

Determination of the Unique Orientation of Two Bodies Connected by a Ball-and-Socket Joint from Four Measured Displacements

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Received March 24, 1997; accepted December 29, 1997

In previous analyses, it has been shown that a maximum of eight relative orientations exist for two bodies connected by a ball-and-socket joint when the distances between three pairs of points are measured. In this paper it is shown how a unique orientation can be determined if the distances between four pairs of points are known. At the outset, the introduction of redundant information is an attractive method for obtaining the unique orientation result. However, this paper demonstrates that high accuracy in the measurement of the linear displacements must be maintained to obtain an accurate result for the orientation. © 1998 John Wiley & Sons, Inc.

1. INTRODUCTION

This work was motivated by the desire to determine the orientation of a trailer relative to a vehicle where the vehicle and trailer are connected by a ball-and-socket joint. It is desired to determine this orientation without making any modification to the ball-and-socket joint (i.e., the hitch), since the hitch is a simple device that withstands the interconnection forces that occur during operation.

For this analysis, the distances between four pairs of points are measured, and a unique relative orientation is to be determined. Figure 1 shows the system configuration where points 1, 2, 3, and 4 are attached to the vehicle and points 1', 2', 3', and 4' are attached to the trailer.

The device depicted in Figure 1 is a parallel mechanism. The analysis of these types of mechanisms has been the focus of much recent research. Stewart¹ introduced his platform mechanism in 1965 as an aircraft simulator. Hunt,^{2,3} Mohamed and Duffy,⁴ Fichter,⁵ Sugimoto,^{6,7} Rees-Jones,⁸ and Kerr⁹ all suggest the use of platform mechanisms. The

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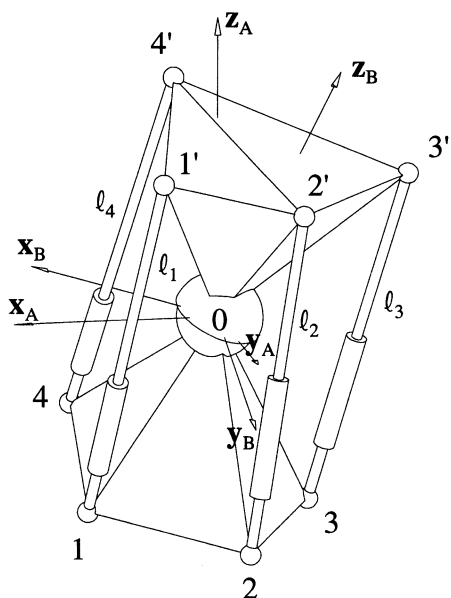


Figure 1. Trailer hitch with four measured displacements.

forward analysis of various parallel mechanisms has been investigated by Griffis and Duffy,¹⁰ Raghavan,¹¹ Wampler,¹² and Innocenti.¹³

Innocenti and Parenti-Castelli¹⁴ and Wohlhart¹⁵ showed that eight configurations exist for a mechanism similar to that shown in Figure 1 when only three connector lengths are given. This paper will focus on the forward analysis problem for the specific geometry shown in Figure 1, and it will be shown that a unique orientation of the top platform can be obtained for the given set of four displacements.

2. PROBLEM STATEMENT

In Figure 1, coordinate systems A and B have been attached to the lower body (the vehicle) and the upper body (the trailer), respectively. The origin of both of these coordinate systems is located at the center of the ball-and-socket joint which connects the two bodies. The coordinates of points 1, 2, 3, and 4 are known in terms of the A coordinate system, whereas the coordinates of points 1', 2', 3', and 4' are known in terms of the B coordinate system. The distances between the points 1-1', 2-2', 3-3', and 4-4', i.e., $l_1, l_2, l_3,$ and l_4 , are also known. The objective is to determine the rotation matrix, ${}^A_B R$, which defines the orientation of the B

coordinate system relative to the A coordinate system. ${}^A_B R$ is a 3×3 matrix whose columns are the unit vectors of the B coordinate system as measured in the A coordinate system.

3. PROBLEM FORMULATION

It is apparent that the distances of the points 1, 2, 3, and 4 from the origin of the coordinate systems, i.e., $r_1, r_2, r_3,$ and r_4 , are known. The distances of the points 1', 2', 3', and 4' from the origin are also known and are named $q_1, q_2, q_3,$ and q_4 .

Two new coordinate systems, labeled C and D, are attached, respectively, to the lower body and upper body. The origin points of both coordinate systems are located at the center of the ball-and-socket joint. The C coordinate system (see Fig. 2) is oriented such that its x axis is along S_1 , its y axis is along $S_1 \times S_2$, and its z axis is along $S_1 \times (S_1 \times S_2)$, where S_1 and S_2 are unit vectors from point 0 through points 1 and 2. The D coordinate system is oriented such that its x axis is along S_1 , its y axis is along $S_1 \times S_2$, and its z axis is along $S_1 \times (S_1 \times S_2)$.

