1.

	Stiffness (N/mm)	Free lengths (mm)
Top ties	$k_T = 0.5$	$T_0 = 35$
Bottom ties	$k_B = 0.3$	$B_0 = 52$
Connecting ties	$k_L = 1$	$L_0 = 80$

For this example $n = 3$ then (35) and (36) yield $\gamma = 120°$, $\alpha = 15°$, $R_{T0} = 20.207mm$ and $R_{B0} = 32.02mm$.

The solution of (32), (33) and (34) yields $R_B = 33.0568mm$, $R_T = 22.8422mm$ and from (41), $H = 84.1287mm$. The initial values of the generalized coordinates are obtained from (37) through (40) and are listed in Table 2.

Table 2. Initial values of the generalized coordinates for the structure of the example 1.

	Strut 1	Strut 2	Strut 3
a (mm)	33.0568	-16.5284	-16.5284
b (mm)	0	28.6280	-28.6280
ε (rad)	-0.1349	0.5491	-0.4443
β (rad)	-0.5567	0.1660	0.3716

4.2 ANALYSIS FOR THE LOADED POSITION

It is required to evaluate the final equilibrium position of the structure when the external forces listed in Table 3 are applied vertically at the upper end of the struts and there are no constraints acting on the struts.

Table 3. External forces and their application points acting on the structure of the example 1.

	Strut 1	Strut 2	Strut 3
F_x (N)	0	0	0
F_y (N)	0	0	0
F_z (N)	-10	-10	-10
L_F (mm)	100	100	100

Since the system has 3 struts and there are no constraints then there are 12 degrees of freedom and therefore 12 equations are required, one per each generalized coordinate. Equation (27) yields f_1, f_2 and

f_3, equation (28) yields f_4, f_5 and f_6, equation (29) yields f_7, f_8 and f_9, and equation (30) yields f_{10}, f_{11} and f_{12}. Each f_i is equated to zero and then the system is solved numerically using the software developed and the initial values listed in Table 2. The values of the generalized coordinates for the structure when the external forces are applied are shown in Table 4.

Table 4. Generalized coordinates for the final position for the structure of the example 1.

	Strut 1	Strut 2	Strut 3
a (mm)	40.8573	-20.4241	-20.4332
b (mm)	-0.0053	35.3861	-35.3808
ε (rad)	-0.0269	0.6808	-0.6643
β (rad)	-0.7434	0.3271	0.3635

Using the values of Tables 2 and 4 equations (7) and (8) yield the coordinates of the ends of the struts for the initial and final position. The results are summarized in Table 5. Figure 3 shows the structure in its initial and final equilibrium position.

Table 5. Lower and upper coordinates for the unloaded and final position for the structure of the example 1 (mm).

	Strut 1		Strut 2		Strut 3	
	Initial Pos.	Final Pos.	Initial Pos.	Final Pos.	Initial Pos.	Final Pos.
E_x	33.0568	40.8573	-16.5284	-20.4241	-16.5284	-20.4332
E_y	0	-0.0053	28.6280	35.3861	-28.6280	-35.3808
E_z	0	0	0	0	0	0
A_x	-19.7819	-26.8241	0	11.7016	19.7819	15.1224
A_y	11.4211	1.9750	-22.8422	-24.2179	11.4211	22.2429
A_z	84.1287	73.5888	84.1287	73.5888	84.1287	73.5888

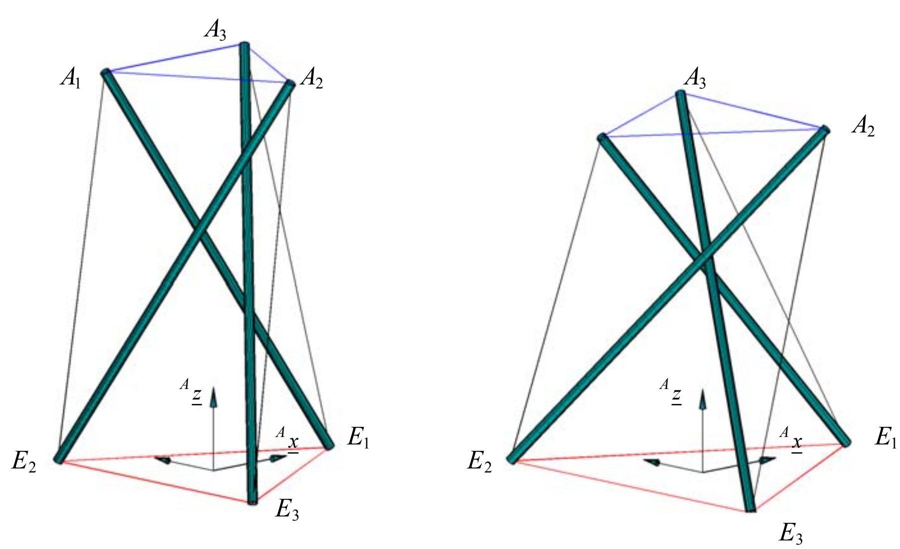

Figure 3. Unloaded and final equilibrium position for the structure of the example 1.

