Closed-Form Equilibrium Analysis of Planar Tensegrity Mechanisms

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ABSTRACT

This paper presents a closed-form analysis of a series of planar tensegrity mechanisms to determine all possible equilibrium configurations for the device when no external forces or moments are applied. The equilibrium position is determined by identifying the configurations at which the potential energy stored in the two springs is a minimum. For a two-spring system, a 28th degree polynomial expressed in terms of the length of one of the springs is developed where this polynomial identifies the cases where the change in potential energy with respect to a change in the spring length is zero. Three spring systems are also analyzed. This more complex systems was solved using the Continuation Method. Numerical examples are presented.

Keywords

Keywords: Tensegrity, Planar Mechanisms

1. INTRODUCTION

The word tensegrity is a combination of the words tension and integrity (Edmondson, 1987 and Fuller, 1975). Tensegrity structures are spatial structures formed by a combination of rigid elements in compression (struts) and connecting elements that are in tension (ties). No pair of struts touch and the end of each strut is connected to three non-coplanar ties (Yin et al, 2002). The entire configuration stands by itself and maintains its form solely because of the internal arrangement of the struts and ties (Tobie, 1976).

The development of tensegrity structures is relatively new and the works related have only existed for approximately twenty five years. Kenner, 1976, established the relation between the rotation of the top and bottom ties. Tobie, 1976, presented procedures for the generation of tensile structures by physical and graphical means. Yin, 2002, obtained Kenner's results using energy considerations and found the equilibrium position for unloaded tensegrity prisms. Stern, 1999, developed generic design equations to find the lengths of the struts and elastic ties needed to create a desired geometry for a symmetric case. Knight, 2000, addressed the problem of stability of tensegrity structures for the design of deployable antennae.

2. TWO-SPRING SYSTEM

A planar tensegrity system is shown in Figure 1. The device consists of two rigid struts and four ties. For the two-spring system, two of the ties are compliant (the ties between points 4 and 2 and points 1 and 3) and two of the ties are non-compliant. The objective is to determine all equilibrium configurations for the system when given the lengths of the struts and non-compliant ties as well as the free length and spring constant for each of the compliant ties.

The specific problem statement for the two-spring system is written as follows:

given:	a_{12}, a_{34}	lengths of struts,
	a_{23}, a_{41}	lengths of non-compliant ties,
	k_1, L_{01}	spring constant and free length of compliant tie between points 4 and 2,
	k_2, L_{02}	spring constant and free length of compliant tie between points 3 and 1.
find:	L_1	length of spring 1 at equilibrium position,
	L_2	length of spring 2 at equilibrium position corresponding to length of spring 1, i.e. L_1.

It should be noted that the problem statement could be

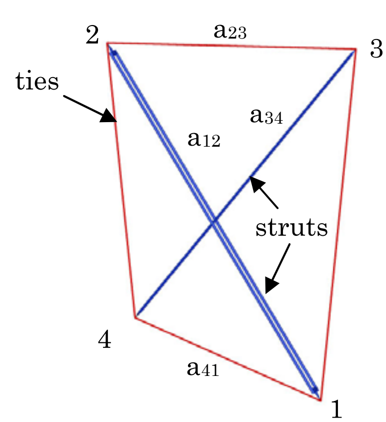

Figure 1: Planar Tensegrity System

formulated in a variety of ways, i.e. a different variable (such as the relative angle between strut a_{34} and tie a_{41}) could have been selected as the generalized parameter for this problem. Attempts at using this alternate approach did not successfully yield a closed form solution for the equilibrium positions.

2.1 Development of Geometry and Energy Equations

Figure 3 shows the nomenclature that is used. L_1 and L_2 are the extended lengths of the compliant ties between points 4 and 2 and points 3 and 1. Figure 4 shows the triangle formed by sides a_{34}, a_{23}, and L_1. A cosine law for this triangle can be written as

$$\frac{L_1^2}{2}+\frac{a_{34}^2}{2}+L_1\,a_{34}\cos\theta_4' = \frac{a_{23}^2}{2}\ . \tag{1}$$

Solving for $\cos\theta_4'$ yields

$$\cos\theta_4' = \frac{a_{23}^2 - L_1^2 - a_{34}^2}{2\,L_1\,a_{34}}\ . \tag{2}$$

Figure 5 shows the triangle formed by sides a_{41}, a_{12}, and L_1. A cosine law for this triangle can be written as

$$\frac{L_1^2}{2}+\frac{a_{41}^2}{2}+L_1\,a_{41}\cos\theta_4'' = \frac{a_{12}^2}{2}\ . \tag{3}$$

Solving for $\cos\theta_4''$ yields

$$\cos\theta_4'' = \frac{a_{12}^2 - L_1^2 - a_{41}^2}{2\,L_1\,a_{41}}\ . \tag{4}$$

Figure 6 shows the triangle formed by sides a_{41}, a_{34}, and L_2.