Since the vectors S_1 and S_2 are known in the A coordinate system, the directions of unit vectors along the coordinate axes of the C coordinate system may be calculated in terms of the A coordinate system and are written as ${}^A x_C, {}^A y_C,$ and ${}^A z_C$. The

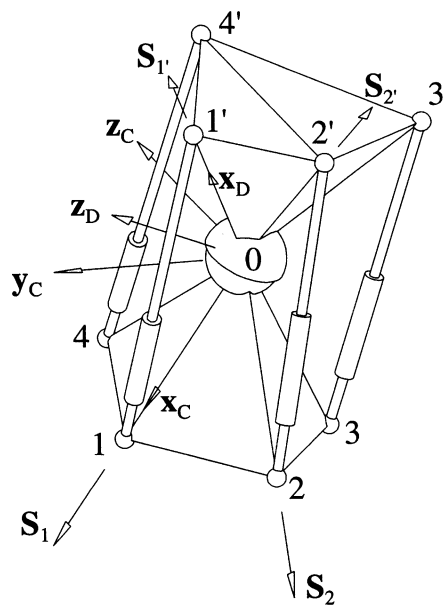


Figure 2. Coordinate systems C and D.

3×3 rotation matrix which transforms a point from the C coordinate system to the A system may be written as

$${}^A_C \mathbf{R} = [{}^A x_C \quad {}^A y_C \quad {}^A z_C]. \quad (1)$$

Similarly, the vectors \mathbf{S}_1 and \mathbf{S}_2 are known in the B coordinate system, and the directions of unit vectors along the coordinate axes of the D coordinate system may be calculated in terms of the B coordinate system and are written as ${}^B x_D$, ${}^B y_D$, and ${}^B z_D$. The 3×3 rotation matrix which transforms a point from the D coordinate system to the B system may be written as

$${}^B_D \mathbf{R} = [{}^B x_D \quad {}^B y_D \quad {}^B z_D]. \quad (2)$$

The rotation matrix which transforms a point from the B coordinate system to the A coordinate system, i.e., ${}^A_B \mathbf{R}$, may be written as

$${}^A_B \mathbf{R} = {}^A_C \mathbf{R} {}^C_D \mathbf{R} {}^B_D \mathbf{R}^T. \quad (3)$$

The matrix ${}^C_D \mathbf{R}$ must still be determined. Once this is accomplished, the solution will be complete.

4. DEFINITION OF ${}^C_D \mathbf{R}$

The D coordinate system may be obtained by initially aligning it with the C coordinate system and then rotating it by an angle θ_1 about the x axis until the y axis is in the plane defined by points 0, 1, and 1', and the scalar product of the y axis with the vector \mathbf{S}_1 is positive. The coordinate system is next rotated by an angle α about its current z axis, which causes the x axis to point along the vector \mathbf{S}_1 . Finally, the coordinate system is rotated about its x axis by an angle θ_2 . The transformation ${}^C_D \mathbf{R}$ may thus be written as

$${}^C_D \mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & -\sin \theta_1 \\ 0 & \sin \theta_1 & \cos \theta_1 \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_2 & -\sin \theta_2 \\ 0 & \sin \theta_2 & \cos \theta_2 \end{bmatrix}. \quad (4)$$

Expanding (4) gives

$${}^C_D \mathbf{R} = \begin{bmatrix} c_\alpha & -s_\alpha c_2 & s_\alpha s_2 \\ s_\alpha c_1 & -s_1 s_2 + c_1 c_2 c_\alpha & -s_1 c_2 - c_1 s_2 c_\alpha \\ s_\alpha s_1 & c_1 s_2 + s_1 c_2 c_\alpha & c_1 c_2 - s_1 s_2 c_\alpha \end{bmatrix} \quad (5)$$

where s_α and c_α represent the sine and cosine of α , and s_i, c_i , ($i = 1, 2$) represent the sine and cosine of θ_i .

The angle α is shown in Figure 3 as the angle between the vectors \mathbf{S}_1 and \mathbf{S}_1' . Since the first rotation of angle θ_1 about the x axis caused the y axis to be in the plane formed by the points 0, 1, and 1', and the scalar product of the y axis with the vector \mathbf{S}_1 is positive, the angle α is constrained to lie in the range of 0 to π . The cosine of α may be determined from a planar cosine law as

$$\cos \alpha = \frac{q_1^2 + r_1^2 - \ell_1^2}{2q_1 r_1} \quad (6)$$

and the value of α is determined as the inverse cosine value in the range of 0 to π .

The matrix ${}^C_D \mathbf{R}$ has been written as a function of the parameters θ_1 and θ_2 . The objective now is to determine the unique values for these parameters that will locate points 1', 2', 3', and 4' such that they are, respectively, a distance of ℓ_1, ℓ_2, ℓ_3 , and ℓ_4 away from points 1, 2, 3, and 4.

