The Optimum Quality Index for a Spatial Redundant 8-8 In-Parallel Manipulator

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Abstract

The quality index for a redundant 4-4 in-parallel manipulator with a square platform and a square base was obtained in [1]. Following this, the quality index for a redundant 4-8 manipulator with a square platform and an octagonal base was determined in [2]. In this paper the optimal quality index for a redundant 8-8 manipulator is determined. The device has an octagonal platform and a similar octagonal base connected by eight legs. The quality index is defined as a dimensionless ratio which takes a maximum value of 1 at a central symmetrical configuration that is shown to correspond to the maximum value of the square root of the determinant of the product of the manipulator Jacobian by its transpose. The Jacobian matrix is none other than the normalized coordinates of the eight leg lines. It is shown that the quality index can be used as a constructive measure of not only acceptable and optimum design proportions but also an acceptable operating workspace (in the static stability sense). The analysis of the redundant 8-8 manipulator described here can be used to model and design a self-deployable space structure that has a pair of flexible octagonal antenna platforms in the base and top platform.

1. Introduction

The quality index for redundant manipulators has been defined in [1] for a redundant 4-4 in-parallel manipulator (see Fig. 1) by the dimensionless ratio

$$\lambda = \sqrt{\frac{\det \mathbf{J}\mathbf{J}^{\mathbf{T}}}{\det \mathbf{J_m}\mathbf{J_m^T}}} \qquad (1)$$

where **J** is the six-by-eight Jacobian matrix of the normalized coordinates of the eight leg lines. It takes a maximum value of λ=1 at a central symmetrical configuration that is shown to correspond to the maximum value of the square root of the determinant of the product of the manipulator Jacobian by its transpose (i.e. $\sqrt{\det \mathbf{J}\mathbf{J}^{\mathbf{T}}} = \sqrt{\det \mathbf{J_m}\mathbf{J_m^T}}$). When the manipulator is actuated so that the moving platform departs from its central configuration, the determinant always diminishes and it becomes zero when a special configuration is reached (The platform then gains one or more uncontrollable freedoms). Following this a redundant 4-8 in-parallel manipulator with an octagonal base is studied in [2].

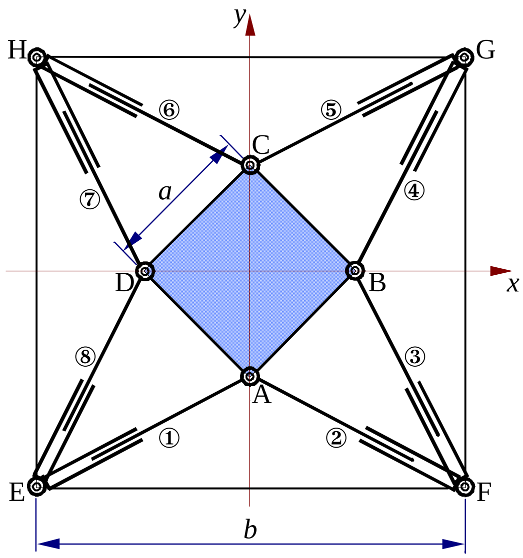


Fig. 1 Plan view of a redundant 4-4 in-parallel manipulator

The quality index was defined initially for a planar 3-3 in-parallel device by the dimensionless ratio [3]

$$\lambda = \frac{|\det \mathbf{J}|}{|\det \mathbf{J_m}|} \tag{2}$$

where $\mathbf{J}$ is the three-by-three Jacobian matrix of the normalized coordinates of three leg lines. After that it was defined for an octahedral in-parallel manipulator [4], 3-6, and 6-6 in-parallel devices [5]. For these cases $\mathbf{J}$ is the six-by-six matrix of the normalized coordinates of the six leg lines.

It has been shown in [4] that by using the Grassmann-Cayley algebra [6], for a general octahedron, when the leg lengths are not normalized, $\det \mathbf{J}$ has dimension of $(\text{volume})^3$ and it is directly related to the products of volumes of tetrahedra which form the octahedron. In this way $\det\ \mathbf{J}$ has geometrical meaning. Similarly, $\sqrt{\det \mathbf{J}\mathbf{J}^{\mathbf{T}}}$ has geometrical meaning because by the Cauchy-Binet theorem, $\det \mathbf{J}\mathbf{J}^{\mathbf{T}} = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_m^2$, has geometrical meaning. Here, each $\Delta_i\,(1 \le i \le m = \binom{n}{6})$ is simply the determinant of the 6×6 submatrices of $\mathbf{J}$ which is a 6×n matrix. Clearly when n = 6, (1) reduces to (2).