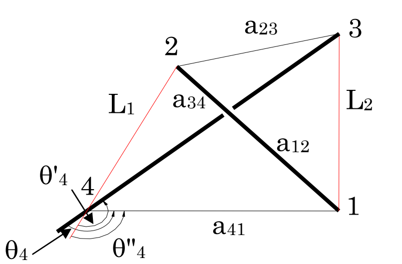

Figure 3: Planar Tensegrity Structure

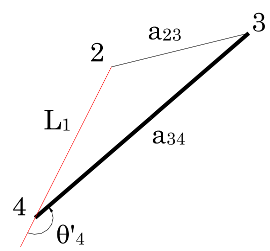

Figure 4: Triangle 4-3-2

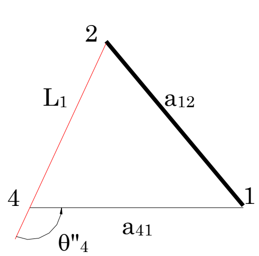

Figure 5: Triangle 4-1-2

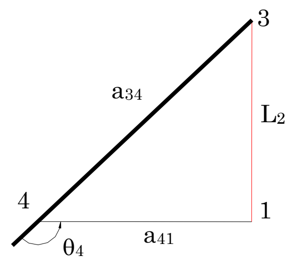

Figure 6: Triangle 4-1-3

A cosine law for this triangle can be written as

$$\frac{a_{34}^2}{2}+\frac{a_{41}^2}{2}+a_{34}\,a_{41}\cos\theta_4 = \frac{L_2^2}{2}\ . \tag{5}$$

Solving for $\cos\theta_4$ yields

$$\cos\theta_4 = \frac{L_2^2 - a_{34}^2 - a_{41}^2}{2\,a_{34}\,a_{41}}\ . \tag{6}$$

From Figure 3 it is apparent that

$$\theta_4 + \theta_4' = \pi + \theta_4'' \tag{7}$$

Equating the cosine of the left and right sides of (7) yields

$$\cos(\theta_4 + \theta_4') = \cos(\pi + \theta_4'') \tag{8}$$

and expanding this equation yields

$$\cos\theta_4\cos\theta_4' - \sin\theta_4\sin\theta_4' = -\cos\theta_4''\ . \tag{9}$$

Rearranging (9) yields

$$\cos\theta_4\cos\theta_4' + \cos\theta_4'' = \sin\theta_4\sin\theta_4'\ . \tag{10}$$

Squaring both sides of (10) gives

$$(\cos\theta_4)^2(\cos\theta_4')^2 + 2\cos\theta_4\cos\theta_4'\cos\theta_4'' + (\cos\theta_4'')^2 = (\sin\theta_4)^2(\sin\theta_4')^2\ . \tag{11}$$

Substituting for $(\sin\theta_4)^2$ and $(\sin\theta_4')^2$ in terms of $\cos\theta_4$ and $\cos\theta_4'$ gives

$$(\cos\theta_4)^2(\cos\theta_4')^2 + 2\cos\theta_4\cos\theta_4'\cos\theta_4'' + (\cos\theta_4'')^2 = (1-\cos^2\theta_4)(1-\cos^2\theta_4')\ . \tag{12}$$

Equations (2), (4), and (6) are substituted into (12) to yield a single equation in the parameters L_1 and L_2 which can be written as

$$A\,L_2^4 + B\,L_2^2 + C = 0 \tag{13}$$

where

$$A = L_1^2,\quad B = L_1^4 + B_2 L_1^2 + B_0,\quad C = C_2 L_1^2 + C_0 \tag{14}$$

and where

$$\begin{aligned} B_2 &= -(a_{23}^2 + a_{34}^2 + a_{41}^2 + a_{12}^2),\\ B_0 &= (a_{12} - a_{41})(a_{12} + a_{41})(a_{23} - a_{34})(a_{23} + a_{34}),\\ C_2 &= (a_{34} - a_{41})(a_{34} + a_{41})(a_{23} - a_{12})(a_{23} + a_{12}),\\ C_0 &= (a_{41}a_{23} + a_{34}a_{12})(a_{41}a_{23} - a_{34}a_{12})\\ &\quad(a_{41}^2 + a_{23}^2 - a_{12}^2 - a_{34}^2). \end{aligned} \tag{15}$$

