Static Analysis of Tensegrity Structures
Part 1. Equilibrium Equations

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Abstract

In this paper the mathematical model to perform the static analysis of an antiprism tensegrity structure subjected to a wide variety of external loads is addressed. The virtual work approach is used to deduce the equilibrium equations and a method based on the Newton's Third Law to verify the numerical results is presented. In the second part of the paper several numerical examples are given.

1. Introduction

Tensegrity structures are spatial structures formed by a combination of rigid elements (the struts) and elastic elements (the ties). No pair of struts touch and the end of each strut is connected to three non-coplanar ties [1]. The struts are always in compression and the ties in tension. The entire configuration stands by itself and maintains its form solely because of the internal arrangement of the ties and the struts [2]. Tensegrity is an abbreviation of tension and integrity.

The development of tensegrity structures is relatively new and the works related have only existed for the 25 years. Kenner [3] established the relation between the rotation of the top and bottom ties. Tobie [2] presented procedures for the generation of tensile structures by physical and graphical means. Yin [1] obtained Kenner's results using energy considerations and found the equilibrium position for the unloaded tensegrity prisms. Stern [4] developed generic design equations to find the lengths of the struts and elastic ties needed to create a desired geometry. Since no external forces are considered his results are referred to the unloaded position of the structure. Knight [5] addressed the problem of stability of tensegrity structures for the design of a deployable antenna.

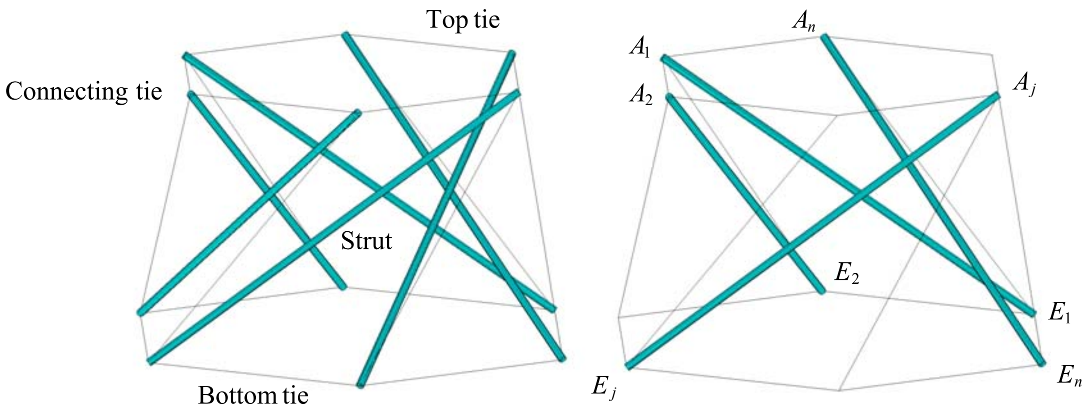
In this paper the problem of the determination of the equilibrium position of a tensegrity structure when external forces and external moments act on the structure is addressed.

2. Nomenclature

Figure 1a shows a tensegrity structure formed by *n* struts each one of length L_S . In every structure it is possible to identify the top ties, the bottom ties and the lateral or connecting ties which are denoted as *T, B* and *L* respectively. Figure 1b shows the same structure. The bottom ends of each strut is labeled consecutively as $E_1, E_2, \cdots, E_j, \cdots, E_n$ where 1 identifies the first strut and *n* stands for the last strut. Similarly the top ends of the struts are labeled as $A_1, A_2, \cdots, A_j, \cdots, A_n$. The selection of the first strut is arbitrary but once it is chosen it should not be changed.

3. Generalized Coordinates and Transformations Matrices

Figure 2a shows an arbitrary point *P* located on a strut of length L_s. In a reference system *D* whose *z* axis is along the axis of the strut and with its origin located at the lower end of the strut, the coordinates of ${}^{D}\underline{P}$ are simply *(0,0,l)*. However frequently is more convenient for purposes of analysis to express the location of $\underline{P}$ in the global reference system *A*.



(a) (b)

Figure 1. Nomenclature for tensegrity structures.
a) Components; b) Strut ends.