There are several advantages by having redundancy in parallel manipulators as proposed by Merlet [7], and Dasgupta and Mruthyunjaya [8]. Redundancy in actuation can be used to increase the reliability of parallel manipulators, eliminate certain type of singularities, and determine the unique position of the platform.

In this paper a redundant 8-8 in-parallel manipulator is studied. The device has an octagonal platform and a similar octagonal base connected by eight legs as shown in Fig. 2. Both the octagonal platform and the octagonal base are formed by separating the connection points from each vertex of a square by a small distance. First, this paper will be focused on obtaining the maximum value of the square root of the determinant of the product of the manipulator Jacobian by its transpose. Then the quality index of the 8-8 manipulator will be determined by employing this maximum value. Finally, by using the quality index, variable motions are investigated for which a moving platform rotates about a central axis or moves parallel to the base.

2. The determination of $\sqrt{\det \mathbf{J_m}\mathbf{J_m^T}}$ for

a redundant 8-8 in-parallel manipulator

The plan view of a redundant 8-8 parallel manipulator is shown in Fig. 2, which is derived simply by separating the double ball-and-socket joints in the base and top platform of a 4-4 manipulator shown in Fig. 1. The device has eight legs connecting an octagonal platform and a similar octagonal base. The octagonal top platform is formed by 4 pairs of joints A_1 and A_2, B_1 and B_2, C_1 and C_2, D_1 and D_2. Each pair of joints is separated from a vertex of a square of side a by a distance αa for which $0 \le \alpha \le \frac{1}{2}$. Similarly, the octagonal base is formed by 4 pairs of joints E_D and E_A, F_A and F_B, G_B and G_C, H_C and H_D, and each of them is separated from a vertex of a square of side b by a distance βb for which $0 \le \beta \le \frac{1}{2}$. The moving platform is parallel to the base and their distance apart is h. This manipulator is said to be redundant since the platform and the base are connected by eight actuated legs.

Firstly, it is necessary to determine the Plücker line coordinates of the eight legs of the platform. The Plücker coordinates for the line joining the points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) were expressed by Grassmann by the six 2×2 determinants of the array

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \end{bmatrix} \tag{3}$$

where the direction ratios of the line are

$$L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}, N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix} \tag{4}$$

and the moments of the line segment about the three coordinate axes are

$$P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{5}$$

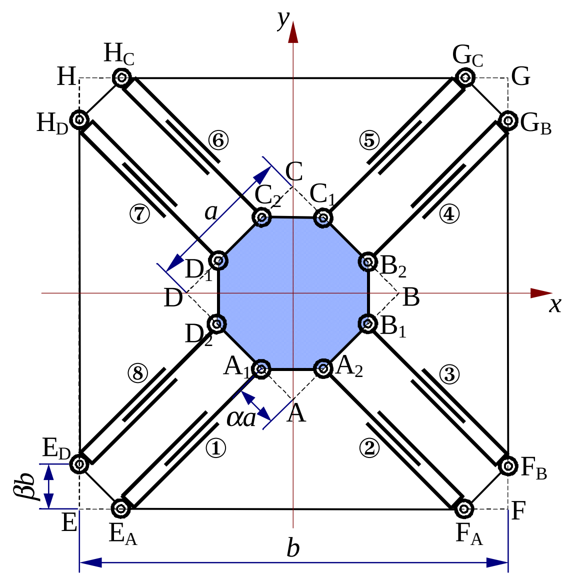


Fig. 2 Plan view of a redundant 8-8 in-parallel manipulator

A fixed coordinate system is placed at the center of the octagonal base as shown in Fig. 2. From the configuration shown in Fig. 2, the coordinates of the points A_1, A_2, B_1, B_2, C_1, C_2, D_1, and D_2 on the platform are

$$\begin{aligned}
&A_1\left(-d_0 \quad d_0 - \frac{\sqrt{2}a}{2} \quad h\right), A_2\left(d_0 \quad d_0 - \frac{\sqrt{2}a}{2} \quad h\right), \\
&B_1\left(\frac{\sqrt{2}a}{2} - d_0 \quad -d_0 \quad h\right), B_2\left(\frac{\sqrt{2}a}{2} - d_0 \quad d_0 \quad h\right), \\
&C_1\left(d_0 \quad \frac{\sqrt{2}a}{2} - d_0 \quad h\right), C_2\left(-d_0 \quad \frac{\sqrt{2}a}{2} - d_0 \quad h\right), \\
&D_1\left(d_0 - \frac{\sqrt{2}a}{2} \quad d_0 \quad h\right), D_2\left(d_0 - \frac{\sqrt{2}a}{2} \quad -d_0 \quad h\right)
\end{aligned} \tag{6}$$

