CLOSED-FORM EQUILIBRIUM ANALYSIS OF A PLANAR TENSEGRITY STRUCTURE

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Abstract This paper presents a closed-form analysis of a two-spring planar tensegrity mechanism to determine all possible equilibrium configurations for the device when no external forces or moments are applied. The equilibrium position is determined by identifying the configurations at which the potential energy stored in the two springs is a minimum. A 28th degree polynomial expressed in terms of the length of one of the springs is developed where this polynomial identifies the cases where the change in potential energy with respect to a change in the spring length is zero. A numerical example is presented.

Keywords: Tensegrity, compliance, equilibrium.

1. Introduction

The word *tensegrity* is a combination of the words *tension* and *integrity* (Edmondson, 1987 and Fuller, 1975). Tensegrity structures are spatial structures formed by a combination of rigid elements in compression (struts) and connecting elements that are in tension (ties). No pair of struts touch and the end of each strut is connected to three non-coplanar ties (Yin et al, 2002). The entire configuration stands by itself and maintains its form solely because of the internal arrangement of the struts and ties (Tobie, 1976).

The development of tensegrity structures is relatively new and the works related have only existed for approximately twenty five years. Kenner, 1976, established the relation between the rotation of the top and bottom ties. Tobie, 1976, presented procedures for the generation of tensile structures by physical and graphical means. Yin, 2002, obtained Kenner's results using energy considerations and found the equilibrium position for unloaded tensegrity prisms. Stern, 1999, developed generic design equations to find the lengths of the struts and elastic ties needed to create a desired geometry for a symmetric case. Knight, 2000, addressed the problem of stability of tensegrity structures for the design of deployable antennae.

2. Problem Statement

Figure 1a shows a tensegrity prism with a three sided base and top. Figure 1b shows a four sided parallel prism where the side ties are parallel and Figure 1c shows a four sided tensegrity prisms. Duffy et al, 2000, based on the works of Tobie, 1976, and Kenner, 1976, states that an n sided tensegrity prism can be obtained from the parallel prism by rotating the top of the parallel prism by an angle α given by

$$\alpha = \frac{\pi}{2} - \frac{\pi}{n} \ . \tag{1}$$

This paper focuses on the analysis of a two-strut tensegrity system. In this case n=2 and α=0 which means that the parallel and tensegrity prisms are the same. Figure 2 shows a two strut tensegrity prism.

The objective of this effort is to determine, in closed-form, all possible equilibrium deployed positions of a planar tensegrity system wherein two of the ties are compliant. Figure 3 shows the system which is comprised of two struts (compression members a_{12} and a_{34}), two non-compliant ties (tension members a_{41} and a_{23}), and two elastic tensile members (springs), one connected between points 1 and 3 and one between points 2 and 4. It should be noted in Figure 2 that strut a_{34} passes through a slit cut in strut a_{12} and as such the two struts do not intersect or collide.

The problem statement can be explicitly written as:

given:	a_{12}, a_{34}	lengths of struts,
	a_{23}, a_{41}	lengths of non-compliant ties,
	k_1, L_{01}	spring constant and free length of compliant tie between points 4 and 2,
	k_2, L_{02}	spring constant and free length of compliant tie between points 3 and 1.
find:	L_1	length of spring 1 at equilibrium position,

L_2	length of spring 2 at equilibrium position corresponding to length of spring 1, i.e. L_1.

It should be noted that the problem statement could be formulated in a variety of ways, i.e. a different variable (such as the relative angle between strut a_{34} and tie a_{41}) could have been selected as the generalized parameter for this problem. Attempts at using this alternate approach did not successfully yield a closed form solution for the equilibrium positions.

3. Development of Geometric and Potential Energy Constraint Equations

Figure 3 shows the nomenclature that is used. L_1 and L_2 are the extended lengths of the compliant ties between points 4 and 2 and points 3 and 1. Figure 4 shows the triangle formed by sides a_{34}, a_{23}, and L_1. A cosine law for this triangle can be written as

$$\frac{L_1^{\ 2}}{2}+\frac{a_{34}^{\ 2}}{2}+L_1\,a_{34}\cos\theta_4{}'=\frac{a_{23}^{\ 2}}{2}\ . \tag{1}$$

Solving for $\cos\theta_4'$ yields

$$\cos\theta_4{}'=\frac{a_{23}^{\ 2}-L_1^{\ 2}-a_{34}^{\ 2}}{2\,L_1\,a_{34}}\ . \tag{2}$$