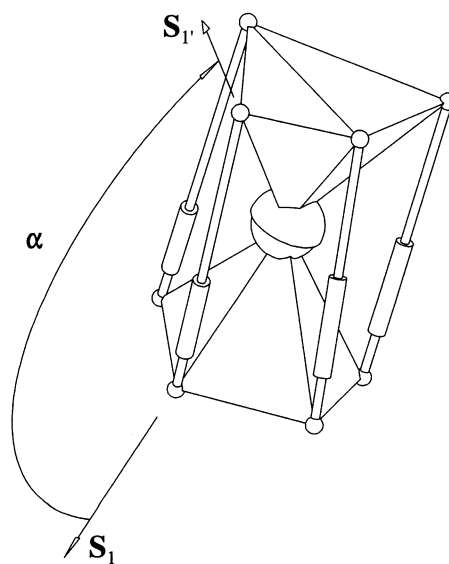


Figure 3. Angle α .

5. DETERMINATION OF θ_1 AND θ_2

The coordinates of points 2', 3', and 4' are known in terms of the B coordinate system and may be written as ${}^B\mathbf{P}_{2'}$, ${}^B\mathbf{P}_{3'}$, and ${}^B\mathbf{P}_{4'}$. The coordinates of these points may be expressed in the D coordinate system as

$${}^D\mathbf{P}_{2'} = {}^D\mathbf{R}^T {}^B\mathbf{P}_{2'}, \quad (7)$$

$${}^D\mathbf{P}_{3'} = {}^D\mathbf{R}^T {}^B\mathbf{P}_{3'}, \quad (8)$$

$${}^D\mathbf{P}_{4'} = {}^D\mathbf{R}^T {}^B\mathbf{P}_{4'}, \quad (9)$$

where the matrix ${}^D\mathbf{R}$ is defined in (2). The coordinates of points 2', 3', and 4' as measured in the D coordinate system may be written as

$${}^D\mathbf{P}_{2'} = \begin{bmatrix} x_{2'D} \\ y_{2'D} \\ z_{2'D} \end{bmatrix}, \quad (10)$$

$${}^D\mathbf{P}_{3'} = \begin{bmatrix} x_{3'D} \\ y_{3'D} \\ z_{3'D} \end{bmatrix}, \quad (11)$$

$${}^D\mathbf{P}_{4'} = \begin{bmatrix} x_{4'D} \\ y_{4'D} \\ z_{4'D} \end{bmatrix}, \quad (12)$$

where the values $x_{2'D}$ through $z_{4'D}$ are known. Similarly, the coordinates of points 2, 3, and 4 are known in terms of the A coordinate system and may be calculated in terms of the C coordinate system as

$${}^C\mathbf{P}_2 = {}^C\mathbf{R}^T {}^A\mathbf{P}_2, \quad (13)$$

$${}^C\mathbf{P}_3 = {}^C\mathbf{R}^T {}^A\mathbf{P}_3, \quad (14)$$

$${}^C\mathbf{P}_4 = {}^C\mathbf{R}^T {}^A\mathbf{P}_4, \quad (15)$$

where ${}^C\mathbf{R}$ is defined in (1). The coordinates of points 2, 3, and 4 as measured in the C coordinate system may be written as

$${}^C\mathbf{P}_2 = \begin{bmatrix} x_{2C} \\ y_{2C} \\ z_{2C} \end{bmatrix}, \quad (16)$$

$${}^C\mathbf{P}_3 = \begin{bmatrix} x_{3C} \\ y_{3C} \\ z_{3C} \end{bmatrix}, \quad (17)$$

$${}^C\mathbf{P}_4 = \begin{bmatrix} x_{4C} \\ y_{4C} \\ z_{4C} \end{bmatrix}, \quad (18)$$

where the values of x_{2C} through z_{4C} are known.

The coordinates of points 2', 3', and 4' may be expressed in the C coordinate system as

$${}^C\mathbf{P}_{2'} = {}^C\mathbf{R}^D {}^D\mathbf{P}_{2'}, \quad (19)$$

$${}^C\mathbf{P}_{3'} = {}^C\mathbf{R}^D {}^D\mathbf{P}_{3'}, \quad (20)$$

$${}^C\mathbf{P}_{4'} = {}^C\mathbf{R}^D {}^D\mathbf{P}_{4'}. \quad (21)$$

Substituting (5) and (10) into (19), (5) and (11) into (20), and (5) and (12) into (21) and expanding gives

$${}^C\mathbf{P}_{2'} = x_{2'D} \begin{bmatrix} c_\alpha \\ s_\alpha c_1 \\ s_\alpha s_1 \end{bmatrix} + y_{2'D} \begin{bmatrix} -s_\alpha c_2 \\ -s_1 s_2 + c_1 c_2 c_\alpha \\ c_1 s_2 + s_1 c_2 c_\alpha \end{bmatrix} + z_{2'D} \begin{bmatrix} s_\alpha s_2 \\ -s_1 c_2 - c_1 s_2 c_\alpha \\ c_1 c_2 - s_1 s_2 c_\alpha \end{bmatrix}, \quad (22)$$