where

$$d_0 = \frac{\sqrt{2}a}{2}.$$

The coordinates of the points E_A, F_A, F_B, G_B,

G_C, H_C, H_D, and E_D on the base are

$$\begin{aligned} &E_A\begin{pmatrix}-d_1 & -\frac{b}{2} & 0\end{pmatrix}, F_A\begin{pmatrix}d_1 & -\frac{b}{2} & 0\end{pmatrix},\\ &F_B\begin{pmatrix}\frac{b}{2} & -d_1 & 0\end{pmatrix}, G_B\begin{pmatrix}\frac{b}{2} & d_1 & 0\end{pmatrix},\\ &G_C\begin{pmatrix}d_1 & \frac{b}{2} & 0\end{pmatrix}, H_C\begin{pmatrix}-d_1 & \frac{b}{2} & 0\end{pmatrix},\\ &H_D\begin{pmatrix}-\frac{b}{2} & d_1 & 0\end{pmatrix}, E_D\begin{pmatrix}-\frac{b}{2} & -d_1 & 0\end{pmatrix}\end{aligned} \quad (7)$$

where

$$d_1 = \frac{(1-2\beta)b}{2}.$$

Counting the 2×2 determinants of the various arrays of the joins of the pairs of points A_1E_A, A_2F_A, …, D_2E_D yields the normalized Jacobian matrix of the eight lines now all reduced to unit length which can be expressed in the form

$$\mathbf{J} = \frac{1}{l}[\$_1 \quad \$_2 \quad \$_3 \quad \$_4 \quad \$_5 \quad \$_6 \quad \$_7 \quad \$_8] \quad (8)$$

where

$$\begin{aligned} \$_1 &= \begin{bmatrix} d_1-d_0 & d_0-d_2 & h & -\frac{bh}{2} & d_1h & d_3\end{bmatrix}^{\mathrm{T}},\\ \$_2 &= \begin{bmatrix} d_0-d_1 & d_0-d_2 & h & -\frac{bh}{2} & -d_1h & -d_3\end{bmatrix}^{\mathrm{T}},\\ \$_3 &= \begin{bmatrix} d_2-d_0 & d_1-d_0 & h & -d_1h & -\frac{bh}{2} & d_3\end{bmatrix}^{\mathrm{T}},\\ \$_4 &= \begin{bmatrix} d_2-d_0 & d_0-d_1 & h & d_1h & -\frac{bh}{2} & -d_3\end{bmatrix}^{\mathrm{T}},\\ \$_5 &= \begin{bmatrix} d_0-d_1 & d_2-d_0 & h & \frac{bh}{2} & -d_1h & d_3\end{bmatrix}^{\mathrm{T}},\\ \$_6 &= \begin{bmatrix} d_1-d_0 & d_2-d_0 & h & \frac{bh}{2} & d_1h & -d_3\end{bmatrix}^{\mathrm{T}},\\ \$_7 &= \begin{bmatrix} d_0-d_2 & d_0-d_1 & h & d_1h & \frac{bh}{2} & d_3\end{bmatrix}^{\mathrm{T}},\\ \$_8 &= \begin{bmatrix} d_0-d_2 & d_1-d_0 & h & -d_1h & \frac{bh}{2} & -d_3\end{bmatrix}^{\mathrm{T}} \end{aligned}$$

where

$$\begin{aligned} d_2 &= \frac{\sqrt{2}a-b}{2},\\ d_3 &= \frac{\sqrt{2}}{4}(2\alpha\beta-2\alpha-2\beta+1)ab. \end{aligned}$$

Here, the device is in a symmetrical position so that the normalization divisor is the same for each leg, namely the leg length $l = A_1E_A = A_2F_A = B_1F_B = B_2G_B = C_1G_C = C_2H_C = D_1H_D = D_2E_D$, and for every leg

$$\begin{aligned} l &= \sqrt{L^2+M^2+N^2}\\ &= \{\frac{1}{2}[(2\alpha^2-2\alpha+1)a^2+\sqrt{2}(2\alpha\beta-1)ab+ \\ &(2\beta^2-2\beta+1)b^2+2h^2]\}^{\frac{1}{2}}. \end{aligned} \quad (9)$$