Figure 5 shows the triangle formed by sides a_{41}, a_{12}, and L_1. A cosine law for this triangle can be written as

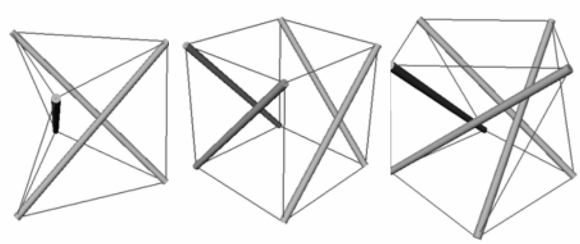
(a) (b) (c)

Figure 1. Tensegrity Structures

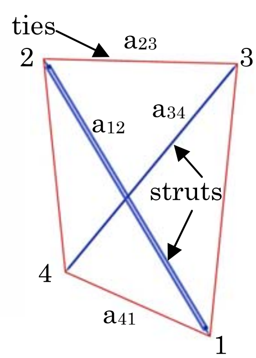

Figure 2. Planar Tensegrity Structure

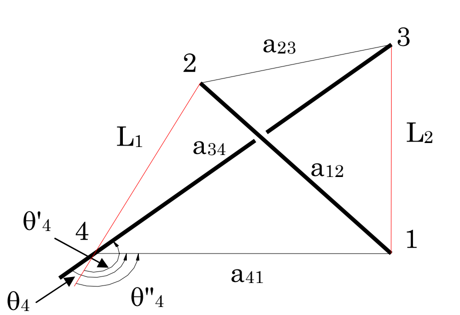

Figure 3. Planar Tensegrity Structure

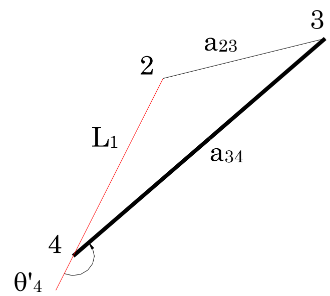

Figure 4. Triangle 4-3-2

$$\frac{L_1^2}{2} + \frac{a_{41}^2}{2} + L_1 a_{41} \cos\theta_4'' = \frac{a_{12}^2}{2} \ . \tag{3}$$

Solving for $\cos\theta_4''$ yields

$$\cos\theta_4'' = \frac{a_{12}^2 - L_1^2 - a_{41}^2}{2 L_1 a_{41}} \ . \tag{4}$$

Figure 6 shows the triangle formed by sides a_{41}, a_{34}, and L_2. A cosine law for this triangle can be written as

$$\frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} \cos\theta_4 = \frac{L_2^2}{2} \ . \tag{5}$$

Solving for $\cos\theta_4$ yields

$$\cos\theta_4 = \frac{L_2^2 - a_{34}^2 - a_{41}^2}{2 a_{34} a_{41}} \ . \tag{6}$$

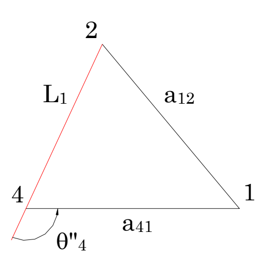

Figure 5. Triangle 4-1-2

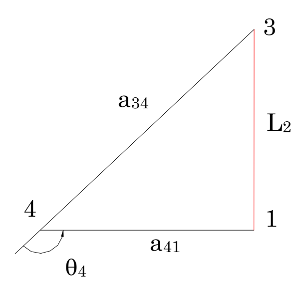

Figure 6. Triangle 4-1-3

From Figure 3 it is apparent that

$$\theta_4 + \theta_4' = \pi + \theta_4'' \tag{7}$$

Equating the cosine of the left and right sides of (7) yields

$$\cos(\theta_4 + \theta_4') = \cos(\pi + \theta_4'') \tag{8}$$

and expanding this equation yields

$$\cos\theta_4 \cos\theta_4' - \sin\theta_4 \sin\theta_4' = - \cos\theta_4'' . \tag{9}$$

Rearranging (9) yields

$$\cos\theta_4 \cos\theta_4' + \cos\theta_4'' = \sin\theta_4 \sin\theta_4' . \tag{10}$$

Squaring both sides of (10) gives

$$(\cos\theta_4)^2 (\cos\theta_4')^2 + 2 \cos\theta_4 \cos\theta_4' \cos\theta_4'' + (\cos\theta_4'')^2 = (\sin\theta_4)^2 (\sin\theta_4')^2 . \tag{11}$$

