Compliance Control of Planar Parallel Mechanisms with Adjustable Springs

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ABSTRACT

This paper presents compliance control of planar mechanisms consisting of rigid bodies connected by line springs. The compliance of the mechanism is described by the stiffness matrix which can be defined in terms of the geometry of the mechanism and the constituent spring properties. To control the compliance of the mechanism, the spring stiffness and free length properties are varied. Numerical examples are presented to support the results.

KEYWORDS: compliance, stiffness, parallel mechanisms

1. INTRODUCTION

Compliant mechanisms can be considered as spatial springs having multiple degrees of freedom. A small force/torque applied to the compliant mechanism generates a small displacement of the mechanism. This relation is well described by the compliance matrix of the mechanism. RCC (Remote Center of Compliance) devices, developed by Whitney [1], are one of the most successfully implemented compliant mechanisms. They have a unique compliant property at a specific operation point and are mainly used to compensate for positional errors during tasks such as inserting a peg into a chamfered hole. Compliant mechanisms can also be employed for force control applications by using the theory of Kinestatic Control which was proposed by Griffis [2]. In Griffis' novel work a passive compliant mechanism is inserted at the end of a robot manipulator, between the manipulator end effector and the environment. The necessary positional move of the manipulator end effector is determined that achieves the desired position and contact wrench between the compliant mechanism and the environment.

Mechanisms with variable compliance are believed to have several advantages over mechanisms having fixed compliance. Since RCC devices typically have a specific operation point, if the length of the peg to be inserted is changed, a different RCC device should be employed to do insertion tasks unless the

RCC device has variable compliance. As for force control tasks, each task may have an optimal compliance. With variable compliant mechanisms, several different tasks involving different force ranges can be accomplished without having to physically change out the compliant mechanism for each different task. Variable compliant mechanisms also can improve the performance of humanoid robot parts such as ankles and wrists. Animals are believed to have physically variable leg compliance which they utilize when running and hopping (Hurst et al. [3]).

Many compliant mechanisms including RCC devices have been designed as parallel kinematic mechanisms. Parallel kinematic mechanisms possess positive features compared to serial mechanisms such as higher stiffness, compactness, and smaller positional errors at the cost of a smaller workspace and increased complexity of analysis. The compliance matrix of a mechanism depends on the geometry of the mechanism and the properties of the constituent springs such as the stiffness coefficient and free length. To control the stiffness matrix, variable geometry and adjustable springs have been investigated. Simaan and Shoham [4] studied the stiffness synthesis problem using a variable geometry planar mechanism. Ryan et al. [5] designed a variable spring by changing the effective number of coils of the spring. Henrie [6] and McLachlan and Hall [7] studied cantilever beam-based variable compliant devices. The concepts of *twist* and *wrench* from screw theory, which was introduced by Ball [4] are employed throughout this paper to describe a small (or instantaneous) displacement of a rigid body and a force/torque applied to a body (Crane [9]).

2. PROBLEM STATEMENT

Planar mechanisms with variable compliance, specifically, compliant parallel mechanisms and mechanisms having two compliant parallel mechanisms in a serial arrangement are investigated (see Figures 1 and 2). These mechanisms may have a trade-off of characteristics relative to traditional parallel and serial mechanisms. The mechanisms consist of rigid bodies joined by adjustable compliant couplings. Each adjustable compliant coupling has a revolute joint at each end and a prismatic joint with an adjustable spring in the middle. The adjustable springs are assumed to be able to change their stiffness coefficient and free length and the mechanisms are in static equilibrium under an external wrench. It is desired to control the compliance of the mechanism while regulating the external wrench. To this end a compliant parallel mechanism is investigated in Section 3 and a mechanism having two compliant parallel mechanisms in a serial arrangement in Section 4. Both of the sections include information about the stiffness matrix of each mechanism, constraints on the stiffness matrix, compliance control, and numerical examples.

3. COMPLIANCE CONTROL OF PLANAR PARALLEL MECHANISM

(1) Stiffness matrix

Figure 1 illustrates a parallel mechanism having *N* compliant couplings. The mechanism is in static equilibrium under the external wrench $\underline{\mathbf{w}}_{ext}$ and thus

$$\begin{aligned}\underline{\mathbf{w}}_{ext} &= \sum_{i=1}^{N} \underline{\mathbf{f}}_i \\ &= \sum_{i=1}^{N} k_i (l_i - l_{oi}) \underline{\$}_i\end{aligned} \tag{1}$$

where $\underline{\mathbf{f}}_i$ is the spring wrench of ith compliant coupling and k_i, l_{oi}, and l_i are the spring constant, the spring free length, and the current spring length, respectively. In addition, $\underline{\$}_i$ represents the unitized planar Plücker coordinates of the line along the ith compliant coupling which may be written explicitly as

$$\underline{\$}_i = \begin{bmatrix} \underline{\mathbf{S}}_i \\ \underline{\mathbf{r}}_{P0,i} \times \underline{\mathbf{S}}_i \end{bmatrix} = \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ r_{x,i} \sin\theta_i - r_{y,i} \cos\theta_i \end{bmatrix} \tag{2}$$

where $r_{x,i}$ and $r_{y,i}$ are the pivot position of the ith compliant coupling in body E and θ_i is the rising angle of the ith compliant coupling (see Figure 3).