$${}^C\mathbf{P}_{3'} = x_{3'D} \begin{bmatrix} c_\alpha \\ s_\alpha c_1 \\ s_\alpha s_1 \end{bmatrix} + y_{3'D} \begin{bmatrix} -s_\alpha c_2 \\ -s_1 s_2 + c_1 c_2 c_\alpha \\ c_1 s_2 + s_1 c_2 c_\alpha \end{bmatrix} + z_{3'D} \begin{bmatrix} s_\alpha s_2 \\ -s_1 c_2 - c_1 s_2 c_\alpha \\ c_1 c_2 - s_1 s_2 c_\alpha \end{bmatrix}, \quad (23)$$

$${}^C\mathbf{P}_{4'} = x_{4'D} \begin{bmatrix} c_\alpha \\ s_\alpha c_1 \\ s_\alpha s_1 \end{bmatrix} + y_{4'D} \begin{bmatrix} -s_\alpha c_2 \\ -s_1 s_2 + c_1 c_2 c_\alpha \\ c_1 s_2 + s_1 c_2 c_\alpha \end{bmatrix} + z_{4'D} \begin{bmatrix} s_\alpha s_2 \\ -s_1 c_2 - c_1 s_2 c_\alpha \\ c_1 c_2 - s_1 s_2 c_\alpha \end{bmatrix}. \quad (24)$$

The distance between points 2 and 2', 3 and 3', and 4 and 4' must be $l_{2'}$, $l_{3'}$, and $l_{4'}$, respectively. The coordinates of points 2, 2', 3, 3', 4, and 4' have all been expressed in terms of the C coordinate system (see Eqs. (16)–(18) and (22)–(24)), and the distances between these points may be expressed in

the C coordinate system as

$$\begin{aligned} & [x_{2'D}c_\alpha - y_{2'D}s_\alpha c_2 + z_{2D}s_\alpha s_2 - x_{2C}]^2 \\ & + [x_{2'D}s_\alpha c_1 + y_{2'D}(-s_1s_2 + c_1c_2c_\alpha) \\ & + z_{2'D}(-s_1c_2 - c_1s_2c_\alpha) - y_{2C}]^2 \\ & + [x_{2'D}s_\alpha s_1 + y_{2'D}(c_1s_2 + s_1c_2c_\alpha) \\ & + z_{2'D}(c_1c_2 - s_1s_2c_\alpha) - z_{2C}]^2 = \ell_2^2, \quad (25) \end{aligned}$$

$$\begin{aligned} & [x_{3'D}c_\alpha - y_{3'D}s_\alpha c_2 + z_{3'D}s_\alpha s_2 - x_{3C}]^2 \\ & + [x_{3'D}s_\alpha c_1 + y_{3'D}(-s_1s_2 + c_1c_2c_\alpha) \\ & + z_{3'D}(-s_1c_2 - c_1s_2c_\alpha) - y_{3C}]^2 \\ & + [x_{3'D}s_\alpha s_1 + y_{3'D}(c_1s_2 + s_1c_2c_\alpha) \\ & + z_{3'D}(c_1c_2 - s_1s_2c_\alpha) - z_{3C}]^2 = \ell_3^2, \quad (26) \end{aligned}$$

$$\begin{aligned} & [x_{4'D}c_\alpha - y_{4'D}s_\alpha c_2 + z_{4'D}s_\alpha s_2 - x_{4C}]^2 \\ & + [x_{4'D}s_\alpha c_1 + y_{4'D}(-s_1s_2 + c_1c_2c_\alpha) \\ & + z_{4'D}(-s_1c_2 - c_1s_2c_\alpha) - y_{4C}]^2 \\ & + [x_{4'D}s_\alpha s_1 + y_{4'D}(c_1s_2 + s_1c_2c_\alpha) \\ & + z_{4'D}(c_1c_2 - s_1s_2c_\alpha) - z_{4C}]^2 = \ell_4^2. \quad (27) \end{aligned}$$

Note that a constraint equation is not written for the distance between points 1 and 1'. The distance between these points will be equal to ℓ_1 , since the coordinate system transformation which relates the D and C coordinate systems included a rotation by the angle α about an axis that is perpendicular to the plane formed by points 0, 1, and 1'.

Equations (25) through (27) may be expanded and factored into the form

$$\begin{aligned} & (A_i c_1 + B_i s_1 + D_i) c_2 + (E_i c_1 + F_i s_1 + G_i) s_2 \\ & + (H_i c_1 + I_i s_1 + J_i) = 0, \quad i = 1 \cdots 3 \quad (28) \end{aligned}$$

after recognizing that $s_1^2 + c_1^2 = 1$, $s_2^2 + c_2^2 = 1$, and $s_\alpha^2 + c_\alpha^2 = 1$ and where

$$\begin{aligned} A_1 &= -2(z_{j'D}z_{jC} + y_{j'D}y_{jC}c_\alpha), \\ B_i &= 2(z_{j'D}y_{jC} - y_{j'D}z_{jC}c_\alpha), \\ D_i &= 2y_{j'D}x_{jC}s_\alpha, \\ E_i &= 2(z_{j'D}y_{jC}c_\alpha - y_{j'D}z_{jC}), \\ F_i &= 2(z_{j'D}z_{jC}c_\alpha + y_{j'D}y_{jC}), \quad (29) \end{aligned}$$

$$\begin{aligned} G_i &= -2z_{j'D}x_{jC}s_\alpha, \\ H_i &= -2x_{j'D}y_{jC}s_\alpha, \\ I_i &= -2x_{j'D}z_{jC}s_\alpha, \\ J_i &= x_{j'D}^2 + y_{j'D}^2 + z_{j'D}^2 + x_{jC}^2 + y_{jC}^2 \\ & + z_{jC}^2 - 2x_{j'D}x_{jC}c_\alpha - \ell_j^2 \end{aligned}$$

and where $j = i + 1$.