Using Eq. (8), the determinant of the product $\mathbf{JJ^T}$ can be expressed in the form

$$\det \mathbf{JJ^T} = \frac{\begin{vmatrix} d_4+d_5 & 0 & 0 & 0 & d_5h & 0\\ 0 & d_4+d_5 & 0 & -d_5h & 0 & 0\\ 0 & 0 & 8h^2 & 0 & 0 & 0\\ 0 & -d_5h & 0 & d_6 & 0 & 0\\ d_5h & 0 & 0 & 0 & d_6 & 0\\ 0 & 0 & 0 & 0 & 0 & 8d_3^2 \end{vmatrix}}{l^{12}} \quad (10)$$

where

$$\begin{aligned} d_4 &= 2(2\alpha^2-2\alpha+1)a^2+\sqrt{2}(2\alpha\beta-1)ab,\\ d_5 &= 2(2\beta^2-2\beta+1)b^2+\sqrt{2}(2\alpha\beta-1)ab,\\ d_6 &= (b^2+4d_1^2)h^2. \end{aligned}$$

Expanding (10) and using (9), then extracting the square root yields

$$\sqrt{\det \mathbf{JJ^T}} = \frac{256\sqrt{2}\left|2\alpha\beta-2\alpha-2\beta+1\right|^3 a^3b^3h^3}{(d_4+d_5+4h^2)^3}. \quad (11)$$

Here, it is important to note that

$\sqrt{\det \mathbf{JJ^T}} = 0$ (with a, b, $h \neq 0$) when

$$f(\alpha,\beta) = 2\alpha\beta - 2\alpha - 2\beta + 1 = 0. \qquad (12)$$

The plot of Eq. (12) is illustrated in Fig. 3. Four cases that satisfy the relation $f(\alpha,\beta)=0$ are also drawn in Fig. 3. These 8-8 manipulators with α and β satisfy the relation $f(\alpha,\beta)=0$ have finite mobility when all eight actuators are locked. This type of result is also true for 6-6 non-redundant manipulators in [5] and was discussed in McAree and Hunt [9]. Finally, it is interesting to note that the two end points of the curve $f(\alpha,\beta)=0$, i.e. α=0, $\beta=\frac{1}{2}$, and $\alpha=\frac{1}{2}$, β=0, are two cases for which the manipulator has degenerated.

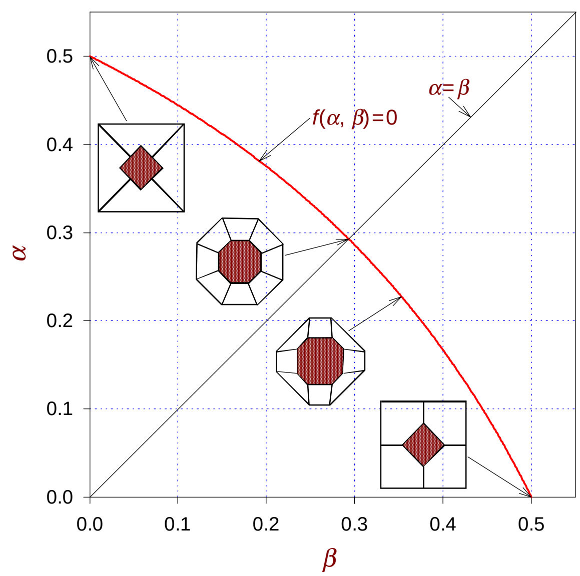


Fig. 3 Plots of $f(\alpha,\beta) = 2\alpha\beta - 2\alpha - 2\beta + 1 = 0$

Assuming the top platform size a is given and taking the partial derivative of (11) with respect to h and b respectively and equating to zero yields

$$\frac{768\sqrt{2}|d_7|^3 a^3 b^3 h^2 (d_4 + d_5 - 4h^2)}{(d_4 + d_5 + 4h^2)^4} = 0 \qquad (13)$$

and

$$\frac{768\sqrt{2}|d_7|^3 a^3 b^2 h^3 (d_4 - d_5 + 4h^2)}{(d_4 + d_5 + 4h^2)^4} = 0 \qquad (14)$$

where

$$d_7 = f(\alpha,\beta) = 2\alpha\beta - 2\alpha - 2\beta + 1$$

When a, b, and h are not equal to zero and $2\alpha\beta - 2\alpha - 2\beta + 1 \neq 0$, Eqs. (13) and (14) yield

$$d_4 + d_5 - 4h^2 = 2[(2\alpha^2 - 2\alpha + 1)a^2 + \sqrt{2}(2\alpha\beta - 1)ab + (2\beta^2 - 2\beta + 1)b^2 - 2h^2] = 0, \qquad (15)$$

$$d_4 - d_5 + 4h^2 = 2[(2\alpha^2 - 2\alpha + 1)a^2 - (2\beta^2 - 2\beta + 1)b^2 + 2h^2] = 0. \qquad (16)$$

