AUTONOMOUS RECTILINEAR MOTION PLANNING
Part II: The Geometry of the End-Effector Workspace
and Determination of a Free Path

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ABSTRACT

In this second paper the analysis of the Reachable Area (RA) of the wrist point in the
previous paper is extended to determine the RA of a point in the end effector for a single
circular obstacle.
A time-efficient algorithm was subsequently developed using the geometry of the manipulator
workspace for determining a collision free rectilinear motion of the end effector of a
planar 3R manipulator with a single circular obstacle inside the workspace.
The algorithms developed for this work were successfully implemented using the Silicon
Graphics 4D system.
The execution time for determining the possibility of a rectilinear motion from an
initial to a final position was within 1 second.
If such rectilinear motion was not possible, the
algorithm generated a path autonomously consisting of a sequence of line segments.
The computer time of this was about 1 second.

INTRODUCTION

The Reachable Area (RA) of the end-effector facilitates the determination of motion
capability simply because it enables one to check whether the target position lies within
the RA or not. Obviously, if the target point lies within the RA of the manipulator, a
rectilinear motion from the initial position to the final position is possible.
However, if the target point lies outside the RA, the tip of the end effector of the
manipulator cannot follow a rectilinear motion to the target point. In this case, the RA for
the final configuration is determined, and an intermediate point will be suggested within the
area created by the intersection of the initial and final RAs. However, it can occur that these
two RAs do not intersect, in which case a free-path is determined using several
intermediate points.

The development presented here can be readily extended to determine paths with
multiple circular obstacles present in the workspace.

REACHABLE AREA OF END-EFFECTOR

It is assumed that the manipulator executes a rectilinear motion with a constant
orientation of the end-effector. The RA of the end-effector (point 4) (see Fig.1) is identical
to that of the wrist (point 3) but is translated a distance \( a_{w}a_{w} \) (see Lipkin, Torfason, and Duffy
[1]). The shaded area, which is inaccessible to the end-effector because of the obstacle,
must be subtracted from the translated RA.

![Figure 1 RA of the End-Effector.](image)

This shaded area is covered by the union of three shaded areas \((m), (n), \) and \((s)\) (see
Fig.2).

**Shaded Area \((m)\)**
The shaded area \((m)\) is bounded by four lines \(L_5, L_4, L_7, \) and \(L_8\) (Fig.2(a)). Lines \(L_5\)
and \( L_2 \) passing through the centers of the obstacle \( C_w \) and the virtual obstacle \( C^{'}_w \), respectively, are selected to be perpendicular to line \( O_0O_1' \) (or \( a_{xy} \)). Lines \( L_3 \) and \( L_4 \) are two common tangent lines to the obstacles. The origin is moved from \( O \) to \( O_0 = 0.5 \times (O + O_1') \) (the midpoint of line \( O_0O_1' \)), the line equations \( I_3, I_4, I_5, \) and \( I_6 \) become \( I_3', I_4', I_5', \) and \( I_6' \). The shaded area \( (m) \) can be expressed by

\[
N = (L_1'(X) > 0) \wedge (L_2'(X) < 0) \wedge (L_3'(X) < 0) \wedge (L_4'(X) < 0) = 1. \tag{1}
\]

**Shaded Area (n)**

The shaded area \( (n) \) is the umbra of the real obstacle with respect to point 4 (Fig.2(b)). \( C_w \) is the boundary of the obstacle. \( L_1 \) is the polar of point \( P_1 \) with respect to the obstacle. \( K_1 \) is a degenerate conic, which represents a pair of tangent lines with respect to the obstacle. When the origin is moved from \( O \) to \( O_0 \), the parameters \( C_w, I_3, K_2, \) and \( K_1 \) are changed into \( C_{w}', I_{3}', K_{2}', \) and \( K_{1}' \). The shaded area \( (n) \) is determined by

\[
N = (C_{w}'(X) < 0) \wedge (I_{3}'(X) < 0) \wedge (K_{1}'(X) < 0) = 1. \tag{2}
\]

**Shaded Area (s)**

The shaded area \( (s) \) is bounded by three lines \( N_1 (p_{1}O_{1}''), N_2 (p_{2}O_{2}''), N_3 (p_{3}O_{3}'') \) and \( C_{w}' \) (Fig.2(c)). The line equations are transformed into \( N_1', N_2', \) and \( N_3' \) after the origin is shifted from \( O \) to \( O_0 = 0.5 \times (0.5 \times (O + O_1') + P_0) \) (the midpoint of the ray through point 4 which bisects \( O_0O_1' \)). The shaded area \( (s) \) is defined by

\[
N = (N_1'(X) < 0) \wedge (N_2'(X) < 0) \wedge (N_3'(X) < 0) = 1. \tag{3}
\]

**Figure 2 Determination of NAS (m), (n), and (s).**

Subtracting these three areas from the translated RA in (Fig.1) yields the RA of the end-effector (point 4). The calculations for these three shaded areas are thus completely analogous to those for the RA of the wrist (point 3) (see part I) and hence no further equations are required for this computation. Also, the translation of the whole RA of the wrist is determined by replacing \( X \) with \( X - a_{xy} \). The fast determination of the RA of the end-effector simplifies the motion planning problem among obstacles.