Figure 4 shows the free body diagram of the strut 2 for the final position. The summation of moments with respect to the lower end E_2 is given by the following equation

$$\underline{r} \times \underline{F} + \underline{r} \times \underline{F}_{A2A1} + \underline{r} \times \underline{F}_{A2A3} + \underline{r} \times \underline{F}_{A2E3} = \underline{0}$$

where

$$\underline{r} = \underline{A}_2 - \underline{E}_2$$

$$\underline{F}_{A2A1} = k_T(|\underline{A}_1 - \underline{A}_2| - T_0)\frac{\underline{A}_1 - \underline{A}_2}{|\underline{A}_1 - \underline{A}_2|}$$

$$\underline{F}_{A2A3} = k_T(|\underline{A}_3 - \underline{A}_2| - T_0)\frac{\underline{A}_3 - \underline{A}_2}{|\underline{A}_3 - \underline{A}_2|}$$

$$\underline{F}_{A2E3} = k_L(|\underline{E}_3 - \underline{A}_2| - L_0)\frac{\underline{E}_3 - \underline{A}_2}{|\underline{E}_3 - \underline{A}_2|}$$

The numerical values are obtained from Tables 1 and 5. In addition from Table 2, F=[0 0 –10]^T . It can be verified that when the numerical values are replaced in the expression for the equilibrium of moments the result is $\underline{0}$ what means the current position is indeed an equilibrium position.

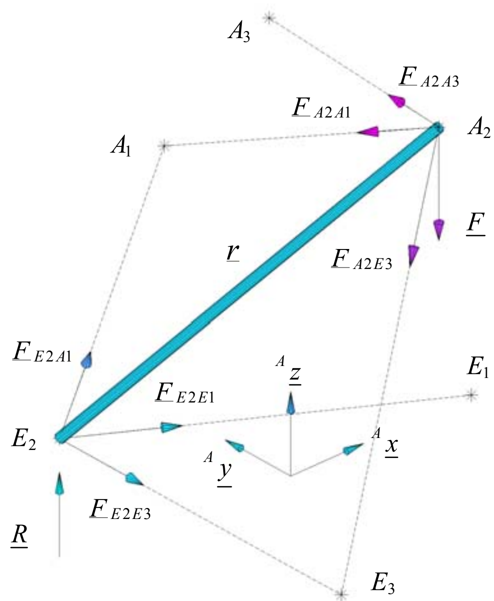

Figure 4. Free body diagram for the second strut of the structure of example 1 in the last position.

5. Example 2: Analysis of a Tensegrity Structure with 4 Struts

5.1 ANALYSIS FOR THE UNLOADED POSITION

It is required to evaluate the unloaded equilibrium position of a tensegrity structure with 4 struts with the stiffness and free lengths shown in Table 6. Each of its struts has a length $L_s = 100mm$.

Table 6. Stiffness and free lengths for the structure of example 2.

	Stiffness (N/mm)	Free lengths (mm)
Top ties	$k_T = 0.5$	$T_0 = 40$
Bottom ties	$k_B = 0.5$	$B_0 = 40$
Connecting ties	$k_L = 0.5$	$L_0 = 40$

For this example $n = 4$ then (35) and (36) yield $\gamma = 90°$, $\alpha = 22.55°$, $R_{T0} = 28.2843mm$ and $R_{B0} = 28.2843mm$.

The solution of (32), (33) and (34) yields $R_B = 41.2528mm$, $R_T = 41.2528mm$ and from (41), $H = 64.7280mm$. The parameters that define the location of the struts at the initial position are obtained from (37) through (40) and are listed in Table 7.

Table 7. Parameters for the location of the struts for the structure of example 2 in the unloaded position.

	Strut 1	Strut 2	Strut 3	Strut 4
a (mm)	41.2528	0	-41.2528	0
b (mm)	0	41.2528	0	-41.2528
ε (rad)	-0.4234	0.8275	0.4234	-0.8275
β (rad)	-0.7813	-0.2960	0.7813	0.2960

5.2 ANALYSIS FOR THE LOADED POSITION

It is required to evaluate the final equilibrium position of the structure when the external moments listed in Table 8 are applied along the axes of the universal joints that model the structure, see Figure 5, and the lower ends of the struts are constrained in such a way that they cannot move in the horizontal plane.