Adding above two equations gives

$$2d_4 = \left[2(2\alpha^2 - 2\alpha + 1)a^2 + \sqrt{2}(2\alpha\beta - 1)ab\right] = 0. \qquad (17)$$

Thus,

$$b = \frac{\sqrt{2}a(2\alpha^2 - 2\alpha + 1)}{1 - 2\alpha\beta}. \qquad (18)$$

Substituting (18) into Eq. (16), then solving for h we obtain two solutions, here we only take the positive solution (the negative solution is simply a reflection through the base)

$$h = \frac{\sqrt{2(2\alpha^2 - 2\alpha + 1)|d_6|a}}{2(1 - 2\alpha\beta)}. \qquad (19)$$

Finally substituting (18) and (19) into (11) we obtain

$$\sqrt{\det \mathbf{J_m J_m^T}} = \left(\sqrt{\det \mathbf{JJ^T}}\right)_{max} = 4\sqrt{2}(2\alpha^2 - 2\alpha + 1)^{\frac{3}{2}} a^3 \qquad (20)$$

where $\mathbf{J_m}$ denotes the Jacobian matrix for the configuration at which $\sqrt{\det \mathbf{JJ^T}}$ has a maximum value. It is interesting to note that this value is dependent on the value of α only and not on the value of β. A similar result has already been obtained for the 6-6 parallel manipulator in [5]. When α = 0, $\sqrt{\det \mathbf{J_m J_m^T}}$ has a maximum value $4\sqrt{2}a^3$, and when $\alpha = \frac{1}{2}$, $\sqrt{\det \mathbf{J_m J_m^T}}$ becomes minimal, $2a^3$. Fig. 4 plots the variation of $\sqrt{\det \mathbf{J_m J_m^T}}$ with respect to α for a = 1 and shows that $\sqrt{\det \mathbf{J_m J_m^T}}$ decreases as α increases.

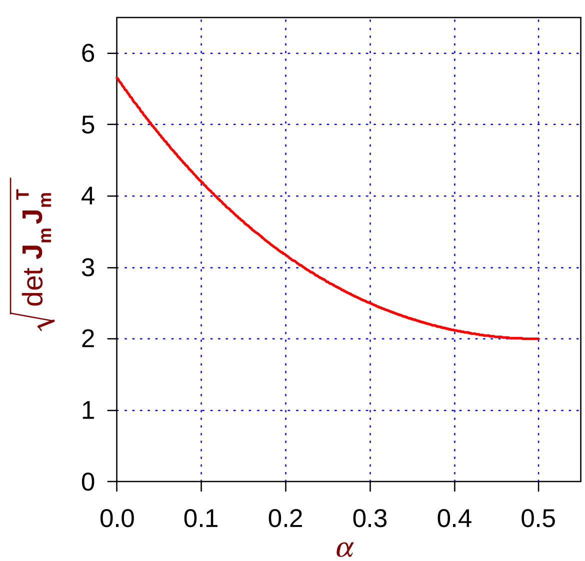


Fig. 4 Plots of $\sqrt{\det \mathbf{J_m J_m^T}}$ vs. α with a = 1

3. Implementation of the quality index

From (1) and (20), the quality index for the redundant 8-8 parallel manipulator will be

$$\lambda = \frac{\sqrt{\det \mathbf{JJ^T}}}{4\sqrt{2}(2\alpha^2 - 2\alpha + 1)^{\frac{3}{2}} a^3}. \qquad (21)$$

The variation of the quality index will now be investigated for a number of simple motions of the top platform. Here, we will consider the case $\alpha = \beta$ and the top side a = 1. The base side b of the manipulator is then calculated using (18).

Firstly, consider a pure translation of the platform from the initial position along the z axis while remaining parallel to the base. For such movement, from (11) and (21), the quality index is given by

$$\lambda = \frac{64\left|2\alpha\beta - 2\alpha - 2\beta + 1\right|^3 b^3 h^3}{(d_4 + d_5 + 4h^2)^3 (2\alpha^2 - 2\alpha + 1)^{\frac{3}{2}}}. \qquad (22)$$