If the lower end of one strut is constrained to move on the horizontal plane and also the rotation about its longitudinal axis is constrained, the strut can be modeled by a universal joint. In this way the joint provides the 4 degrees of freedom associated with the strut. The total system has $4*n$ degrees of freedom which means there are $4*n$ generalized coordinates.

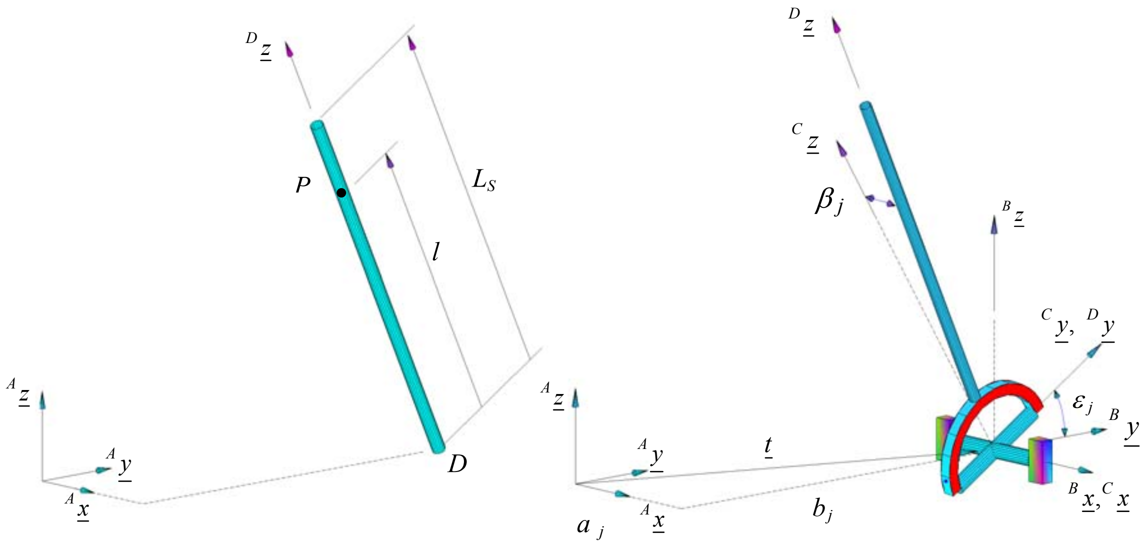
For each strut the generalized coordinates are the horizontal displacements a_j, b_j of the lower end of the strut together with two rotations about the axes of the universal joint, ε_j and β_j. ε_j corresponds to the rotation of the strut about ${}^{B}\underline{x}$ axis and β_j corresponds to the rotation about ${}^{C}\underline{y}$ axis, see Figure 2b.

The alignment of the z axis on the fixed system with the axis of the rod can be accomplished using the following three consecutive transformations, [7]: translation, $\underline{t}=(a_j, b_j, 0)$, rotation ε about the current x axis axis (${}^{B}\underline{x}$) and rotation β about the current y axis (${}^{C}\underline{y}$).

The coordinates of P expressed in the global reference system are

$$ {}^{A}\underline{P} = {}^{A}_{B}T_{a,b,0}\; {}^{B}_{C}T_{\varepsilon}\; {}^{C}_{D}T_{\beta}\; {}^{D}\underline{P} \qquad (1) $$

where



(a) (b)

Figure 2. Degrees of freedom associated with one of the struts of a tensegrity structure.
a) Arbitrary point on the strut; b) Strut modeled as a universal joint.

$$ {}^{A}_{B}T_{a,b,0} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2} $$

$$ {}^{B}_{C}T_{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon & 0 \\ 0 & \sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3} $$

$$ {}^{C}_{D}T_{\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4} $$

$$ {}^{D}\underline{P} = [0 \ \ 0 \ \ l \ \ 1]^{T} \tag{5} $$

Substituting the above three expressions into (1) yields

$$ {}^{A}\underline{P} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l\sin\beta + a \\ -l\sin\varepsilon\cos\beta + b \\ l\cos\varepsilon\cos\beta \\ 1 \end{bmatrix} \tag{6} $$

In addition to the constraint imposed that the lower ends are to remain in the horizontal plane and for each strut to avoid the rotation about its longitudinal axis the following assumptions are made without loss of generality:

- The external moments are applied along the axes of the universal joints.
- The struts are massless.
- All the struts have the same length.
- Only one external force is applied per strut.
- There are no dissipative forces acting on the system.
- All the ties are in tension at the equilibrium position, i.e., the initial lengths of the ties are greater than their respective free lengths.
- The free lengths of the top ties are equal.
- The free lengths of the bottom ties are equal.
- The free lengths of the connecting ties are equal.
- The stiffness of all the top ties is the same.
- The stiffness of all the bottom ties is the same.
- The stiffness of all the connecting ties is the same.