A derivative of the external wrench $\underline{\mathbf{w}}_{ext}$ may be written as

$$\begin{aligned}\delta \underline{\mathbf{w}}_{ext} &= \sum_{i=1}^{N} \delta \underline{\mathbf{f}}_i \\ &= \sum_{i=1}^{N} \left[k_i \delta\, l_i\, \underline{\$}_i + k (l_i - l_{oi}) \delta\, \underline{\$}_i \right] \\ &= \sum_{i=1}^{N} \left[k_i \delta\, l_i\, \underline{\$}_i + k_i (1 - \frac{l_{oi}}{l_i}) \frac{\partial \underline{\$}_i}{\partial \theta_i}\, l_i\ \delta\, \theta_i \right]\end{aligned} \tag{3}$$

Further δl_i and $l_i \delta\theta_i$ can be written as (see Jung et al. [10] for a detailed explanation)

$$\delta l_i = \underline{\$}_i^{T\ E} \delta \underline{\mathbf{D}}^A \tag{4}$$

$$l_i \delta\theta_i = \frac{\partial \underline{\mathbf{\$}}_i'^T}{\partial \theta_i} \, {}^E\delta\underline{\mathbf{D}}^A \tag{5}$$

where ${}^E\delta\underline{\mathbf{D}}^A$ is a small twist applied to body A and

$$\frac{\partial \underline{\mathbf{\$}}_i'}{\partial \theta_i} = \begin{bmatrix} \dfrac{\partial \underline{\mathbf{S}}_i}{\partial \theta_i} \\ {}^E\underline{\mathbf{r}}_{P1,i}^A \times \dfrac{\partial \underline{\mathbf{S}}_i}{\partial \theta_i} \end{bmatrix} \tag{6}$$

where ${}^E\underline{\mathbf{r}}_{P1,i}^A$ is the pivot position of the i^th^ compliant coupling in body A.

Eqs. (3)-(5) then lead to

$$\delta \underline{\mathbf{w}}_{ext} = \left(\sum_{i=1}^{N} [K_F]_i \right) {}^E\delta\underline{\mathbf{D}}^A \tag{7}$$

where

$$[K_F]_i = k_i \underline{\mathbf{\$}}_i \underline{\mathbf{\$}}_i^T + k_i (1 - \frac{l_{oi}}{l_i}) \frac{\partial \underline{\mathbf{\$}}_i}{\partial \theta_i} \frac{\partial \underline{\mathbf{\$}}_i'^T}{\partial \theta_i} \tag{8}$$

$$\frac{\partial \underline{\mathbf{\$}}_i}{\partial \theta_i} = \begin{bmatrix} -\sin\theta_i \\ \cos\theta_i \\ r_{x,i}\cos\theta_i + r_{y,i}\sin\theta_i \end{bmatrix}. \tag{9}$$

The stiffness matrix of the mechanism $[K]$ which maps a small twist of body A ${}^E\delta\underline{\mathbf{D}}^A$ into a small wrench variation $\delta\,\underline{\mathbf{w}}_{ext}$ can be written from Eq. (7) as

$$[K] = \sum_{i=1}^{N} [K_F]_i \, . \tag{10}$$

(2) Constraints on the stiffness matrix

Ciblak and Lipkin [11] showed that the stiffness matrix of a compliant parallel mechanism can be decomposed into a symmetric and a skew symmetric part and that the skew symmetric part is negative one-half the externally applied load expressed as a spatial cross product operator (see Featherstone [14]). For planar mechanisms, the skew symmetric part can be written as

$$
\begin{aligned}
[K] &= \frac{[K]+[K]^T}{2} + \frac{[K]-[K]^T}{2} \\
&= [K]_{Symmetric} + [K]_{Skew\ Symmetric}
\end{aligned} \tag{11}
$$

$$
[K]_{Skew\ Symmetric} = \frac{[K]-[K]^T}{2} = -\frac{1}{2}\begin{bmatrix} 0 & 0 & f_y \\ 0 & 0 & -f_x \\ -f_y & f_x & 0 \end{bmatrix} \tag{12}
$$

where $\underline{\mathbf{w}}_{ext} = \left[f_x,\ f_y, m_z \right]^T$ is the external wrench.