With $\alpha = \beta$, a = 1, and (18), d_4 becomes zero and d_5 reduces to

$$d_5 = \frac{2(2\alpha^2 - 2\alpha + 1)(1 - 4\alpha + 2\alpha^2)^2}{(2\alpha^2 - 1)^2}$$

$$= \frac{2d_8^2}{(2\alpha^2 - 1)^2}$$

where

$$d_8 = \sqrt{2\alpha^2 - 2\alpha + 1}\left|1 - 4\alpha + 2\alpha^2\right|$$

Thus, the quality index becomes

$$\lambda = \frac{16\sqrt{2}d_8^3 (1 - 2\alpha^2)^3 h^3}{\left[d_8^2 + 2(1 - 2\alpha^2)^2 h^2\right]^3} \qquad (23)$$

and is plotted in Fig. 5 as a function of h for different values of α. The variation of quality index with α (= β) and h is interesting. When $\alpha(=\beta)=0$ we obtain the 4-4 platform shown by Fig. 1, when $\alpha(=\beta)=\frac{1}{2}$ we obtain another 4-4 platform shown by Fig. 6, when $\alpha(=\beta)$ is the solution of (12) for which $\alpha=\beta=1-\frac{1}{\sqrt{2}}$, the platform is degenerate and h=0 (see also Fig. 3). It follows that as α increases from 0 to $1-\frac{1}{\sqrt{2}}$ the value of h for λ=1 decreases whereas when α increases from $1-\frac{1}{\sqrt{2}}$ to $\frac{1}{2}$ the value of h for λ=1 increases. Each value of α designates the distance between the separation points in the top platform and base and is our initial design parameter. Clearly, $\alpha\,(=\beta) = 0$ is the best overall design.

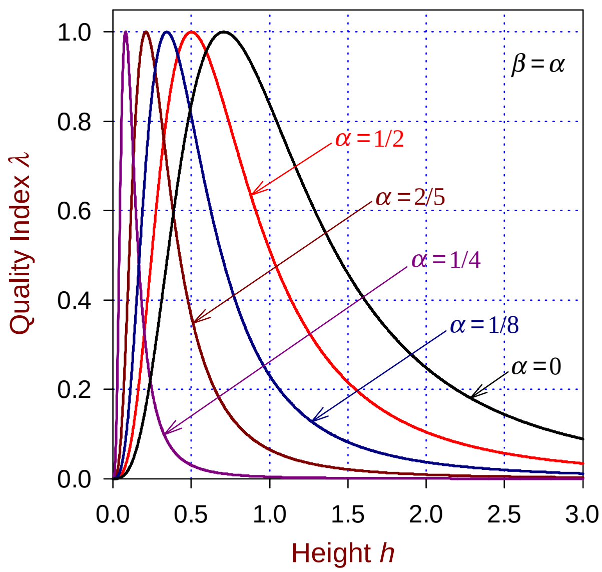


Fig. 5 Quality index for platform vertical movement

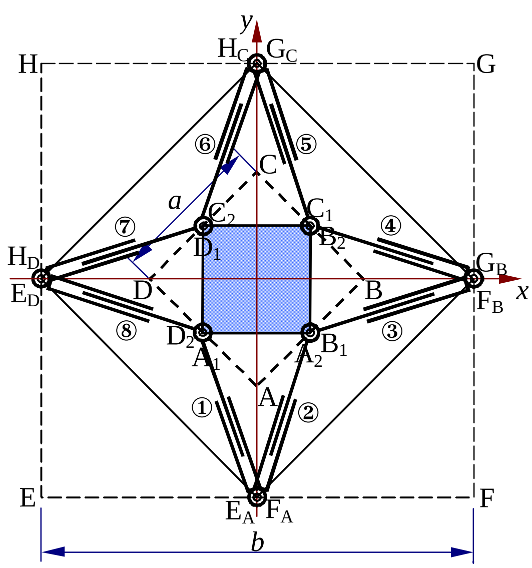
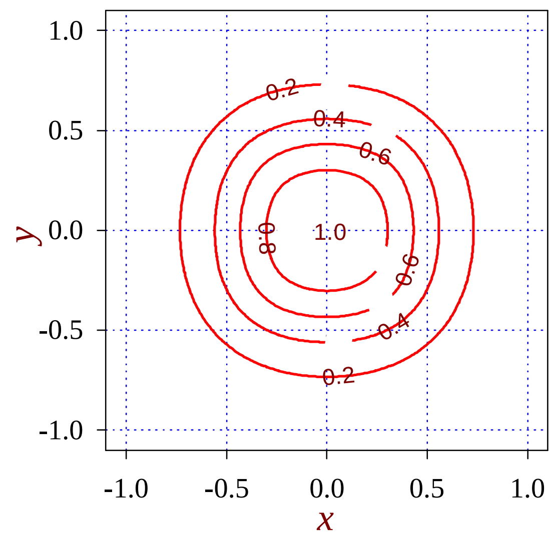
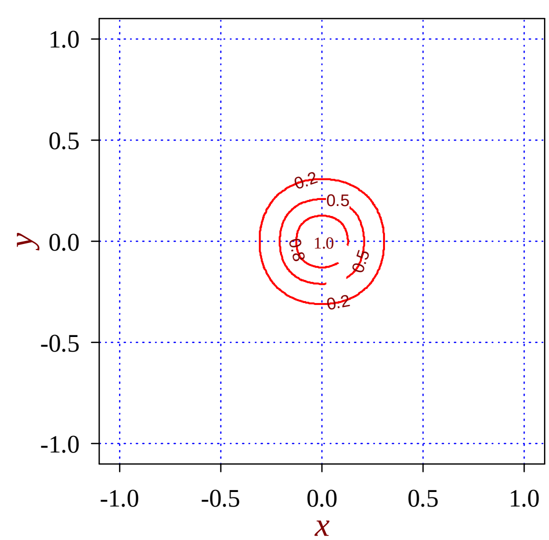