Table 8. External moments acting on the structure of example 2.

	Strut 1	Strut 2	Strut 3	Strut 4
M_ε (N.mm)	450	-900	450	-900
M_β (N.mm)	450	450	450	450

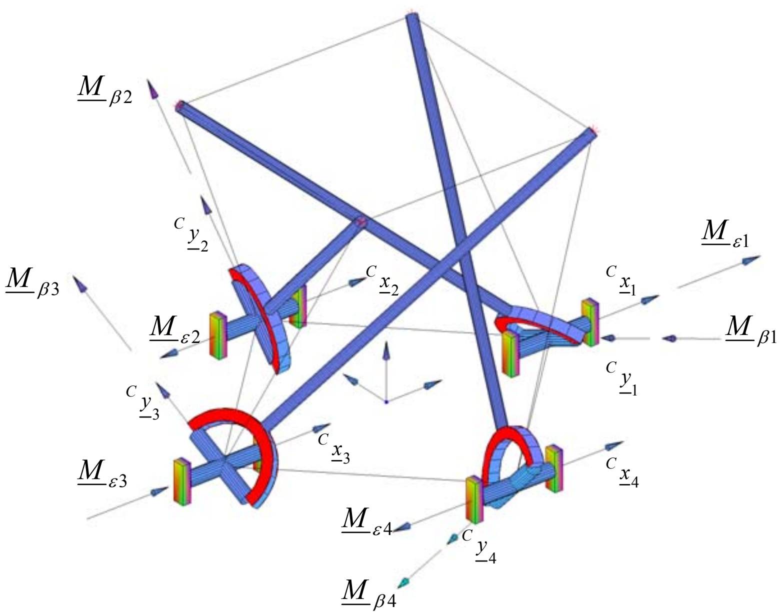

Figure 5. Directions of the external moments for the structure of example 2.

Because there are 2 constraints per strut there are 8 degrees of freedom for this system, and they are associated with the rotations of the struts. The generalized coordinates are ε_1, β_1, ε_2, β_2, ε_3, β_3 and ε_4, β_4, where the subscript indicates the number of the strut. The required 8 equations are obtained as follows: equation (29) yields f_9, f_{10}, f_{11} and f_{12} and equation (30) yields f_{13}, f_{14}, f_{15} and f_{16}. The solution of this system yields the generalized coordinates for the final position listed in Table 9.

Table 9. Generalized coordinates for the final position for the structure of example 2.

	Strut 1	Strut 2	Strut 3	Strut 4
ε (rad)	-0.4689	0.6311	0.8140	-1.1716
β (rad)	-0.2916	0.1598	1.2136	0.5948

Using the values of Tables 7 and 9 equations (7) and (8) yield the coordinates of the ends of the struts for the initial and final position. The results are summarized in Table 10. Figure 6 shows the structure in its initial and final equilibrium position. It should be noted that the coordinates *aj* and *bj* are the same for the unloaded and final position due to the constraints imposed to the lower ends.

Table 10. Lower and upper coordinates for the unloaded and final position for the structure of the example 2 (mm).

	Strut 1		Strut 2		Strut 3		Strut 4	
	Init. Pos.	Fin. Pos.	Init.Pos.	Fin. Pos.	Init.Pos.	Fin. Pos.	Init.Pos.	Fin. Pos.
E_x	41.2528	41.2528	0	0	-41.2528	-41.2528	0	0
E_y	0	0	41.2528	41.2528	0	0	-41.2528	-41.2528
E_z	0	0	0	0	0	0	0	0
A_x	-29.1701	12.5079	-29.1701	15.9151	29.1701	52.4370	29.1701	56.0322
A_y	29.1701	43.2859	-29.1701	-16.9992	-29.1701	-25.4176	29.1701	35.0631
A_z	64.7280	85.4404	64.7280	79.7083	64.7280	24.0035	64.7280	32.1911