The potential energy of the system can be evaluated as

$$U = \frac{1}{2}k_1(L_1 - L_{01})^2 + \frac{1}{2}k_2(L_2 - L_{02})^2\ . \tag{16}$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the length L_1 equals zero, i.e.

$$\frac{dU}{dL_1} = k_1 (L_1 - L_{01}) + k_2 (L_2 - L_{02}) \frac{dL_2}{dL_1} = 0 \quad . \tag{17}$$

The derivative dL_2/dL_1 can be determined via implicit differentiation from equation (13) as

$$\frac{dL_2}{dL_1} = \frac{-L_1 [L_2^2 (L_2^2 + 2L_1^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{23}^2)(a_{41}^2 - a_{34}^2)]}{L_2 [L_1^2 (L_1^2 + 2L_2^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{41}^2)(a_{23}^2 - a_{34}^2))]} \, . \tag{18}$$

Substituting (18) into (17) and regrouping gives

$$D L_2^5 + E L_2^4 + F L_2^3 + G L_2^2 + H L_2 + J = 0 \tag{19}$$

where

$D = D_1 L_1$,

$E = E_1 L_1$,

$F = F_3 L_1^3 + F_2 L_1^2 + F_1 L_1$,

$G = G_3 L_1^3 + G_1 L_1$,

$H = H_5 L_1^5 + H_4 L_1^4 + H_3 L_1^3 + H_2 L_1^2 + H_1 L_1 + H_0$,

$J = J_1 L_1$ (20)

and where,

$D_1 = k_2$, $E_1 = -k_2 L_{02}$,

$F_3 = 2 (k_2 - k_1)$, $F_2 = 2 k_1 L_{01}$,

$F_1 = - k_2 (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2)$,

$G_3 = -2 k_2 L_{02}$, $G_1 = k_2 L_{02} (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2)$,

$H_5 = - k_1$, $H_4 = k_1 L_{01}$, $H_3 = k_1 (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2)$,

$H_2 = - k_1 L_{01} (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2)$,

$H_1 = - k_1 (a_{34}^2 - a_{23}^2) (a_{41}^2 - a_{12}^2)$
$\quad + k_2 (a_{34}^2 - a_{41}^2) (a_{23}^2 - a_{12}^2)$,

$H_0 = k_1 L_{01} (a_{34}^2 - a_{23}^2) (a_{41}^2 - a_{12}^2)$,

$J_1 = k_2 L_{02} (a_{34}^2 - a_{41}^2) (a_{12}^2 - a_{23}^2)$. (21)

2.2 Solution of Geometry and Energy Equations

Equations (19) and (13) represent two equations in the two unknowns L_1 and L_2. These equations can be solved by using Sylvester's variable elimination procedure by multiplying equation (13) by L_2, L_2^2, L_2^3, and L_2^4 and equation (19) by L_2, L_2^2, L_2^3 to yield a total of nine equations that can be written in matrix form as

$$\begin{bmatrix} 0 & 0 & 0 & D & E & F & G & H & J \\ 0 & 0 & 0 & 0 & A & 0 & B & 0 & C \\ 0 & 0 & 0 & A & 0 & B & 0 & C & 0 \\ 0 & 0 & D & E & F & G & H & J & 0 \\ 0 & 0 & A & 0 & B & 0 & C & 0 & 0 \\ 0 & D & E & F & G & H & J & 0 & 0 \\ 0 & A & 0 & B & 0 & C & 0 & 0 & 0 \\ D & E & F & G & H & J & 0 & 0 & 0 \\ A & 0 & B & 0 & C & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} L_2^8 \\ L_2^7 \\ L_2^6 \\ L_2^5 \\ L_2^4 \\ L_2^3 \\ L_2^2 \\ L_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \, . \tag{22}$$

A solution to this set of equations can only occur if the determinant of the 9×9 coefficient matrix is equal to zero. Expansion of this determinant yields a 30th degree polynomial in the variable L_1. When the determinant was expanded symbolically, it was seen that the two lowest order coefficients were identically zero. Thus the polynomial can be divided throughout by L_1^2 to yield a 28th degree polynomial. The coefficients of the 28th degree polynomial were obtained symbolically in terms of the given quantities, but are not presented here due to their length and complexity.

Values for L_2 that correspond to each value of L_1 can be determined by first solving (13) for four possible values of L_2. Only one of these four values also satisfies equation (19).

2.3 Numerical Example

The following parameters were selected to show the results of a numerical example:

strut lengths:
$a_{12} = 3$ in. $a_{34} = 3.5$ in.

non-compliant tie lengths:
$a_{41} = 4$ in. $a_{23} = 2$ in.

spring 1 free length & spring constant:
$L_{01} = 0.5$ in. $k_1 = 4$ lbf/in.

spring 2 free length & spring constant:
$L_{02} = 1$ in. $k_2 = 2.5$ lbf/in.