Fig. 6 Reduction of the size of the redundant 8-8 manipulator when $\alpha\,(=\beta) = \dfrac{1}{2}$

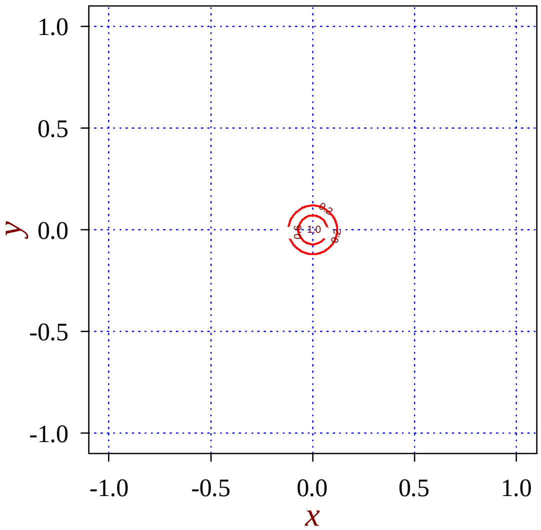
Fig. 7 (a)-(e) illustrate the contours of quality index for various values of α as the platform of the redundant 8-8 parallel manipulator is translated away from the central location while remaining parallel to the base. The values of $h = h_{\mathrm{m}}$ for a maximum quality index for each value $\alpha(=\beta)$ is obtained from (19). Clearly cases (b), (c), and (d) are unacceptable designs.



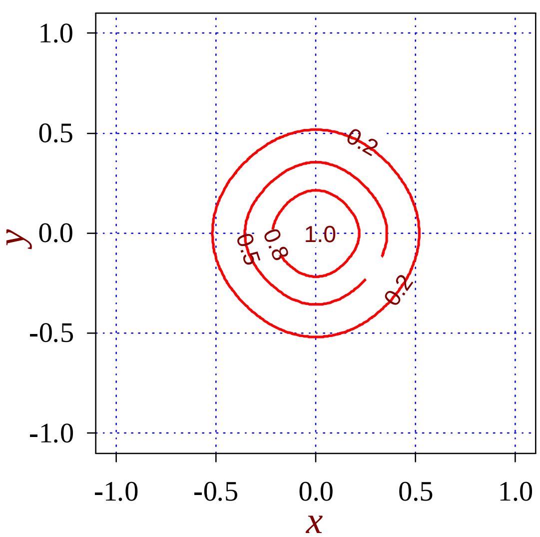
(a) $\alpha\,(=\beta) = \dfrac{1}{2}$ at $h_m = \dfrac{1}{2}$



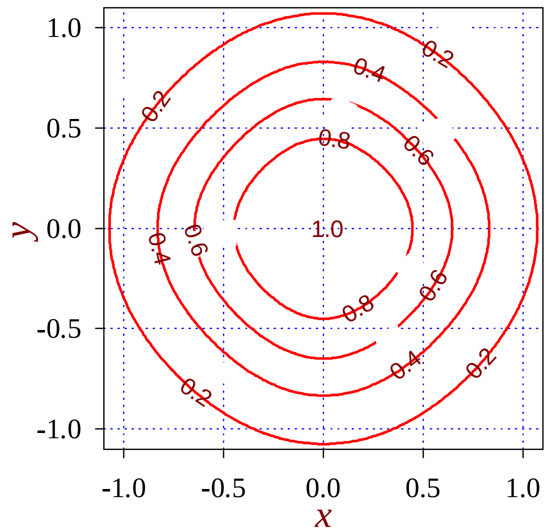
(b) $\alpha\,(=\beta) = \dfrac{2}{5}$ at $h_m = \dfrac{7\sqrt{26}}{170} \cong 0.21$