4. Coordinates of The Ends of the Struts

The Cartesian coordinates of the lower ends $\underline{E}_j$, expressed in the global reference system A, are obtained in terms of the generalized coordinates substituting l in (6) by 0, and replacing a and b by a_j, b_j

$$ {}^{A}\underline{E}_j = \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} \tag{7} $$

Similarly the coordinates of the upper end of the struts A_j are evaluated by replacing l by the length of the struts L_S in (6)

$$ {}^{A}\underline{A}_{j} = \begin{bmatrix} L_s \sin\beta_j + a_j \\ -L_s \sin\varepsilon_j \cos\beta_j + b_j \\ L_s \cos\varepsilon_j \cos\beta_j \end{bmatrix} \tag{8} $$

Equations (7) and (8) permit one to obtain expressions for the lengths of the top, bottom and lateral ties in terms of the generalized coordinates as follows

$$ T_j = \left(\left(A_{j+1,x} - A_{j,x}\right)^2 + \left(A_{j+1,y} - A_{j,y}\right)^2 + \left(A_{j+1,z} - A_{j,z}\right)^2 \right)^{1/2} \tag{9} $$

$$ B_j = \left(\left(E_{j+1,x} - E_{j,x}\right)^2 + \left(E_{j+1,y} - E_{j,y}\right)^2 + \left(E_{j+1,z} - E_{j,z}\right)^2 \right)^{1/2} \tag{10} $$

$$ L_j = \left(\left(A_{j,x} - E_{j+1,x}\right)^2 + \left(A_{j,y} - E_{j+1,y}\right)^2 + \left(A_{j,z} - E_{j+1,z}\right)^2 \right)^{1/2} \tag{11} $$

where if $j = n$ then $j+1 = 1$

5. The Principle of Virtual Work for Tensegrity Structures

The virtual work for systems able to store potential energy can be stated from [6] by

$$ \delta W = \delta W_{nc} + \delta W_c \tag{12} $$

where δW is the total virtual work, δW_{nc} is the virtual work performed for non-conservative forces and moments and δW_c is the virtual work performed by conservative forces. δW_{nc} can be represented as

$$ \delta W_{nc} = \delta W_F + \delta W_M \tag{13} $$

where δW_F is the total virtual work performed by non-conservative forces and δW_M is the total virtual work performed by non-conservative moments.

The virtual work performed by the conservative force j, δW_{cj} is $\delta W_{cj} = -\delta V_j$ where δV_j is the potential energy associated with the conservative force j, therefore the total contribution of the conservatives forces δW_c is

$$ \delta W_c = -\delta V \tag{14} $$

where δV is the summation over all the δV_j present in the structure.

Substituting (13) and (14) into (12) yields

$$ \delta W = \delta W_F + \delta W_M - \delta V \tag{15} $$

In equilibrium the virtual work described by (15) must be zero [6], then the equilibrium conditions can be deduced from

$$ \delta W_F + \delta W_M - \delta V = 0 \tag{16} $$

6. The Virtual Work Due to the External Forces

As it is assumed that there is only one external force acting on each strut, the virtual work δW_F performed by all the external forces is given by

$$ \delta W_F = \sum_{j=1}^{n} \underline{F}_j \cdot \delta \underline{r}_j \tag{17} $$

where $\underline{F}_j$ is the external force acting in the strut j and $\delta \underline{r}_j$ is the virtual displacement of $\underline{r}_j$, this is the vector that goes from the origin of the global reference system to the point of application of the external force. If the distance between the point of application of the force and the lower end of the strut is L_{Fj}, see Figure 3, then an expression for $\underline{r}_j$ in the global system can be obtained from (6) replacing l by L_{Fj} and its rectangular coordinates are

$$ {}^{A}\underline{r}_j = \begin{bmatrix} {}^{A}r_{j\,x} \\ {}^{A}r_{j\,y} \\ {}^{A}r_{j\,z} \end{bmatrix} = \begin{bmatrix} L_{Fj} \sin\beta_j + a_j \\ -L_{Fj} \sin\varepsilon_j \cos\beta_j + b_j \\ L_{Fj} \cos\varepsilon_j \cos\beta_j \end{bmatrix} \tag{18} $$