Eight real and twenty complex roots were obtained for L_1. The real values for L_1 and the corresponding values of L_2 are shown in Table 1.

Table 1: Eight Real Solutions

Case	1	2	3	4
L_1, in.	-5.485	-5.322	-1.741	-1.576
L_2, in.	2.333	-2.901	-1.495	1.870

Case	5	6	7	8
L_1, in.	1.628	1.863	5.129	5.476
L_2, in.	1.709	-1.354	-3.288	2.394

The values of L_1 and L_2 listed in Table 1 satisfy the geometric constraints defined by equation (13) and the energy condition defined by equation (19). Each of these eight cases was analyzed to determine whether it represented a minimum or maximum potential energy condition and cases 3, 4, 5, and 6 were found to be minimum states. A free body analysis of struts a_{12} and a_{34} was performed to show that these bodies were indeed in equilibrium. Figure 7 shows the four equilibrium configurations for the numerical case under consideration.

Efforts were undertaken to obtain a numerical example that would yield more than eight real roots. One example of each type of Grashof and non-Grashof mechanism was analyzed, yet for all these cases eight real roots were found. The complex values of L_1 were analyzed and corresponding complex values of L_2 were determined. In every case it was possible to obtain complex pairs of L_1 and L_2 that satisfied the geometric constraint equation, (13), and the derivative of potential energy equation (19). Thus no extraneous roots were introduced during the variable elimination procedure.

3. THREE-SPRING SYSTEM

Figure 8 shows a three-spring tensegrity system. Two parameters must be specified, in addition to the constant mechanism parameters, in order to define the configuration of the device. These two parameters will be referred to as the *descriptive parameters* for the system. One obvious set of descriptive parameters are the angles θ_4 and θ_1. Considering the non-compliant member a_{41} as being fixed to ground, specification of θ_4 will define the location of point 3. Similarly, specification of θ_1 will define the location of point 2.

Two approaches to solve this problem have been analyzed. Both aim to find a set of descriptive parameters that minimize the potential energy in the system. In the first approach, the lengths of the compliant members L_1 and L_2 are chosen as the descriptive parameters. Derivatives of the potential energy equation are obtained with respect to L_1 and L_2 and values for the descriptive parameters are obtained such that these derivatives are zero, corresponding to either a minimum or maximum potential energy state. In the second approach, the cosines of the angles θ_4 and θ_1 were chosen as the descriptive parameters. The cosines of the angles were chosen rather than the angles themselves in the hope that the resulting equations would be simpler in that, for example, a single value of $\cos\theta_4$ accounts for the obvious symmetry in solutions that will occur with respect to the fixed member a_{41}.

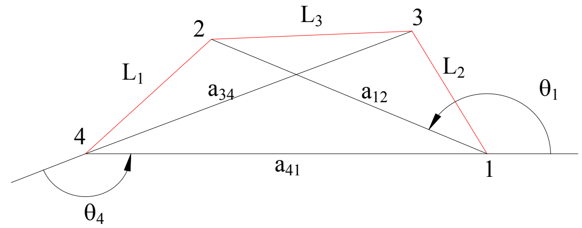

Figure 8: Three-Spring Tensegrity System

The first approach will be presented here.

3.1 Approach 1 – Descriptive Parameters L_1 and L_2

The problem statement can be explicitly written as:

Given: length of struts (a_{12}, a_{34}) ; spring constants and free lengths of three springs (k_1, L_{01} ; k_2, L_{02} ; k_3, L_{03})

Find: length of springs 1 and 2 (L_1, L_2) and corresponding length of spring 3 (L_3) at equilibrium

The analysis for this case can proceed in a manner similar to that presented for the two-spring system. Specifically, the term a_{23} in (15) can be replaced by L_3 and equation (13) can be factored into the form

$$G_1 L_3^4 + (G_2 L_2^2 + G_3) L_3^2 + (G_4 L_2^4 + G_5 L_2^2 + G_6) = 0 \quad (23)$$

where

$$G_1 = a_{41}^2,$$
$$G_2 = G_{2a} L_1^2 + G_{2b},$$
$$G_3 = G_{3a} L_1^2 + G_{3b},$$
$$G_4 = G_{4a} L_1^2,$$
$$G_5 = G_{5a} L_1^4 + G_{5b} L_1^2 + G_{5c},$$
$$G_6 = G_{6a} L_1^2 + G_{6b} \quad (24)$$