**INTERSECTION OF TWO REACHABLE WORKSPACES WITH RESPECT TO THE INITIAL AND THE FINAL CONFIGURATIONS**

When a starting point and a target point are specified together with the orientation of the end-effector, two configurations (the initial and the final) can be determined as shown in (Fig.3). An RA of the end-effector with respect to the final configuration is constructed (see Fig.4) by using the algorithms given in the previous section. It is apparent that in this example the target point lies outside the RA of the initial configuration so that a single rectilinear motion of the end-effector is not possible and one or several intermediate points are needed to construct a path to the target position.

In order to solve the problem, an RA of the final configuration is determined (see Fig.5). The intersection of these two RAs (Fig.6) is the region where intermediate points can be selected. When there is no intersection of these two RAs, it is necessary to move the robot to an intermediate position in the initial RA and to determine a corresponding intermediate RA which intersects the final RA.

**Figure 3 Initial and Final Configurations of a Planar 3R Manipulator.**

**GENERATION OF A FREE PATH BETWEEN TWO POINTS**

An automatic search method to determine the intermediate positions is presented in this section. There are three distinct cases which are similar to the Cases (1), (2), and (3) discussed in Part I.

**Computation of a free path for case (1)**

In this case the second link cannot be a tangent to the obstacle (see Case (1), Part I). There are three subcases which must be considered.
respectively are drawn (lines \( L_4' \) and \( L_5' \)) which intersect at point \( q_i \). A two sided path \( p \), \( q_i \) thus satisfies the required motion. The two other tangents from \( p \) and \( q_i \) intersecting at \( q_i \) also satisfy the required motion. The choice of the path may be determined by, for example, selecting the shorter of the lengths \( p \), \( q_i \) and \( p \), \( q_i \).

**Figure 4** RA of the Initial Configuration.

**Figure 5** RA of the Final Configuration.

**Figure 6** Intersection of the Initial and Final RAs.

**Case (1.1):** The target position \( q_i \) lies inside the shadow of the obstacle \( C_{ob} \). In this case an intermediate point \( q_i \) (Fig. 8) is chosen so that the robot can move from point \( p \) to \( q \) through this point. Point \( q_i \) is chosen at the intersection of two tangent lines \( (L_4' \) and \( L_5' \)) to the obstacle \( C_{ob} \). These two lines pass through points \( p \) and \( q_i \), respectively.

**Figure 7** Determination of the Collision Free Paths of Case (1.1).

**Figure 8** Determination of the Collision Free Path of Case (1.2).
Case (1.3): Points \( p \) and \( q \) and the center of the virtual obstacle \( (O',') \) are on the same side of line \( L_0' \). The first step is to move the end-effector away from the obstacle (from \( p \) to \( p' \)) (see Fig. 9). The next step is to move the end-effector from \( p' \) to \( q' \) (see Case (1.1) and (1.2)). The final step is to move the end-effector from \( q' \) to \( q \). Points \( p' \) and \( q' \) are two intersections of line \( L_0' \) and circle \( C_{\infty}' \).

\[ \text{Figure 9 Determination of the Collision Free Path of Case (1.3).} \]

Computation of a free path for case (2)

In this case the second link can be a tangent to the obstacle and the obstacle lies outside the path of the first link (see Case (2), Part I). There are two subcases which must be considered.

Case (2.1): The target position \( q \) is in the shadow of the circle \( C_{\infty}' \) or in the shadow of the obstacle \( C_{\infty} \). In this case the algorithm is identical to that in Cases (1.1) and (1.2).

Case (2.2): Point \( p \) (or \( q \)) and the center of the virtual obstacle \( (O',') \) are on the same side of line \( L_0' \). Points \( h_3 \) and \( h_4 \) are the intersections of line \( L_0' \) and the circle \( C_{\infty}' \) (see Fig. 10). Point \( h_5 \) is chosen to be an intermediate point so that the end-effector of the robot can move from \( h_3 \) to \( q \) in the final step.

There are three possible choices for the intermediate point \( p' \). The first one is point \( B' \) (an intersection of line \( L_0' \) and circle \( C_{\infty}' \)). The second one is point \( h_5 \), which is the middle point of line segment \( h_3 \). This can be chosen to be an intermediate point when point \( B' \) is not available, viz. when \( B' \) lies inside the obstacle \( C_{\infty} \). The third choice is point \( h_4 \). This can be chosen as an intermediate point when both points \( B' \) and \( h_4 \) are inside the obstacle.

If point \( p \) cannot move directly to any of these three points, points \( h_3 \) and \( h_4 \) on the circular arc \( e_h \) (see Fig. 10) can be determined as the intermediate points of the free path \( p, h_3, h_4, q \) to point \( q \), where \( e, h_3 = h_4, q \).

\[ \text{Figure 10 Determination of the Collision Free Path of Case (2.2).} \]

Computation of a free path for case (3)

In this case link \( a_2 \) can interfere with the obstacle during a specified motion (see Case (3), Part I). There are six subcases which must be considered.