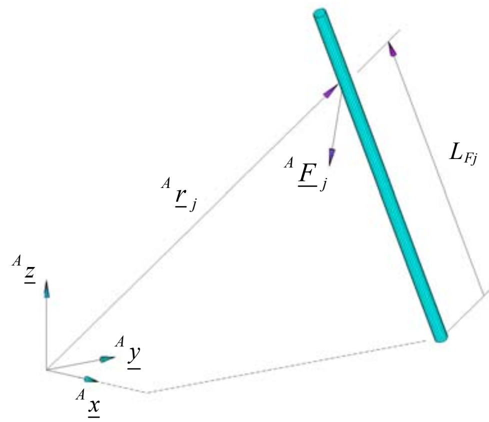


Figure 3. Location of the external force acting on the strut j.

The virtual displacements can be deduced from (18) where the derivatives are taken with respect to the generalized coordinates εj, βj, aj and bj

$$\delta\,{}^{A}\underline{r}_{j} \;=\; \begin{bmatrix} {}^{A}\delta r_{j\;x} \\ {}^{A}\delta r_{j\;y} \\ {}^{A}\delta r_{j\;z} \end{bmatrix} \;=\; \begin{bmatrix} \delta a_j \;+\; L_{Fj}\cos\beta_j\delta\beta_j \\ \delta b_j \;-\; L_{Fj}\cos\varepsilon_j\cos\beta_j\delta\varepsilon_j \;+\; L_{Fj}\sin\varepsilon_j\sin\beta_j\delta\beta_j \\ -L_{Fj}\sin\varepsilon_j\cos\beta_j\delta\varepsilon_j \;-\; L_{Fj}\cos\varepsilon_j\sin\beta_j\delta\beta_j \end{bmatrix} \tag{19}$$

Substituting (19) into (17), the general expression for the virtual work performed by external forces is given by

$$\begin{aligned} \delta W_F \;=\; \sum_{j=1}^{n} \;& (L_{Fj}\left[-{}^{A}F_{j\;y}\cos\varepsilon_j\cos\beta_j - {}^{A}F_{j\;z}\sin\varepsilon_j\cos\beta_j\right]\delta\varepsilon_j \\ & + L_{Fj}\left[{}^{A}F_{j\;x}\cos\beta_j + {}^{A}F_{j\;y}\sin\varepsilon_j\sin\beta_j - {}^{A}F_{j\;z}\cos\varepsilon_j\sin\beta_j\right]\delta\beta_j \;+ {}^{A}F_{j\;x}\delta a_j \;+ {}^{A}F_{j\;y}\delta b_j) \end{aligned} \tag{20}$$

7. The Virtual Work Due to the External Moments

The virtual work performed by the external moments is given by

$$\delta W_M \;=\; \sum_{j=1}^{n} \underline{M}_{\varepsilon\;j} \cdot \delta\underline{\varepsilon}_j \;+\; \underline{M}_{\beta\;j} \cdot \delta\underline{\beta}_j \tag{21}$$

Provided that in this model of the tensegrity structure the external moments can be exerted only along the axis of the universal joint, $\underline{M}_{\varepsilon j}$ is collinear with $\delta\underline{\varepsilon}_j$ and $\underline{M}_{\beta j}$ is collinear with $\delta\underline{\beta}_j$, see Figure 4, therefore (21) is simplified to

$$\delta W_M \;=\; \sum_{j=1}^{n} M_{\varepsilon\;j}\delta\varepsilon_j \;+\; M_{\beta\;j}\delta\beta_j \tag{22}$$

8. The Potential Energy

Since the struts are considered massless the term related to the potential energy in the principle of virtual work is the resultant of the elastic potential energy contributions given by the ties. The potential elastic energy for a general tie *j* is given by, [6]