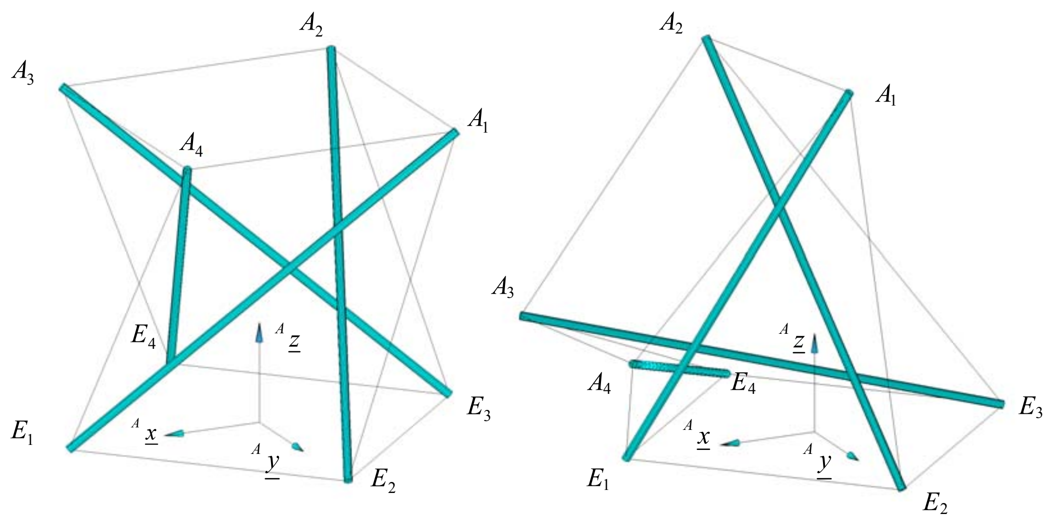

Figure 6. Unloaded and final equilibrium position for the structure of example 2.

Figure 7 shows the free body diagram for the fourth strut in its final position modeled with a universal joint. In addition to the forces in the ties and the external moments, all known, there are a reaction force $\underline{R}$ and a reaction moment $\underline{RM}$, these both of which are unknowns. The equilibrium equation expressed in Plücker coordinates in the C system, this is the system defined by the axes of the universal joint, see (51) Part 1, is

$$k_T(|\underline{A}_4-\underline{A}_1|-T_0)\,{}^C\hat{\$}_{A4A1} + k_T(|\underline{A}_4-\underline{A}_3|-T_0)\,{}^C\hat{\$}_{A4A3} + k_L(|\underline{A}_4-\underline{E}_1|-L_0)\,{}^C\hat{\$}_{A4E1} +$$

$$k_B(|\underline{E}_4-\underline{E}_1|-B_0)\,{}^C\hat{\$}_{E4E1} + k_B(|\underline{E}_4-\underline{E}_3|-B_0)\,{}^C\hat{\$}_{E4E3} + k_L(|\underline{E}_4-\underline{A}_3|-L_0)\,{}^C\hat{\$}_{E4A3} +$$

$$M_\varepsilon\,{}^C\hat{\$}_{M\varepsilon} + M_\beta\,{}^C\hat{\$}_{M\beta} + {}^C\underline{\$}_R + {}^C\underline{\$}_{RM} = \underline{0}$$

${}^C\hat{\$}_{A4A1}$, ${}^C\hat{\$}_{A4A3}\cdots$ are the unitized Plücker coordinates of the lines joining the points A_4A_1, $A_4A_3\cdots$ They are calculated in the A system with the aid of (42) through (44) of Part 1 and using the data of Table 10 for the structure in its final position. Then they are converted to the C system using (47) through (50) of Part 1. It should be noted that *aj* and *bj* in (44) correspond to the coordinates *x* and *y* for the lower end of the strut 4. They do not change for this example and their numerical values are listed in Table 7. The angle εj in (50) is one of the generalized coordinates associated with the rotations of strut 4 in the last position and is listed in Table 9.

$M\varepsilon\,{}^C\underline{\$}_{M\varepsilon}$ are the Plücker coordinates of the moment acting along axis ${}^C\underline{x}_4$ and according to the initial data is given by 900 $[0\ 0\ 0\ {-1}\ 0\ 0]^T$, see Table 4.

Similarly $M\beta\,{}^{C}\underline{\$}_{M\beta}$ are the Plücker coordinates of the moment acting along axis ${}^{C}\underline{y}_4$ and is given by 450 $[0\ 0\ 0\ 0\ 1\ 0]^T$, see Table 4.

${}^{C}\underline{\$}_R$ is the reaction force through E_4 which components are $[Rx\ Ry\ Rz\ 0\ 0\ 0]^T$ and ${}^{C}\underline{\$}_{RM}$ is the reaction moment along the axes of the C system which components are $[0\ 0\ 0\ RMx\ RMy\ RMz]^T$.