At this point the trigonometric identities $\sin \theta_i = 2x_i/(1+x_i^2)$ and $\cos \theta_i = (1-x_i^2)/(1+x_i^2)$ are introduced where $x_i = \tan(\theta_i/2)$ and $i = 1, 2$. Substituting these identities into equation set (28), multiplying throughout by $(1+x_1^2)(1+x_2^2)$, and then regrouping gives

$$\begin{aligned} & (a_i x_1^2 + b_i x_1 + d_i) x_2^2 + (e_i x_1^2 + f_i x_1 + g_i) x_2 \\ & + (h_i x_1^2 + i_i x_1 + j_i) = 0, \quad i = 1 \cdots 3 \quad (30) \end{aligned}$$

where

$$\begin{aligned} a_i &= A_i - D_i - H_i + J_i, & b_i &= 2(I_i - B_i), \\ d_i &= -A_i - D_i + H_i + J_i, & e_i &= 2(G_i - E_i), \\ f_i &= 4F_i, & g_i &= 2(G_i + E_i), \quad (31) \\ h_i &= -A_i + D_i - H_i + J_i, & i_i &= 2(I_i + B_i), \\ j_i &= A_i + D_i + H_i + J_i. \end{aligned}$$

The objective now is to determine values for x_1 and x_2 which will simultaneously satisfy the three equations represented by (30). It will be shown that only one value of x_1 and one corresponding value of x_2 will simultaneously satisfy the three equations. The Cayley-Dixon formulation¹⁶ will be used as the method of solution for this problem.

The coefficients in set (31) will now be used to write three polynomials in terms of the variables u and v as

$$\begin{aligned} f_i(u, v) &= (a_i u^2 + b_i u + d_i) v^2 + (e_i u^2 + f_i u + g_i) v \\ & + (h_i u^2 + i_i u + j_i), \quad i = 1 \cdots 3. \quad (32) \end{aligned}$$

When $u = x_1$ and $v = x_2$, these polynomials will all equal zero, and it is these special values of x_1 and x_2 that must be obtained.

Evaluating the polynomials of set (32) when u equals some value λ_1 gives the three additional polynomials

$$f_i(\lambda_1, v), \quad i = 1 \cdots 3. \quad (33)$$

Furthermore, evaluating the polynomials of set (32) when u equals λ_1 and v equals some value λ_2 gives the following three additional polynomials

$$f_i(\lambda_1, \lambda_2), \quad i = 1 \dots 3. \quad (34)$$

The determinant

$$\Delta(u, v, \lambda_1, \lambda_2) = \begin{vmatrix} f_1(u, v) & f_2(u, v) & f_3(u, v) \\ f_1(\lambda_1, v) & f_2(\lambda_1, v) & f_3(\lambda_1, v) \\ f_1(\lambda_1, \lambda_2) & f_2(\lambda_1, \lambda_2) & f_3(\lambda_1, \lambda_2) \end{vmatrix} \quad (35)$$

must vanish whenever λ_1 or λ_2 is substituted for u or v , respectively. This implies that $(u - \lambda_1)(v - \lambda_2)$ is a factor of the above determinant. The expression

$$\delta(u, v, \lambda_1, \lambda_2) = \frac{\Delta(u, v, \lambda_1, \lambda_2)}{(u - \lambda_1)(v - \lambda_2)} \quad (36)$$

is a polynomial of degree 1 in u , degree 3 in v , degree 3 in λ_1 , and degree 1 in λ_2 , and may be written as

$$\delta(u, v, \lambda_1, \lambda_2) = (M_1\lambda_2 + M_2)\lambda_1^3 + (M_3\lambda_2 + M_4)\lambda_1^2 + (M_5\lambda_2 + M_6)\lambda_1 + (M_7\lambda_2 + M_8) \quad (37)$$

where

$$M_i = (N_{i1}v^3 + N_{i2}v^2 + N_{i3}v + N_{i4})u + (N_{i5}v^3 + N_{i6}v^2 + N_{i7}v + N_{i8}), \quad i = 1 \dots 8 \quad (38)$$

and where N_{i1} through N_{i8} ($i = 1 \dots 8$) are defined in terms of the constant coefficients a_1 through j_3 as