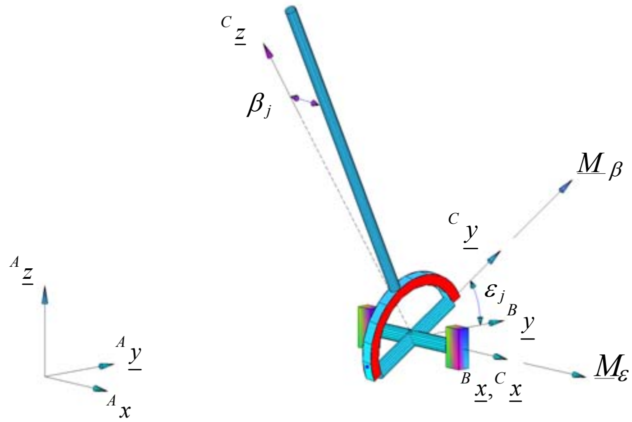


Figure 4. External moments applied to one of the struts of a tensegrity structure.

$$V_j = \frac{1}{2}k \ (w_j - w_{j0})^2 \tag{23}$$

where V_j is the elastic potential energy for tie j, k the tie stiffness, w_j the current length of the tie j and w_{jo} the free length of the tie j. Therefore the differential of the potential energy for tie j is

$$\delta V_j = k \ (w_j - w_{j0}) \ \delta w_j \tag{24}$$

The differential of the potential energy for all the tensegrity structure, δV, is the resultant of the contributions of the top ties, the bottom ties and the lateral ties and can be expressed as

$$\delta V = \sum_{j=1}^{n} k_T \left(T_j - T_o\right)\delta T_j + \sum_{j=1}^{n} k_B \left(B_j - B_o\right)\delta B_j + \sum_{j=1}^{n} k_L \left(L_j - L_o\right)\delta L_j \tag{25}$$

where k_T, k_B, k_L are the stiffness of the top, bottom and connecting ties respectively, T_0, B_0 and L_0 are the free lengths of the top, bottom and connecting ties respectively and T_j, B_j and L_j are given by (9), (10) and (11) and are functions of some of the generalized coordinates.

9. The General Equations

Now that each one of the terms contributing to the virtual work has been evaluated, the equilibrium condition for the general tensegrity structure can be established. Substituting (20), (22) and (25) into (16) and re-grouping yields

$$\begin{aligned} &f_1\delta a_1 + f_2\delta a_2 + \ldots\ldots f_n\delta a_n + f_{n+1}\delta b_1 + f_{n+2}\delta b_2 + \ldots\ldots f_{2n}\delta b_n + \\ &f_{2n+1}\delta\varepsilon_1 + f_{2n+2}\delta\varepsilon_2 + \ldots\ldots f_{3n}\delta\varepsilon_n + f_{3n+1}\delta\beta_1 + f_{3n+2}\delta\beta_2 + \ldots\ldots f_{4n}\delta\beta_n = 0 \end{aligned} \tag{26}$$

where

$$f_i = {}^{A}F_{i\ x} - \sum_{j=1}^{n} k_T \ \left(T_j - T_o\right) \frac{\partial T_j}{\partial a_i} - \sum_{j=1}^{n} k_B \ \left(B_j - B_o\right) \frac{\partial B_j}{\partial a_i} - \sum_{j=1}^{n} k_L \ \left(L_j - L_o\right) \frac{\partial L_j}{\partial a_i} \tag{27}$$

$$f_{n+i} = {}^{A}F_{i\ y} - \sum_{j=1}^{n} k_T \ \left(T_j - T_o\right) \frac{\partial T_j}{\partial b_i} - \sum_{j=1}^{n} k_B \ \left(B_j - B_o\right) \frac{\partial B_j}{\partial b_i} - \sum_{j=1}^{n} k_L \ \left(L_j - L_o\right) \frac{\partial L_j}{\partial b_i} \tag{28}$$