(c) $\alpha\,(=\beta) = \frac{1}{4}$ at $h_m = \frac{\sqrt{5}}{28} \cong 0.08$



(d) $\alpha\,(=\beta) = \frac{1}{8}$ at $h_m = \frac{85}{248} \cong 0.34$



(e) $\alpha\,(=\beta) = 0$ at $h_m = \frac{\sqrt{2}}{2} \cong 0.71$

Fig. 7 Quality index for platform parallel translation with different values of $\alpha\,(=\beta)$

Fig. 8 illustrates the variation of the quality index for rotations about x' and y' axes located in the moving platform in its central configuration and which are parallel to the x and y axes located in the base platform respectively. The $x'y'$ and xy planes are parallel with a distance h_m. Rotations about any line in the $x'y'$ plane passing through the intersection point of x' and y' axes are simply linear combination of rotations about the x' and y' axes.

Fig. 9 shows how the quality index varies as the platform is rotated about the vertical axis z through its center. The legs being adjusted in length to keep the platform parallel to the base at a distance h_m. It is shown in the figure that the manipulator has the highest quality index $\lambda = 1$ when $\theta = 0°$, and $\lambda = 0$ (singularity) when $\theta = \pm 90°$.

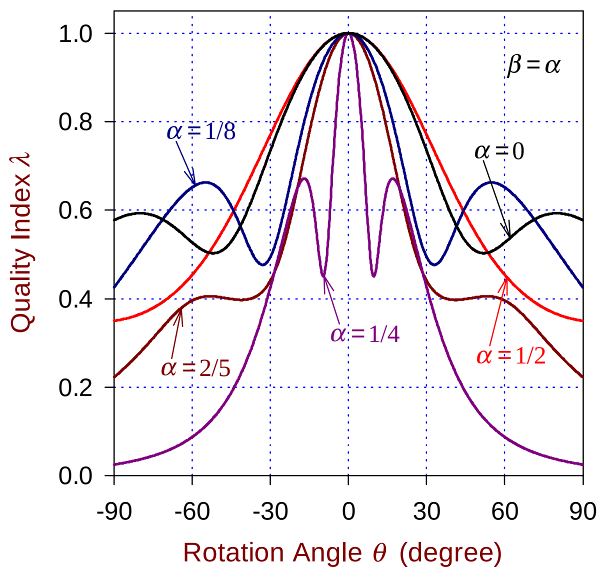


Fig. 8 Quality index for platform rotations about *x'* axis or *y'* axis

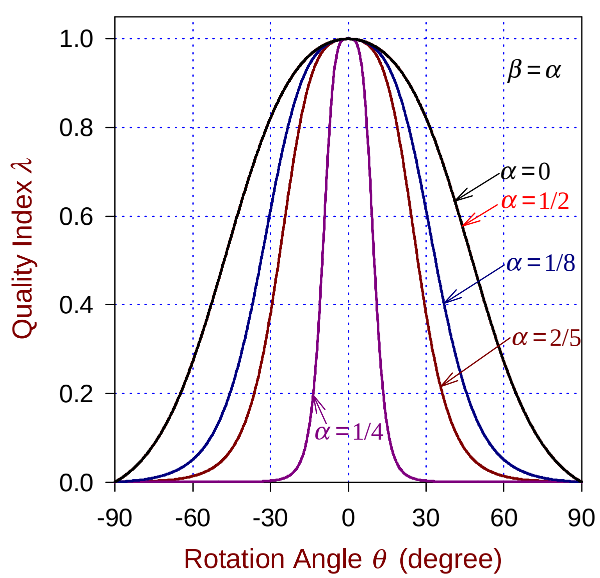


Fig. 9 Quality index for platform rotation about *z* axis

Again, from Figs. 8 and 9 we can clearly see that better designs are obtained as $\alpha\,(=\beta)$ reduces to zero. Hence the best 8-8 parallel manipulator design is obtained when the pair of joints in the base and top platform are as close as possible.

4. Conclusion

In this paper, the maximum value of the square root of the determinant of the product of Jacobian matrix by its transpose, which is used for quality index for a redundant 8-8 manipulator, has been obtained. It is shown in this paper how the quality index varies for various motions of the platform. Comparing those quality index curves with different joint separations, we have obtained useful design information for the 8-8 redundant parallel manipulator, i.e., after avoiding the use of concentric ball-joints so as to overcome the interference problems, we should keep two base joints as close as possible.

5. Acknowledgements

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