and where the remaining coefficients are written in terms of the

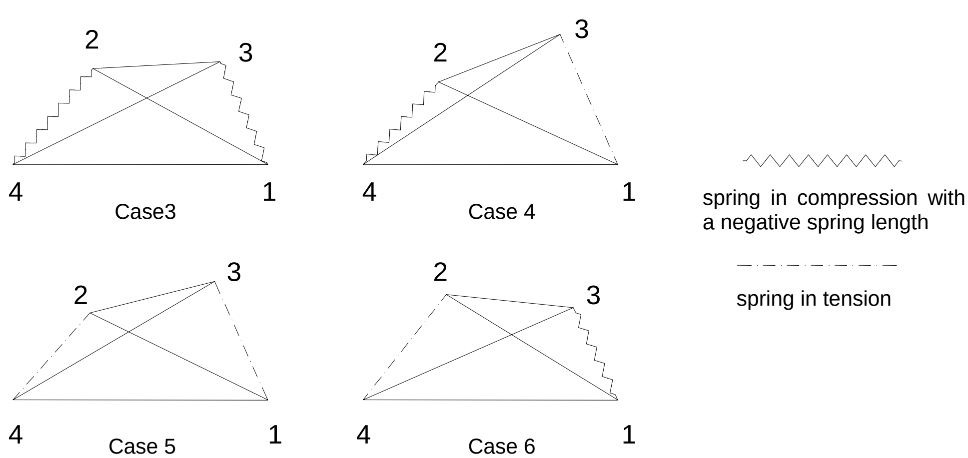

Figure 7: Four Equilibrium Configurations for Two-Spring Tensegrity System

constant mechanism parameters as

$G_{2a} = -1,$
$G_{2b} = a_{12}^2 - a_{41}^2,$
$G_{3a} = (a_{34}^2 - a_{41}^2),$
$G_{3b} = a_{41}^2 (a_{41}^2 - a_{12}^2 - a_{34}^2) - a_{12}^2 a_{34}^2,$
$G_{4a} = 1,$
$G_{5a} = 1,$
$G_{5b} = (-a_{12}^2 - a_{34}^2 - a_{41}^2),$
$G_{5c} = a_{34}^2 (a_{41}^2 - a_{12}^2),$
$G_{6a} = a_{12}^2 (a_{41}^2 - a_{34}^2), G_{6b} = a_{12}^2 a_{34}^2 (a_{12}^2 + a_{34}^2 - a_{41}^2)$. (25)

The potential energy of the system can be written as

$$U = \frac{1}{2} k_1 (L_1 - L_{01})^2 + \frac{1}{2} k_2 (L_2 - L_{02})^2 + \frac{1}{2} k_3 (L_3 - L_{03})^2 . \quad (26)$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the mechanism whereby the derivative of the potential energy taken with respect to the descriptive parameters L_1 and L_2 both equal zero. The geometric constraint equation, Equation 23, contains three unknown terms, L_1, L_2, and L_3. From this equation, L_3 can be considered as a dependent variable of L_1 and L_2. The following two expressions may be written:

$$\frac{\partial U}{\partial L_1} = k_1 (L_1 - L_{01}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_1} = 0 , \quad (27)$$

$$\frac{\partial U}{\partial L_2} = k_2 (L_2 - L_{02}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_2} = 0 . \quad (28)$$

The derivatives $\delta L_3/\delta L_1$ and $\delta L_3/\delta L_2$ can be determined via implicit differentiation from Equation 23 as

$$\frac{\partial L_3}{\partial L_1} = \frac{-L_1[2L_1^2 L_2^2 + G_{2a} L_2^2 L_3^2 + G_{3a} L_3^2 + L_2^4 + G_{5b} L_2^2 + G_{6a}]}{L_3[2G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} \quad (29)$$

$$\frac{\partial L_3}{\partial L_2} = \frac{-L_2[2L_1^2 L_2^2 + G_{2a} L_1^2 L_3^2 + G_{2b} L_3^2 + L_1^4 + G_{5b} L_1^2 + G_{5c}]}{L_3[2G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} \quad (30)$$