$$\begin{aligned} f_{2n+i} = {} & L_{Fi} \ \left[-{}^{A}F_{i\ y}\cos\varepsilon_i\cos\beta_i - {}^{A}F_{i\ z}\sin\varepsilon_i\cos\beta_i\right] + M_{\varepsilon\ i} - \sum_{j=1}^{n} k_T \ \left(T_j - T_o\right)\frac{\partial T_j}{\partial\varepsilon_i} \\ & - \sum_{j=1}^{n} k_B \ \left(B_j - B_o\right)\frac{\partial B_j}{\partial\varepsilon_i} - \sum_{j=1}^{n} k_L \ \left(L_j - L_o\right)\frac{\partial L_j}{\partial\varepsilon_i} \end{aligned} \tag{29}$$

$$f_{3n+i} = L_{Fi} \left[{}^{A}F_{i\ x} \cos\beta_i + {}^{A}F_{i\ y} \sin\varepsilon_i \sin\beta_i - {}^{A}F_{i\ z} \cos\varepsilon_i \sin\beta_i \right] + M_{\beta\ i}$$
$$- \sum_{j=1}^{n} k_T \left(T_j - T_o\right) \frac{\partial T_j}{\partial \beta_i} - \sum_{j=1}^{n} k_B \left(B_j - B_o\right) \frac{\partial B_j}{\partial \beta_i} - \sum_{j=1}^{n} k_L \left(L_j - L_o\right) \frac{\partial L_j}{\partial \beta_i} \tag{30}$$

$i = 1, 2, \ . \ . \ . \ , n$

Equation (26) must be satisfied for all the values of the virtual displacements which in general are different from zero, then

$$\begin{aligned} f_1 &= 0 \\ f_2 &= 0 \\ &\vdots \\ f_{4n} &= 0 \end{aligned} \tag{31}$$

where f_i is given by equations (27) to (30). Equations (31) represent a strongly coupled system of *4*n* equations depending only on the *4*n* generalized coordinates. The equilibrium position for a general tensegrity structure is obtained by solving numerically the set (31) for $a_1, b_1, \varepsilon_1, \beta_1, \ldots\ldots, a_n, b_n, \varepsilon_n, \beta_n$. After that equations (7) and (8) yield explicitly expressions for the coordinates of the ends of the struts in the global coordinate system.

10. Initial Conditions

To be able to solve (31) it is necessary to find a proper set of values for the generalized coordinates in the unloaded position. This is accomplished using Yin's method [1], which is presented here without proof.

$$k_L \left(1 - \frac{L_o}{L}\right) R_B - 2k_T \left(R_T - R_{To}\right) \sin\frac{\gamma}{2} = 0 \tag{32}$$

$$k_L \left(1 - \frac{L_o}{L}\right) R_T - 2k_B \left(R_B - R_{Bo}\right) \sin\frac{\gamma}{2} = 0 \tag{33}$$

$$L - \sqrt{L_s^2 + 2R_B R_T \left[\cos(\alpha+\gamma) - \cos\alpha\right]} = 0 \tag{34}$$

where

$$R_{T0} = \frac{T_o}{2\sin\frac{\gamma}{2}} \text{ and } R_{B0} = \frac{B_o}{2\sin\frac{\gamma}{2}} \tag{35}$$

And the angles γ and α are given by

$$\gamma = \frac{2\pi}{n} \text{ and } \alpha = \frac{\pi}{2} - \frac{\pi}{n} \tag{36}$$

where n is the number of struts.

The three unknowns R_B, R_T and the length of the connecting ties L are solved using equations (32) through (34). These values are used to solve the following set of generalized coordinates for the unloaded position.

$$a_{j,0} = R_B \cos\left(\left(j-1\right) \gamma\right) , \quad j = 1, 2, \ldots, n \tag{37}$$

$$b_{j,0} = R_B \sin\left(\left(j-1\right) \gamma\right) , \quad j = 1, 2, \ldots, n \tag{38}$$

$$\tan\varepsilon_{j,0} = \frac{b_{j,0} - R_T \sin(\ (j-1)\ \gamma + \alpha)}{H} \tag{39}$$

$$\tan\beta_{j,0} = \frac{R_T \cos(\ (j-1)\ \gamma + \alpha) - a_{j,0}}{\left(\dfrac{b_{j,0} - R_T \sin(\ (j-1)\ \gamma + \alpha)}{\sin\varepsilon_{j,0}}\right)} \tag{40}$$

where $H = \sqrt{L_s^2 - R_B^2 - R_T^2 - 2R_B R_T \sin\frac{\gamma}{2}}$ (41)

and if $j=1$ then $j-1=n$

11. Verification of the Numerical Results

Because of the complexity of the equilibrium equations it is essential to verify the answers obtained. An independent validation of the results can be accomplished using Newton's Third Law. If there are no external moments acting on an isolated strut, it is sufficient to perform the summation of moments with respect to the lower end of the strut. If there are external moments the verification process involves more steps and needs some additional concepts.