Case (3.1): The initial position \( p \) and the target position \( q \) belong to two different zones or point \( q \) lies in the shadow of circle \( C_{\infty}' \). Point \( q' \), an intersection of the two tangent lines \( L_0' \) and \( L_n' \), is chosen to be an intermediate point (see Fig. 11(a)). These two lines pass through points \( p \) and \( q \) respectively.

Case (3.2): Obstacle \( C_{\infty} \) lies in the path of the motion. Point \( q' \) (see Fig. 11(b)) is selected to be an intermediate point which is the intersection of the two tangent lines \( L_0' \) and \( L_n' \).

Case (3.3): Point \( q \) lies in the shadow of the virtual obstacle \( C_{\infty}' \). Point \( q' \) (see Fig. 11(c)) is chosen to be an intermediate point so that the robot can move from \( p \) to \( q \) through point \( q' \). Point \( q' \) is the intersection of the two tangent lines \( L_0' \) and \( L_n' \).

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1 Line \( L_0' \) (see Fig 9) is a tangent line to the virtual obstacle \( C_{\infty}' \) with the tangency between points \( O' \) and \( O' \), and is perpendicular to line \( O'Q_0' \).

2 \( L_0' \) is tangent to the virtual obstacle \( C_{\infty}' \) at point \( e' \) and intersects circle \( C_{\infty}' \) at \( B' \) where \( e'B' = a_3 \) (see Case (2), Part I).

3 Point \( e \) is an intersection of line \( L_0' \) and the circle \( C_{\infty}' \), which is closest to the virtual obstacle \( C_{\infty}' \). Line \( L_0' \) is drawn from \( O' \) and tangent to \( C_{\infty}' \) on the right side.
when the initial and final configurations are in zone (I) (see Fig. 12), point \( B' \) is the best choice for an intermediate point. Then the determination of the rectilinear motion from \( B' \) to \( q_i \) is identical to Case (3.1) through Case (3.3). The second choice is point \( h_i \), which is the middle point of line segment \( B_i h_i \). This can be chosen to be an intermediate point when point \( B' \) lies inside the obstacle \( C_{ob} \).

If the initial and final configurations belong to two different zones, point \( h_i \), is the best choice for an intermediate point.

![Figure 12 Determination of the Collision Free Path of Case (3.4).](image)

**Case (3.5):** Points \( q_i \) and \( O' \) are on the same side of line \( L_{o'} \). Point \( q'_i \), an intersection of line \( L_{o'} \) and circle \( C_{o'} \), is chosen to be an intermediate point, which is the nearest to the target point \( q_i \) (see Fig. 13). When the robot cannot move from \( q'_i \) directly to point \( q_i \), an intermediate point \( h_i \) on circle \( C_{o'} \) is chosen to generate a free path from \( q'_i \) to \( q_i \), where \( q'_i h_i \) = \( h_i q_i \).

![Figure 13 Determination of the Collision Free Path of Case (3.5).](image)

**Case (3.6):** Line \( L_{o'} \) does not exist. When \( 0 < a_1 = (a_0 + r')^2 \), line \( L_{o'} \) does not exist.

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4 Point \( q'_i \) is an intersection of line \( L_{o'} \) and the circle \( C_{o'} \), which is closest to the virtual obstacle \( C_{ob} \). Line \( L_{o'} \) is drawn from \( O' \) and tangent to \( C_{ob} \) on the left side.
Line $L_4$ is determined (Fig.14), which is tangent to the virtual obstacle $C_{ob'}$ and is perpendicular to line $O'Q_1$ with the tangency between points $O'$ and $Q_1'$. The determination of the intermediate points is identical to those in Cases (3.1) through (3.8).

**Figure 14** Determination of the Collision Free Path of Case (3.6).

**SIMULATION OF THE ALGORITHM OF THE OBSTACLE AVOIDANCE OF A PLANAR 3R ROBOT**

The obstacle avoidance algorithms developed in this paper have been successfully implemented using the Silicon Graphics 4D system in CIMAR Laboratory. The link lengths of the planar 3R robot were selected as follows: $a_3 = 30.0$ units, $a_2 = 15.0$ units, and $a_1 = 10.0$ units.

Figs. 15 and 16 show the results of the rectilinear motion planning for case(2) and case(3) respectively. The algorithm generates a path autonomously and it takes about 1 second of the computer time in the Silicon Graphics 4D system.

**CONCLUSION AND FUTURE WORK**

A time-efficient algorithm concerning the motion planning of a planar 3R robot with an obstacle inside its workspace has been developed. An effective checking procedure without using the stepped-move approach and the swept-volume calculation is applied to enable a fast determination of a free path around the obstacle. Further studies could elaborate this algorithm; for example, multiple obstacles inside the workspace. In this case the RA of a robot configuration can be defined as the common area of the RAs of different obstacles.

An important extension of this algorithm is its applications in articulating the links of some spatial robots to guide the end-effector through horizontal tubes or piping.

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**REFERENCES**