It is important to note that no matter how many compliant couplings are connected and no matter how the spring constants and the free lengths of the constituent compliant couplings are changed, the stiffness matrix of a compliant parallel mechanism contains only six independent variables and considering Eqs. (13) and (14) the stiffness matrix may be rewritten as

$$
[K] = \begin{bmatrix} K_{11} & K_{12} & K_{13} \\ K_{12} & K_{22} & K_{32}+f_x \\ K_{13}+f_y & K_{32} & K_{33} \end{bmatrix}. \tag{15}
$$

From Eqs. (6)-(10) the six independent elements of the stiffness matrix $[K]$ can be explicitly written as

$$
K_{11} = \sum_{i=1}^{N}\left(k_i - k_i \frac{l_{oi}}{l_i} \sin^2\theta_i \right) \tag{16}
$$

$$
K_{12} = \sum_{i=1}^{N}\left(k_i \frac{l_{oi}}{l_i} \sin\theta_i \cos\theta_i \right) \tag{17}
$$

$$
K_{13} = \sum_{i=1}^{N}\left(-k_i (l_i \sin\theta_i + r_{y,i}) + k_i \frac{l_{oi}}{l_i}(l_i \sin\theta_i + r_{x,i}\sin\theta_i\cos\theta_i + r_{y,i}\sin^2\theta_i) \right) \tag{18}
$$

$$
K_{22} = \sum_{i=1}^{N}\left(k_i - k_i \frac{l_{oi}}{l_i} \cos^2\theta_i \right) \tag{19}
$$

$$
K_{32} = \sum_{i=1}^{N}\left(k_i r_{x,i} - k_i \frac{l_{oi}}{l_i}(r_{x,i}\cos^2\theta_i + r_{y,i}\sin\theta_i\cos\theta_i) \right) \tag{20}
$$

$$
K_{33} = \sum_{i=1}^{N}\left(\begin{aligned} & k_i r_{x,i}(l_i\cos\theta_i + r_{x,i}) + k_i r_{y,i}(l_i\sin\theta_i + r_{y,i}) \\ & \qquad - k_i \frac{l_{oi}}{l_i} r_{x,i}(l_i\cos\theta_i + r_{y,i}\sin\theta_i\cos\theta_i + r_{x,i}\cos^2\theta_i) \\ & \qquad - k_i \frac{l_{oi}}{l_i} r_{y,i}(l_i\sin\theta_i + r_{x,i}\sin\theta_i\cos\theta_i + r_{y,i}\sin^2\theta_i) \end{aligned} \right). \tag{21}
$$

(3) Compliance control with the pose of the mechanism fixed

In this case it is desired to find an appropriate set of spring constants and free lengths of the constituent compliant couplings of the mechanism shown in Figure 1 to implement a given stiffness matrix and to regulate a given external wrench.

It is important to note that the stiffness matrix contains only six independent variables and the equations for the independent variables are linear in terms of k_i 's and $k_i l_{oi}$'s as shown in Eqs. (15)-(21) since all the geometrical terms are constant. In addition to the equations for the stiffness matrix, the system should satisfy the static equilibrium equations and from Eqs. (1) and (2) it can be written that

$$\underline{\mathbf{w}}_{ext} = \begin{bmatrix} f_x \\ f_y \\ m_z \end{bmatrix} = \sum_{i=1}^{N} \left(k_i \left(l_i - l_{oi} \right) \begin{bmatrix} \cos\theta_i \\ \sin\theta_i \\ r_{x,i} \sin\theta_i - r_{y,i} \cos\theta_i \end{bmatrix} \right). \tag{22}$$

Eq. (22) consists of three equations which are also linear in terms of k_i 's and $k_i l_{oi}$'s. Since there are nine linear equations to be satisfied and each adjustable compliant coupling possesses two control variables such as the spring constant and free length, at least five adjustable compliant couplings are required.