Substituting for $(\sin\theta_4)^2$ and $(\sin\theta_4')^2$ in terms of $\cos\theta_4$ and $\cos\theta_4'$ gives

$$\begin{aligned}(\cos\theta_4)^2 (\cos\theta_4')^2 + 2 \cos\theta_4 \cos\theta_4' \cos\theta_4'' + (\cos\theta_4'')^2 \\ = (1-\cos^2\theta_4)(1-\cos^2\theta_4') .\end{aligned} \tag{12}$$

Equations (2), (4), and (6) are substituted into (12) to yield a single equation in the parameters L_1 and L_2 which can be written as

$$A L_2^4 + B L_2^2 + C = 0 \tag{13}$$

where

$$A = L_1^2, \quad B = L_1^4 + B_2 L_1^2 + B_0, \quad C = C_2 L_1^2 + C_0 \tag{14}$$

and where

$$\begin{aligned}B_2 &= -(a_{23}^2 + a_{34}^2 + a_{41}^2 + a_{12}^2),\\ B_0 &= (a_{12} - a_{41})(a_{12} + a_{41})(a_{23} - a_{34})(a_{23} + a_{34}) ,\\ C_2 &= (a_{34} - a_{41})(a_{34} + a_{41})(a_{23} - a_{12})(a_{23} + a_{12}) ,\\ C_0 &= (a_{41}a_{23} + a_{34}a_{12})(a_{41}a_{23} - a_{34}a_{12})(a_{41}^2 + a_{23}^2 - a_{12}^2 - a_{34}^2).\end{aligned} \tag{15}$$

The potential energy of the system can be evaluated as

$$U = \frac{1}{2} k_1 (L_1 - L_{01})^2 + \frac{1}{2} k_2 (L_2 - L_{02})^2 . \tag{16}$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the length L_1 equals zero, i.e.

$$\frac{dU}{dL_1} = k_1 (L_1 - L_{01}) + k_2 (L_2 - L_{02}) \frac{dL_2}{dL_1} = 0 . \tag{17}$$

The derivative dL_2/dL_1 can be determined via implicit differentiation from equation (13) as

$$\frac{dL_2}{dL_1} = \frac{-L_1\,[L_2^{\,2}(L_2^{\,2} + 2L_1^{\,2} - a_{23}^{\,2} - a_{41}^{\,2} - a_{34}^{\,2} - a_{12}^{\,2}) + (a_{12}^{\,2} - a_{23}^{\,2})(a_{41}^{\,2} - a_{34}^{\,2})]}{L_2[L_1^{\,2}(L_1^{\,2} + 2L_2^{\,2} - a_{23}^{\,2} - a_{41}^{\,2} - a_{34}^{\,2} - a_{12}^{\,2}) + (a_{12}^{\,2} - a_{41}^{\,2})(a_{23}^{\,2} - a_{34}^{\,2}))]}\,. \quad (18)$$

Substituting (18) into (17) and regrouping gives

$$D\,L_2^5 + E\,L_2^4 + F\,L_2^3 + G\,L_2^2 + H\,L_2 + J = 0 \quad (19)$$

where

$$D = D_1\,L_1\,,\ E = E_1\,L_1\,,\ F = F_3\,L_1^3 + F_2\,L_1^2 + F_1\,L_1,\ G = G_3\,L_1^3 + G_1\,L_1,$$
$$H = H_5\,L_1^5 + H_4\,L_1^4 + H_3\,L_1^3 + H_2\,L_1^2 + H_1\,L_1 + H_0,\ J = J_1\,L_1 \quad (20)$$

and where,

$$D_1 = k_2,\ \ E_1 = -k_2\,L_{02},$$
$$F_3 = 2\,(k_2 - k_1),\ F_2 = 2\,k_1\,L_{01},\ F_1 = -\,k_2\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$
$$G_3 = -2\,k_2\,L_{02},\ G_1 = k_2\,L_{02}\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$
$$H_5 = -\,k_1,\ H_4 = k_1\,L_{01},\ H_3 = k_1\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$
$$H_2 = -\,k_1\,L_{01}\,(a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),$$
$$H_1 = -\,k_1\,(a_{34}^2 - a_{23}^2)\,(a_{41}^2 - a_{12}^2) + k_2\,(a_{34}^2 - a_{41}^2)\,(a_{23}^2 - a_{12}^2),$$
$$H_0 = k_1\,L_{01}\,(a_{34}^2 - a_{23}^2)\,(a_{41}^2 - a_{12}^2),$$
$$J_1 = k_2\,L_{02}\,(a_{34}^2 - a_{41}^2)\,(a_{12}^2 - a_{23}^2)\,. \quad (21)$$