Substituting (29) into (27) and rearranging gives

$$(D_1 L_2^2 + D_2) L_3^3 + (D_3 L_2^2 + D_4) L_3^2 + (D_5 L_2^4 + D_6 L_2^2 + D_7) L_3 + (D_8 L_2^4 + D_9 L_2^2 + D_{10}) = 0 \quad (31)$$

where

$D_1 = D_{1a} L_1 ,$
$D_2 = D_{2a} L_1 + D_{2b} ,$
$D_3 = D_{3a} L_1 ,$
$D_4 = D_{4a} L_1 ,$
$D_5 = D_{5a} L_1 ,$
$D_6 = D_{6a} L_1^3 + D_{6b} L_1^2 + D_{6c} L_1 + D_{6d} ,$
$D_7 = D_{7a} L_1^3 + D_{7b} L_1^2 + D_{7c} L_1 + D_{7d} ,$
$D_8 = D_{8a} L_1 ,$
$D_9 = D_{9a} L_1^3 + D_{9b} L_1$
$D_{10} = D_{10a} L_1$ (32)

and

$D_{1a} = - G_{2a} k_3 ,$
$D_{2a} = 2 G_1 k_1 - G_{3a} k_3, \quad D_{2b} = - 2 G_1 k_1 L_{01} ,$
$D_{3a} = G_{2a} k_3 L_{03} ,$
$D_{4a} = G_{3a} k_3 L_{03} ,$
$D_{5a} = - k_3 ,$
$D_{6a} = G_{2a} k_1 - 2 k_3 ,$
$D_{6b} = -G_{2a} k_1 L_{01} , D_{6c} = G_{2b} k_1 - G_{5b} k_3 , D_{6d} = -G_{2b} k_1 L_{01} ,$
$D_{7a} = G_{3a} k_1 , D_{7b} = -G_{3a} k_1 L_{01} , D_{7c} = G_{3b} k_1 - G_{6a} k_3 ,$
$D_{7d} = -G_{3b} k_1 L_{01} ,$
$D_{8a} = k_3 L_{03} ,$
$D_{9a} = 2 k_3 L_{03} , \quad D_{9b} = G_{5b} k_3 L_{03}$
$D_{10a} = G_{6a} k_3 L_{03}$ (33)

Substituting (30) into (28) and rearranging gives

$$(E_1 L_2 + E_2) L_3^3 + (E_3 L_2) L_3^2 + (E_4 L_2^3 + E_5 L_2^2 + E_6 L_2 + E_7) L_3 + (E_8 L_2^3 + E_9 L_2) = 0 \quad (34)$$

where

$E_1 = E_{1a} L_1^2 + E_{1b} ,$
$E_2 = - 2 G_1 k_2 L_{02} ,$
$E_3 = E_{3a} L_1^2 + E_{3b} ,$
$E_4 = E_{4a} L_1^2 + E_{4b} ,$
$E_5 = E_{5a} L_1^2 + E_{5b} ,$
$E_6 = E_{6a} L_1^4 + E_{6b} L_1^2 + E_{6c} ,$
$E_7 = E_{7a} L_1^2 + E_{7b} ,$
$E_8 = E_{8a} L_1^2 ,$
$E_9 = E_{9a} L_1^4 + E_{9b} L_1^2 + E_{9c}$ (35)

and

$E_{1a} = -G_{2a} k_3 , \quad E_{1b} = -G_{2b} k_3 + 2 G_1 k_2 ,$
$E_{3a} = G_{2a} k_3 L_{03} , \quad E_{3b} = G_{2b} k_3 L_{03} ,$
$E_{4a} = G_{2a} k_2 - 2 k_3 , \quad E_{4b} = G_{2b} k_2 ,$
$E_{5a} = -G_{2a} k_2 L_{02} , \quad E_{5b} = -G_{2b} k_2 L_{02} ,$
$E_{6a} = - k_3 , \quad E_{6b} = G_{3a} k_2 - G_{5b} k_3 , E_{6c} = G_{3b} k_2 - G_{5c} k_3 ,$
$E_{7a} = -G_{3a} k_2 L_{02} , \quad E_{7b} = -G_{3b} k_2 L_{02} ,$
$E_{8a} = 2 k_3 L_{03} ,$
$E_{9a} = k_3 L_{03} , E_{9b} = G_{5b} k_3 L_{03} , E_{9c} = G_{5c} k_3 L_{03} .$ (36)

3.2 Solution of Three Simultaneous Equations in Three Unknowns – Sylvester's Method

Equations (23), (31), and (34) are three equations in the three unknowns L_1, L_2, and L_3. Sylvester's method can be applied in order to obtain sets of values for these parameters that simultaneously satisfy all three equations. In this solution, the parameter L_1 is embedded in the coefficients of the three equations to yield three equations in the apparent unknowns L_2 and L_3. Determining the condition that these new coefficients (which contain L_1) must satisfy such that the three equations can have common roots for L_2 and L_3 will yield a single polynomial in L_1.