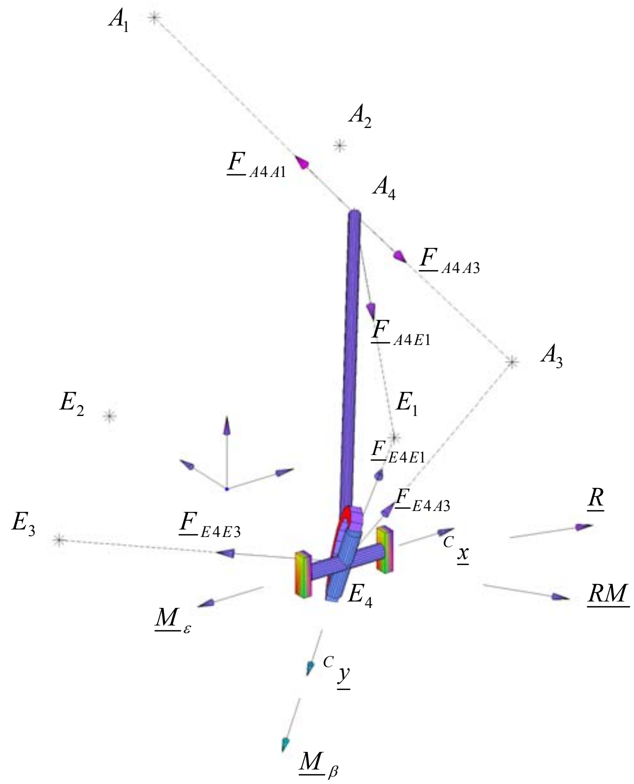

Figure 7. Free body diagram for the fourth strut of example 2 in the last position.

The substitution of numerical values in the last expression yields

$$14.6319\begin{bmatrix}-0.6284\\-0.6622\\0.4082\\54.8493\\-74.9186\\-37.1052\end{bmatrix}+10.5691\begin{bmatrix}-0.0588\\-0.2611\\-0.9635\\21.6248\\49.1177\\-14.6290\end{bmatrix}+4.9205\begin{bmatrix}-0.2965\\0.3217\\-0.8992\\-26.6442\\25.8241\\18.0246\end{bmatrix}+9.1701\begin{bmatrix}0.7071\\0.2748\\0.6515\\0\\0\\0\end{bmatrix}+9.1701\begin{bmatrix}-0.7071\\0.2748\\0.6515\\0\\0\\0\end{bmatrix}+$$

$$9.9022\begin{bmatrix}0.8768\\-0.2669\\0.4\\0\\0\\0\end{bmatrix}-900\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}+450\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}+\begin{bmatrix}{}^{C}R_x\\{}^{C}R_y\\{}^{C}R_z\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\{}^{C}RM_x\\{}^{C}RM_y\\{}^{C}RM_z\end{bmatrix}=\underline{0}$$

The expansion of rows four and five yield $RMx=0$ and $RMy=0$, this is the component of the reaction moment along the axes of the universal joint is zero. Since the universal joint cannot exert any reaction moment along its axes, the foregoing results confirm that the current position is an equilibrium position. The same procedure is executed for the software for all the struts and all the positions of the structure.

6. CONCLUSIONS

The model allows one to analyze a general anti-prism tensegrity structure subjected to a wide variety of external loads and the software developed is able to solve the system of equations generated for the model. The results are presented both numerically and in a three dimensional graphical representation, which permits one to visualize the behavior of the structure.

The model is developed using the virtual work approach and all the results are checked using Newton's Third Law. This verification assures one that the answers produced by the numerical method accurately correspond to equilibrium positions.

Mathematical models for variations of the basic configuration of tensegrity structures such as the reinforced tensegrity prisms might be developed following the same procedure presented in this research.

The mathematical model always assumes that the ties are in tension. If under the action of the external load the distance between two strut ends which are connected by a tie is less than the free length for that tie, the model is no longer a valid representation of the structure and as result no convergence is found and the software cannot yield a solution for that particular situation.

Also when two struts or a tie and a strut intersect the Jacobian for the structure vanishes, Lee et al. [3], and corresponds to a singular configuration that the software cannot solve.

In certain situations even though there are no singularities and all the ties are in tension, a small increase in the external load can make it impossible for the software to converge to a solution, i.e., it is not possible to find a new equilibrium position. The system suffers a sudden change and it jumps from one equilibrium position to another for a smooth transition force. This is known as a catastrophe, Hines [4], and Arnold's [5]. Catastrophe Theory is a well developed classical method. It describes sudden changes caused by a gradually changing input. It offers a better understanding of the phenomena reported here which is beyond the scope of this work.

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