$$\begin{aligned} N_{11} &= |a \ b \ e|, & N_{12} &= |a \ b \ h| + |a \ f \ e|, \\ N_{13} &= |a \ f \ h| + |a \ i \ e|, & N_{14} &= |a \ i \ h|, \\ N_{15} &= |a \ d \ e|, & N_{16} &= |a \ g \ e| + |a \ d \ h|, \\ N_{17} &= |a \ g \ h| + |a \ j \ e|, & N_{18} &= |a \ j \ h|, \\ N_{21} &= |h \ a \ b|, & N_{22} &= |h \ a \ f| + |h \ e \ b|, \\ N_{23} &= |h \ a \ i| + |h \ e \ f|, & N_{24} &= |h \ e \ i|, \\ N_{25} &= |h \ a \ d|, & N_{26} &= |h \ a \ g| + |h \ e \ d|, \\ N_{27} &= |h \ a \ j| + |h \ e \ g|, & N_{28} &= |h \ e \ j|, \end{aligned}$$

$$\begin{aligned} N_{31} &= |a \ b \ f| + |a \ d \ e|, \\ N_{32} &= |a \ d \ h| + |a \ b \ i| + |a \ g \ e| + |b \ f \ e|, \\ N_{33} &= |a \ j \ e| + |a \ g \ h| + |b \ i \ e| + |b \ f \ h|, \\ N_{34} &= |h \ a \ j| + |h \ b \ i|, \\ N_{35} &= |d \ f \ a| + |d \ e \ b|, \\ N_{36} &= |d \ i \ a| + |d \ h \ b| + |g \ f \ a| + |g \ e \ b|, \\ N_{37} &= |a \ j \ f| + |a \ g \ i| + |b \ g \ h| + |b \ j \ e|, \\ N_{38} &= |j \ i \ a| + |j \ h \ b|, \\ N_{41} &= |a \ d \ h| + |a \ b \ i|, \\ N_{42} &= |a \ g \ h| + |a \ f \ i| + |e \ d \ h| + |e \ b \ i|, \\ N_{43} &= |h \ b \ i| + |h \ e \ g| + |h \ a \ j| + |f \ i \ e|, \\ N_{44} &= |h \ f \ i| + |h \ e \ j|, \\ N_{45} &= |d \ i \ a| + |d \ h \ b|, \\ N_{46} &= |d \ i \ e| + |d \ h \ f| + |g \ i \ a| + |g \ h \ b|, \\ N_{47} &= |h \ f \ g| + |h \ b \ j| + |i \ e \ g| + |i \ a \ j|, \\ N_{48} &= |j \ i \ e| + |j \ h \ f|, \\ N_{51} &= |a \ d \ f| + |a \ b \ g|, \\ N_{52} &= |a \ d \ i| + |a \ b \ j| + |e \ d \ f| + |e \ b \ g|, \\ N_{53} &= |b \ g \ h| + |b \ j \ e| + |d \ f \ h| + |d \ i \ e|, \\ N_{54} &= |h \ d \ i| + |h \ b \ j|, \\ N_{55} &= |d \ g \ a| + |d \ f \ b|, \\ N_{56} &= |d \ j \ a| + |d \ g \ e| + |d \ i \ b| + |b \ g \ f|, \\ N_{57} &= |b \ g \ i| + |b \ j \ f| + |d \ j \ e| + |d \ g \ h|, \\ N_{58} &= |j \ h \ d| + |j \ i \ b|, \\ N_{61} &= |a \ d \ i| + |a \ b \ j|, \\ N_{62} &= |a \ f \ j| + |a \ g \ i| + |e \ b \ j| + |e \ d \ i|, \\ N_{63} &= |e \ f \ j| + |e \ g \ i| + |h \ d \ i| + |h \ b \ j|, \\ N_{64} &= |h \ f \ j| + |h \ g \ i|, \\ N_{65} &= |d \ i \ b| + |d \ j \ a|, \\ N_{66} &= |d \ i \ f| + |d \ j \ e| + |g \ i \ b| + |g \ j \ a|, \\ N_{67} &= |g \ j \ e| + |g \ i \ f| + |j \ i \ b| + |j \ h \ d|, \\ N_{68} &= |j \ h \ g| + |j \ i \ f|, & N_{71} &= |a \ d \ g|, \\ N_{72} &= |d \ j \ a| + |d \ g \ e|, \\ N_{73} &= |d \ f \ j| + |d \ g \ i|, \end{aligned} \quad (39)$$

$$\begin{aligned}
 N_{74} &= |d \ j \ h|, & N_{75} &= |d \ g \ b|, \\
 N_{76} &= |d \ j \ b| + |d \ g \ f|, \\
 N_{77} &= |d \ j \ f| + |d \ g \ i|, \\
 N_{78} &= |d \ j \ i|, & N_{81} &= |a \ d \ j|, \\
 N_{82} &= |j \ a \ g| + |j \ e \ d|, \\
 N_{83} &= |j \ h \ d| + |j \ e \ g|, \\
 N_{84} &= |j \ h \ g|, & N_{85} &= |j \ b \ d|, \\
 N_{86} &= |j \ f \ d| + |j \ b \ g|, \\
 N_{87} &= |j \ i \ d| + |j \ f \ g|, \\
 N_{88} &= |j \ i \ g|.
 \end{aligned}$$