4. Solution of Geometry and Energy Equations

Equations (19) and (13) represent two equations in the two unknowns L_1 and L_2. These equations can be solved by using Sylvester's variable elimination procedure by multiplying equation (13) by L_2, L_2^2, L_2^3, and L_2^4 and equation (13) by L_2, L_2^2, L_2^3 to yield a total of nine equations that can be written in matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & D & E & F & G & H & J \\ 0 & 0 & 0 & 0 & A & 0 & B & 0 & C \\ 0 & 0 & 0 & A & 0 & B & 0 & C & 0 \\ 0 & 0 & D & E & F & G & H & J & 0 \\ 0 & 0 & A & 0 & B & 0 & C & 0 & 0 \\ 0 & D & E & F & G & H & J & 0 & 0 \\ 0 & A & 0 & B & 0 & C & 0 & 0 & 0 \\ D & E & F & G & H & J & 0 & 0 & 0 \\ A & 0 & B & 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_2^8 \\ L_2^7 \\ L_2^6 \\ L_2^5 \\ L_2^4 \\ L_2^3 \\ L_2^2 \\ L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (22)$$

A solution to this set of equations can only occur if the determinant of the 9×9 coefficient matrix is equal to zero. Expansion of this determinant yields a 30^{th} degree polynomial in the variable L_1. When the determinant was expanded symbolically, it was seen that the two lowest order coefficients were identically zero. Thus the polynomial can be divided throughout by L_1^2 to yield a 28^{th} degree polynomial. The coefficients of the 28^{th} degree polynomial were obtained symbolically in terms of the given quantities, but are not presented here due to their length and complexity.

Values for L_2 that correspond to each value of L_1 can be determined by first solving (13) for four possible values of L_2. Only one of these four values also satisfies equation (19).

5. Numerical Example

The following parameters were selected to show the results of a numerical example:

strut lengths:	a_{12} = 3 in.	a_{34} = 3.5 in.
non-compliant tie lengths:	a_{41} = 4 in.	a_{23} = 2 in.
spring 1 free length & spring constant:	L_{01} = 0.5 in.	k_1 = 4 lbf/in.
spring 2 free length & spring constant:	L_{02} = 1 in.	k_2 = 2.5 lbf/in.

Eight real and twenty complex roots were obtained for L_1. The real values for L_1 and the corresponding values of L_2 are shown in Table 1.

Table 1. Eight Real Solutions

Case	1	2	3	4	5	6	7	8
L1, in.	-5.485	-5.322	-1.741	-1.576	1.628	1.863	5.129	5.476
L2, in.	2.333	-2.901	-1.495	1.870	1.709	-1.354	-3.288	2.394

The values of L_1 and L_2 listed in Table 1 satisfy the geometric constraints defined by equation (13) and the energy condition defined by equation (19). Each of these eight cases was analyzed to determine whether it represented a minimum or maximum potential energy condition and cases 3, 4, 5, and 6 were found to be minimum states. A free body analysis of struts a_{12} and a_{34} was performed to show that these bodies were indeed in equilibrium.

Efforts were undertaken to obtain a numerical example that would yield more than eight real roots. One example of each type of Grashof and non-Grashof mechanism was analyzed, yet for all these cases eight real roots were found. The complex values of L_1 were analyzed and corresponding complex values of L_2 were determined. In every case it was possible to obtain complex pairs of L_1 and L_2 that satisfied the geometric constraint equation, (13), and the derivative of potential energy equation (19). Thus no extraneous roots were introduced during the variable elimination procedure.

6. Conclusions

This paper presents a technique to obtain all possible equilibrium positions of a planar tensegrity system that incorporates two compliant members. The approach of satisfying geometric constraints while simultaneously finding positions where the derivative of the total potential energy with respect to the generalized coordinate equaled zero resulted in a 28^{th} degree polynomial in a single variable. Although the resulting polynomial was of higher degree than anticipated, an analysis of the real and complex solutions indicates that no extraneous solutions were introduced during the variable elimination procedure.

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