Equation (23) was multiplied by L_2, L_3, L_2L_3, L_3^2, L_2^2, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, L_3^3, L_2^3, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2^3L_3^3$. Equation (31) was multiplied by L_3, L_2, L_3^2, $L_2L_3^2$, L_3^3, L_2^2, $L_2L_3^3$, $L_2^2L_3$, $L_2^2L_3^2$, L_2^3, L_3^4, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2L_3^4$. Equation (34) was multiplied by L_2, L_3, L_2L_3, L_3^2, L_2^2, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, L_3^3, L_2^3, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, L_2^4, L_3^4, $L_2^4L_3$, $L_2^4L_3^2$, $L_2L_3^4$, and $L_2^2L_3^4$. This resulted in a set of 52 equations that can be written in matrix for as

$$\mathbf{M}\,\boldsymbol{\lambda} = \mathbf{0}\,. \qquad (37)$$

The vector $\boldsymbol{\lambda}$ is written as

$$\begin{aligned}\boldsymbol{\lambda} = [&L_2^7L_3^3, L_2^5L_3^5, L_2^3L_3^7, L_2^7L_3^2, L_2^6L_3^3, L_2^5L_3^4, L_2^4L_3^5,\\ &L_2^3L_3^6, L_2^2L_3^7, L_2^7L_3, L_2^6L_3^2, L_2^5L_3^3, L_2^4L_3^4, L_2^3L_3^5, L_2^2L_3^6,\\ &L_2L_3^7, L_2^7, L_2^6L_3, L_2^5L_3^2, L_2^4L_3^3, L_2^3L_3^4, L_2^2L_3^5, L_2L_3^6, L_3^7,\\ &L_2^6, L_2^5L_3, L_2^4L_3^2, L_2^3L_3^3, L_2^2L_3^4, L_2L_3^5, L_3^6, L_2^5, L_2^4L_3,\\ &L_2^3L_3^2, L_2^2L_3^3, L_2L_3^4, L_3^5, L_2^4, L_2^3L_3, L_2^2L_3^2, L_2L_3^3, L_3^4,\\ &L_2^3, L_2^2L_3, L_2L_3^2, L_3^3, L_2^2, L_2L_3, L_3^2, L_2, L_3, 1]^T\,. \end{aligned} \qquad (38)$$

The coefficient matrix **M** is a 52×52 matrix whose elements are the coefficients G_1 through G_6, D_1 through D_{10} and E_1 through E_9 which are polynomials in terms of the variable L_1. This matrix is not presented here due to its size. Since the set of 52 simultaneous equations represented by (37) must be linearly dependent, the determinant of the matrix **M** must equal zero. This will yield an equation in terms of the single variable L_1.

It was not possible to symbolically expand the determinant of matrix **M**. A numerical case was analyzed and a polynomial of degree 158 in the variable L_1 was obtained. It was not possible to solve this high degree polynomial for the values of L_1, although several commercial and in-house written algorithms were attempted. Because of this, a different method was attempted to solve the set of equations (23), (31), and (34).

3.3 Solution of Three Simultaneous Equations in Three Unknowns – Continuation Method

The continuation method (Garcia and Li, 1980, Morgan, 1983, 1986, Wampler et al., 1990) is a numerical technique to solve a set of equations in multiple variables. This is as opposed to Sylvester's method which would lead to a symbolic solution of the problem.

A concise description of the continuity method is presented by Tsai, 1999. Suppose one wishes to solve the set of equations F(**x**) which are defined by

$$F(\mathbf{x}):\begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases}. \qquad (39)$$

F(**x**) is called the target system.

The continuation method begins by first estimating the total number of possible solution sets (sets of values for L_1, L_2, and L_3 for our case) that satisfy the given equations. For example, Bezout's theorem states that a polynomial of total degree *n* has at most *n* isolated solutions in the complex Euclidean space. Including solutions at infinity, the Bezout number of a polynomial system is equal to the total degree of the system.

Next, an initial system, G(**x**) =0, is obtained, whose solution will be of the same degree as that of F(**x**), but whose solution set is known in closed form. In other words, G(**x**) maintains the same polynomial structure as F(**x**).

Finally, a homotopy function H(**x**, t) is prepared such as

$$H(\mathbf{x}, t) = \gamma\,(1\text{-}t)\,G(\mathbf{x}) + t\,F(\mathbf{x}) \qquad (40)$$

where γ is a random complex constant. When t=0, the homotopy function equals the initial system, G(**x**). When t=1, the homotopy function equals the target system, F(**x**). Recall that the solutions to G(**x**) are known. As the parameter *t* is increased in small steps from 0 to 1, the solutions of H(**x**, t) can be tracked (referred to as path tracking) and when t =1, these solutions will be the solutions to the original target system. If the degree of the solution set was overestimated, some of the solutions will track to infinity and these can easily be discarded.