The notation $|a \ b \ c|$ is used above to represent a 3×3 determinant as

$$|a \ b \ c| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}. \quad (40)$$

The terms in the top row of the determinant Δ will vanish when $u = x_1$ and $v = x_2$ (see Eq. (35)), and thus $\Delta(x_1, x_2, \lambda_1, \lambda_2) = 0$. From (36) it is apparent that $\delta(x_1, x_2, \lambda_1, \lambda_2) = 0$, also for any value of λ_1 or λ_2 . For the polynomial δ to vanish when $u = x_1$ and $v = x_2$ for any value of λ_1 or λ_2 , it must be the case that the coefficients M_1 through M_8 all equal zero. This gives eight equations in the unknowns x_1 and x_2 , which may be written in matrix form as

$$\mathbf{N}\mathbf{y} = \mathbf{0} \quad (41)$$

where

$$\mathbf{N} = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} & N_{15} & N_{16} & N_{17} & N_{18} \\ N_{21} & N_{22} & N_{23} & N_{24} & N_{25} & N_{26} & N_{27} & N_{28} \\ N_{31} & N_{32} & N_{33} & N_{34} & N_{35} & N_{36} & N_{37} & N_{38} \\ N_{41} & N_{42} & N_{43} & N_{44} & N_{45} & N_{46} & N_{47} & N_{48} \\ N_{51} & N_{52} & N_{53} & N_{54} & N_{55} & N_{56} & N_{57} & N_{58} \\ N_{61} & N_{62} & N_{63} & N_{64} & N_{65} & N_{66} & N_{67} & N_{68} \\ N_{71} & N_{72} & N_{73} & N_{74} & N_{75} & N_{76} & N_{77} & N_{78} \\ N_{81} & N_{82} & N_{83} & N_{84} & N_{85} & N_{86} & N_{87} & N_{88} \end{bmatrix},$$

$$\mathbf{y} = \begin{bmatrix} x_1 x_2^3 \\ x_1 x_2^2 \\ x_1 x_2 \\ x_1 \\ x_2^3 \\ x_2^2 \\ x_2 \\ 1 \end{bmatrix}. \quad (42)$$

The matrix equation (41) may be thought of as eight homogeneous equations in eight unknowns. The trivial solution of $\mathbf{y} = \mathbf{0}$ is not feasible, since the last element of \mathbf{y} must equal 1. Solutions other than the trivial solution will exist only if the homogeneous equations are linearly dependent, and as such the determinant of the matrix \mathbf{N} must equal zero. Evaluating this determinant and seeing how close it is to zero will provide an indication of the quality of the measured data (i.e., the measured coordinates of points 1-4 and 1'-4' in the A and B coordinate systems) and the sensed data (i.e., the four measured displacements ℓ_1 - ℓ_4). The issue of how close to zero is satisfactory is not addressed in this paper.

The eight equations represented by (41) may now be rearranged into the form

$$\mathbf{A}\mathbf{x} = \mathbf{b} \quad (43)$$

where

$$\mathbf{A} = \begin{bmatrix} N_{11} & N_{12} & N_{13} & N_{14} & N_{15} & N_{16} & N_{17} \\ N_{21} & N_{22} & N_{23} & N_{24} & N_{25} & N_{26} & N_{27} \\ N_{31} & N_{32} & N_{33} & N_{34} & N_{35} & N_{36} & N_{37} \\ N_{41} & N_{42} & N_{43} & N_{44} & N_{45} & N_{46} & N_{47} \\ N_{51} & N_{52} & N_{53} & N_{54} & N_{55} & N_{56} & N_{57} \\ N_{61} & N_{62} & N_{63} & N_{64} & N_{65} & N_{66} & N_{67} \\ N_{71} & N_{72} & N_{73} & N_{74} & N_{75} & N_{76} & N_{77} \\ N_{81} & N_{82} & N_{83} & N_{84} & N_{85} & N_{86} & N_{87} \end{bmatrix},$$

$$\mathbf{x} = \begin{bmatrix} x_1 x_2^3 \\ x_1 x_2^2 \\ x_1 x_2 \\ x_1 \\ x_2^3 \\ x_2^2 \\ x_2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -N_{18} \\ -N_{28} \\ -N_{38} \\ -N_{48} \\ -N_{58} \\ -N_{68} \\ -N_{78} \\ -N_{88} \end{bmatrix}. \quad (44)$$

Equation (44) represents eight linear equations in seven unknowns. The vector \mathbf{x} may be solved for

by selecting any seven of these equations. The terms x_1 and x_2 are, respectively, the fourth and seventh components of the vector x and unique values for these terms are thereby determined. Corresponding values for θ_1 and θ_2 may be obtained from

$$\theta_i = 2 \tan^{-1}(x_i), \quad i = 1, 2. \quad (45)$$

The rotation matrix ${}^A_B\mathbf{R}$ may next be determined from (3), where ${}^A_C\mathbf{R}$ and ${}^B_D\mathbf{R}$ have been defined in (1) and (2), and ${}^C_D\mathbf{R}$ is defined in terms of θ_1 , θ_2 , and α by (5).