The unitized Plücker coordinates of a line joining two finite points *(x1, y1, z1)* and *(x2, y2, z2)*, as is the case of the forces in the ties, can be written in the global reference system as, [8]

$$ {}^{A}\hat{\$} = \frac{1}{\sqrt{L^2+M^2+N^2}} \begin{bmatrix} L & M & N & P & Q & R \end{bmatrix}^T \tag{42}$$

where

$$ L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, \; M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}, \; N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix} \tag{43}$$

and

$$ P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \; Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \; R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \tag{44}$$

The subindex *1* in (43) and (44) identifies the end of the tie attached to the current strut and the subindex *2* is for the remaining end of the tie. Further the coordinates of the ends of the ties can be evaluated using (7) and (8).

If *L, M* and *N* are simultaneously equal to zero (42) must be modified to

$$ {}^{A}\hat{\$} = \frac{1}{\sqrt{P^2+Q^2+R^2}} \begin{bmatrix} 0 & 0 & 0 & P & Q & R \end{bmatrix}^T \tag{45}$$

When an external force ${}^{A}\underline{F}_j$ and its point of application are known, the Plücker coordinates are obtained by

$$ {}^{A}\underline{\$}_F = \begin{bmatrix} {}^{A}\underline{F}_j \\ {}^{A}\underline{r}_j \times {}^{A}\underline{F}_j \end{bmatrix} \tag{46}$$

where ${}^{A}\underline{F}_j$ corresponds to the external force expressed in the global reference system and ${}^{A}\underline{r}_j$ is given by (18).

The Plücker coordinates can be expressed in a new system that is translated and rotated with respect to the global reference system. If the new system is the *C* system, this is the system defined for the axes of the universal joint, the expression that relates the Plücker coordinates in the *A* system and the *C* system is, [8]

$$ {}^{C}\underline{\$} = e^{-1} \; {}^{A}\underline{\$} \tag{47}$$

where

$$ e^{-1} = \begin{bmatrix} {}^{A}_{C}R^T & \underline{O}_3 \\ {}^{A}_{C}R^T A_3^T & {}^{A}_{C}R^T \end{bmatrix} \tag{48}$$

$$ A_3 = \begin{bmatrix} 0 & 0 & b_j \\ 0 & 0 & -a_j \\ -b_j & a_j & 0 \end{bmatrix} \tag{49}$$

$$ {}^{A}_{C}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon_j & -\sin\varepsilon_j \\ 0 & \sin\varepsilon_j & \cos\varepsilon_j \end{bmatrix} \quad (50) $$

and $\underline{0}_3$ is a 3 by 3 zeroes matrix.

Figure 5 shows the free body diagram for an arbitrary strut modeled with a universal joint. In addition to the forces in the ties there is an external force $\underline{F}_j$ which is known, a reaction force $\underline{R}$ passing through the lower end and a reaction moment $\underline{RM}$ at the lower end. Newton's Third Law expressed in Plücker coordinates in the C system is

$$ F_{Aj\,Aj+1}\,{}^{C}\hat{\$}_{Aj\,Aj+1} + F_{Aj\,Aj-1}\,{}^{C}\hat{\$}_{Aj\,Aj-1} + F_{Aj\,Ej+1}\,{}^{C}\hat{\$}_{Aj\,Ej+1} + F_{Ej\,Ej+1}\,{}^{C}\hat{\$}_{Ej\,Ej+1} + F_{Ej\,Ej-1}\,{}^{C}\hat{\$}_{Ej\,Ej-1} + $$
$$ F_{Ej\,Aj-1}\,{}^{C}\hat{\$}_{Ej\,Aj-1} + {}^{C}\underline{\$}_{F} + M_{\varepsilon}\,{}^{C}\hat{\$}_{M\varepsilon} + M_{\beta}\,{}^{C}\hat{\$}_{M\beta} + {}^{C}\underline{\$}_{R} + {}^{C}\underline{\$}_{RM} = \underline{0} \quad (51) $$