3.4 Numerical Example

The following information is given:

strut lengths:
a_{12} = 14 in. a_{34} = 12 in.

non-compliant tie lengths:
a_{41} = 10 in.

spring 1 free length & spring constant:
L_{01} = 8 in. k_1 = 1 lbf/in.

spring 2 free length & spring constant:
L_{02} = 2 in. k_2 = 2.687 lbf/in.

spring 3 free length & spring constant:
L_{03} = 2.5 in. k_3 = 3.465 lbf/in.

Based on these values, the coefficients in equations (23), (31), and (34) were evaluated to yield the three equations

$$\begin{aligned}&100\,L_3^4 + [(-L_1^2 + 96)\,L_2^2 + 44\,L_1^2 - 52224]\,L_3^2 + L_1^2\,L_2^4\\ &+ (L_1^4 - 440\,L_1^2 - 13824)\,L_2^2 - 8624\,L_1^2 + 6773760 = 0 \end{aligned} \qquad (41)$$

$$\begin{aligned}&(2.5\,L_1\,L_2^2 + 90\,L_1 - 1600)\,L_3^3\\ &+ (-8.663\,L_1\,L_2^2 + 381.165\,L_1)\,L_3^2 + [-2.5\,L_1\,L_2^4\\ &+ (-6\,L_1^3 + 8\,L_1^2 + 1196\,L_1 - 768)\,L_2^2 + 44\,L_1^3\\ &- 352\,L_1^2 - 30664\,L_1 + 417792]\,L_3 + 8.663\,L_1\,L_2^4\end{aligned}$$

$$+ (17.326\ L_1^3 - 3811.651\ L_1)\ L_2^2 - 74708.354\ L_1 = 0 \quad (42)$$

$$[(2.5\ L_1^2 + 160)\ L_2 - 1074.637]\ L_3^3$$
$$+ (831.633 - 8.663\ L_1^2)\ L_2\ L_3^2 + [(-7\ L_1^2 + 192)\ L_2^3$$
$$+ (-515.826 + 5.373\ L_1^2)\ L_2^2 + (-2.5\ L_1^4 + 1188\ L_1^2 - 69888)\ L_2$$
$$- 236.420\ L_1^2 + 280609.161]\ L_3 + 17.326\ L_1^2\ L_2^3$$
$$+ (-119755.135 + 8.663\ L_1^4 - 3811.651\ L_1^2)\ L_2 = 0\ . \quad (43)$$

The continuation method was run on this set of three equations in three unknowns to obtain all solution sets for the three spring lengths, L_1, L_2, and L_3, for the particular numerical example. The software PHCpack (Verschelde, 1999) was used to implement the method.

The PHCpack software estimated the number of possible solutions to be 136. Seven real solutions were obtained and these are listed in Table 2.

Table 2: Seven Real Solutions for Three-Spring Planar Tensegrity System

Case	L_1	L_2	L_3
1	13.000	8.000	7.017
2	-11.376	-10.371	-5.333
3	-7.585	9.097	10.106
4	-11.029	12.557	-3.044
5	13.969	-5.800	9.164
6	14.248	-9.373	-4.774
7	13.181	11.599	-2.488

An equilibrium analysis was conducted for the seven cases and only the first case was indeed in equilibrium. Case 1 is shown in Figure 9.

4. CONCLUSIONS

The two-spring planar tensegrity system represents the simplest tensegrity device. It was remarkable that such a simple device would have such a complicated solution, i.e. a 28th degree polynomial was obtained in the variable L_1. It is true that conditions corresponding to maximum as well as minimum potential energy states are obtained in this formulation. The coefficients of this polynomial were obtained symbolically.

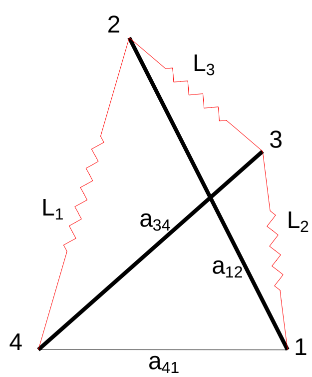

Figure 9: Case 1 ; Equilibrium Solution

The complexity of the three-spring system was such that it could not be solved symbolically. Further, an attempt to solve a numerical example using Sylvester's solution method was not successful. The numerical example was solved using the Continuation Method. However, of the seven real solutions for the three spring lengths, only one solution was found that was in equilibrium.

Future work would focus on examining other formulations of the three-spring tensegrity system to determine if a symbolic solution to the problem can be obtained.

5. ACKNOWLEDGEMENTS

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