6. NUMERICAL EXAMPLES

The coordinates of four points on the lower body are specified in the A coordinate system as

$$\begin{aligned} {}^A\mathbf{P}_1 &= \begin{bmatrix} 8 \\ 0 \\ -10 \end{bmatrix} \text{ cm}, & {}^A\mathbf{P}_2 &= \begin{bmatrix} 2 \\ 7 \\ -11 \end{bmatrix} \text{ cm}, \\ {}^A\mathbf{P}_3 &= \begin{bmatrix} -7 \\ 1 \\ -8 \end{bmatrix} \text{ cm}, & {}^A\mathbf{P}_4 &= \begin{bmatrix} -1 \\ -8 \\ -12 \end{bmatrix} \text{ cm}. \end{aligned} \quad (46)$$

The coordinates of four points on the upper body are specified in the B coordinate system as

$$\begin{aligned} {}^B\mathbf{P}_{1'} &= \begin{bmatrix} 6 \\ 1 \\ 10 \end{bmatrix} \text{ cm}, & {}^B\mathbf{P}_{2'} &= \begin{bmatrix} 0 \\ 8 \\ 12 \end{bmatrix} \text{ cm}, \\ {}^B\mathbf{P}_{3'} &= \begin{bmatrix} -8 \\ -1 \\ 8 \end{bmatrix} \text{ cm}, & {}^B\mathbf{P}_{4'} &= \begin{bmatrix} 1 \\ -7 \\ 12 \end{bmatrix} \text{ cm}. \end{aligned} \quad (47)$$

Figure 4a shows the mechanism in a configuration in which the A and B coordinate systems are aligned. The configuration shown in Figure 4b was achieved by initially aligning the B coordinate system with the A system. The B coordinate system was then rotated 10° about the x axis, followed by -15° about its modified y axis, and then by 5° about its modified z axis.

6.1. Case A

The distances between the four pairs of points for the case shown in Figure 4a are calculated as

$$\begin{aligned} \ell_1 &= 20.1246, & \ell_2 &= 23.1084, \\ \ell_3 &= 16.1555, & \ell_4 &= 24.1039 \text{ cm} \end{aligned} \quad (48)$$

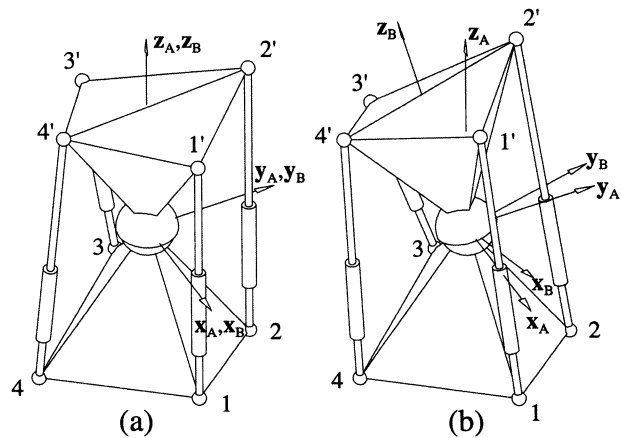


Figure 4. Two configurations of the test case mechanism.

The calculated values for θ_1 , θ_2 , and α were determined to be

$$\begin{aligned} \theta_1 &= 31.95595286^\circ, & \theta_2 &= 39.22517597^\circ, \\ \alpha &= 110.29864361^\circ. \end{aligned} \quad (49)$$

The rotation matrix ${}^A_B\mathbf{R}$ is calculated as

$${}^A_B\mathbf{R} = \begin{bmatrix} 1 & -2.22 \times 10^{-14} & -1.25 \times 10^{-15} \\ 2.24 \times 10^{-14} & 1 & 2.92 \times 10^{-14} \\ 1.03 \times 10^{-15} & -2.93 \times 10^{-14} & 1 \end{bmatrix} \quad (50)$$

which is very close to the expected value of the identity matrix.

6.2. Case B

For this case, the rotation matrix ${}^A_B\mathbf{R}$ is known a priori as

$$\begin{aligned} {}^A_B\mathbf{R} &= \begin{bmatrix} \cos 5^\circ & -\sin 5^\circ & 0 \\ \sin 5^\circ & \cos 5^\circ & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &\times \begin{bmatrix} \cos(-15^\circ) & 0 & \sin(-15^\circ) \\ 0 & 1 & 0 \\ -\sin(-15^\circ) & 0 & \cos(-15^\circ) \end{bmatrix} \\ &\times \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 10^\circ & -\sin 10^\circ \\ 0 & \sin 10^\circ & \cos 10^\circ \end{bmatrix}, \end{aligned} \quad (51)$$

$${}^A_B\mathbf{R} = \begin{bmatrix} 0.96225 & -0.13060 & -0.23878 \\ 0.08419 & 0.97714 & -0.19520 \\ 0.25882 & 0.16773 & 0.95125 \end{bmatrix}. \quad (52)$$