The coefficients $F_{Aj\,Aj+1},\ F_{Aj\,Aj-1},\cdots$ represent the magnitudes of the forces in the ties $A_j A_{j+1},\ A_j A_{j-1}\cdots$ and are given by *k*(w-wo)* where *k* is the stiffness, *w* the current length and *wo* the free length of the tie. The current lengths are given by (9) and (11) for the top ties and connecting ties respectively. It should be noted that the magnitude does not depend of the reference system which is used.

The unitized Plücker coordinates ${}^{C}\hat{\$}$ for each one of the ties can be calculated in the A system using (42) thorough (44) and then converted to the C system using (47) through (50).

The Plücker coordinates of the external force acting on the current strut $\underline{\$}_F$, can be evaluated in the A system using (46) and then converted to the C system with the aid of (47) through (50).

M_ε and M_β are the magnitudes of the external moments and their unitized Plücker coordinates in the C system are given by ${}^{C}\hat{\$}_{M\varepsilon} = [0\ \ 0\ \ 0\ \ 1\ \ 0\ \ 0]^T$ and ${}^{C}\hat{\$}_{M\beta} = [0\ \ 0\ \ 0\ \ 0\ \ 1\ \ 0]^T$.

Since the reaction force ${}^{C}\underline{\$}_R$ expressed in the C system is a pure force and the reaction moment ${}^{C}\underline{\$}_{RM}$ expressed in the C system is a pure moment they have the form $[\,{}^{C}R_x\ \ {}^{C}R_y\ \ {}^{C}R_z\ \ 0\ \ 0\ \ 0\,]^T$ and $[\,0\ \ 0\ \ 0\ \ {}^{C}RM_x\ \ {}^{C}RM_y\ \ {}^{C}RM_z\,]^T$ respectively. Further they are the only unknowns in (51).

After expanding (51), rows four and five represent the components in the x and y directions of the summation of moments about the lower end of the strut. A universal joint cannot exert moment along its own axes. If after a numerical evaluation, RM_x and RM_y are both zero, then the equilibrium of moments is maintained solely due to the forces in the ties and to the external loads (if any) and therefore the current position is an equilibrium position.

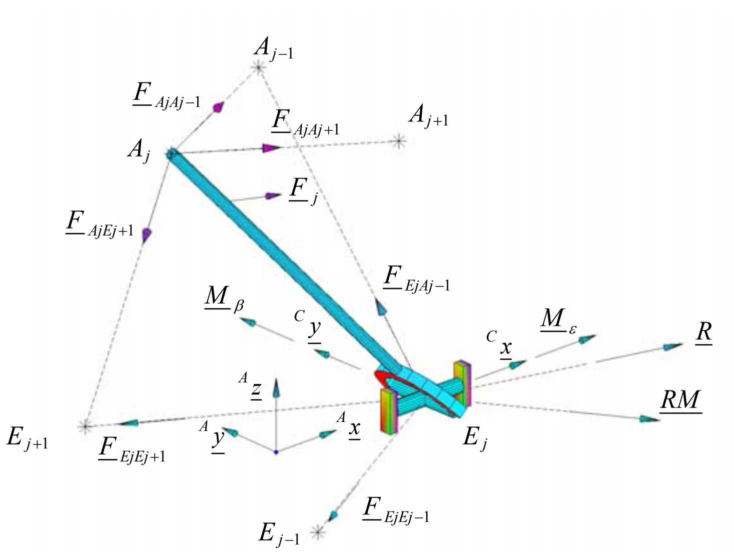


Figure 5. Free Body diagram for an arbitrary strut modeled with a universal joint.

12. Conclusions

The model developed here allows one to analyze a general anti-prism tensegrity structure subjected to a wide variety of external loads.

The model is developed using the virtual work approach and all the results are checked using the Newton's Third Law. This verification assures one that the answers produced by the numerical method accurately correspond to equilibrium positions.

Mathematical models for variations of the basic configuration of tensegrity structures such as the reinforced tensegrity prisms might be developed following the same procedure presented in this research.

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