The nine equations may be written in matrix form as

$$[A]\underline{\mathbf{X}} = \underline{\mathbf{B}} \tag{23}$$

where

$$\underline{\mathbf{B}} = \left[K_{11}, K_{12}, K_{13}, K_{22}, K_{32}, K_{33}, f_x, f_y, m_z \right]^T \tag{24}$$

$$\underline{\mathbf{X}} = \left[k_1, k_2, \cdots, k_N, k_1 l_{o1}, k_2 l_{o2}, \cdots, k_N l_{oN} \right]^T \tag{25}$$

$$[A] = \begin{bmatrix} 1 & 1 & \cdots & 1 & G_{1,1} & G_{1,2} & \cdots & G_{1,N} \\ 0 & 0 & \cdots & 0 & G_{2,1} & G_{2,2} & \cdots & G_{2,N} \\ G_{3,1} & G_{3,2} & \cdots & G_{3,N} & G_{4,1} & G_{4,2} & \cdots & G_{4,N} \\ 1 & 1 & \cdots & 1 & G_{5,1} & G_{5,2} & \cdots & G_{5,N} \\ G_{6,1} & G_{6,2} & \cdots & G_{6,N} & G_{7,1} & G_{7,2} & \cdots & G_{7,N} \\ G_{8,1} & G_{8,2} & \cdots & G_{8,N} & G_{9,1} & G_{9,2} & \cdots & G_{9,N} \\ H_{1,1} & H_{1,2} & \cdots & H_{1,N} & H_{2,1} & H_{2,2} & \cdots & H_{2,N} \\ H_{3,1} & H_{3,2} & \cdots & H_{3,N} & H_{4,1} & H_{4,2} & \cdots & H_{4,N} \\ H_{5,1} & H_{5,2} & \cdots & H_{5,N} & H_{6,1} & H_{6,2} & \cdots & H_{6,N} \end{bmatrix} \tag{26}$$

and where

$$G_{1,i} = -\frac{\sin^2\theta_i}{l_i},\ G_{2,i} = \frac{\sin\theta_i\cos\theta_i}{l_i},\ G_{3,i} = -l_i\sin\theta_i - r_{y,i}$$

$$G_{4,i} = \sin\theta_i + \frac{r_{x,i}\sin\theta_i\cos\theta_i + r_{y,i}\sin^2\theta_i}{l_i},\ G_{5,i} = -\frac{\cos^2\theta_i}{l_i}$$

$$G_{6,i} = r_{x,i},\quad G_{7,i} = \frac{-r_{x,i}\cos^2\theta_i - r_{y,i}\sin\theta_i\cos\theta_i}{l_i}$$

$$G_{8,i} = r_{x,i}{}^2 + r_{y,i}{}^2 + l_i(r_{x,i}\cos\theta_i + r_{y,i}\sin\theta_i)$$

$$G_{9,i} = -r_{x,i}\cos\theta_i - r_{y,i}\sin\theta_i - \frac{r_{x,i}{}^2\cos^2\theta_i + r_{y,i}{}^2\sin^2\theta_i + 2r_{x,i}r_{y,i}\sin\theta_i\cos\theta_i}{l_i}$$

$$H_{1,i} = l_i\cos\theta_i,\ H_{2,i} = -\cos\theta_i,\ H_{3,i} = l_i\sin\theta_i,\ H_{4,i} = -\sin\theta_i$$

$$H_{5,i} = l_i\left(r_{x,i}\sin\theta_i - r_{y,i}\cos\theta_i\right),\ H_{6,i} = -\left(r_{x,i}\sin\theta_i - r_{y,i}\cos\theta_i\right).$$

It is important to note that $\left[A\right]$, $\underline{\mathbf{X}}$, and $\underline{\mathbf{B}}$ are 9 × (2**N*), (2**N*) × 1, and 9 × 1 matrices, respectively, where *N* denotes the number of the adjustable compliant couplings.

It is required to solve Eq. (23) where the number of columns of matrix $\left[A\right]$ is in general greater than the number of rows and the general solution $\underline{\mathbf{X}}_{sol}$ can be written as

$$\begin{aligned} \underline{\mathbf{X}}_{sol} &= \underline{\mathbf{X}}_p + \underline{\mathbf{X}}_h \\ &= \underline{\mathbf{X}}_p + \left[A_{Null}\right]\underline{\mathbf{C}} \end{aligned} \tag{27}$$

where $\underline{\mathbf{X}}_p$, $\underline{\mathbf{X}}_h$, $\left[A_{Null}\right]$, and $\underline{\mathbf{C}}$ are the particular solution, the homogeneous solution, the null space of matrix $\left[A\right]$, and the coefficient column matrix, respectively (see Strang [12]). Once a solution $\underline{\mathbf{X}}_{sol}$ is obtained, the l_{oi} terms are calculated from the k_i's and $k_i l_{oi}$'s in $\underline{\mathbf{X}}_{sol}$. It is important to note that $\left[A_{Null}\right]$ is (2**N*) × (2**N*-9) matrix and $\underline{\mathbf{C}}$ is (2**N*-9) × 1 column matrix.

There might be many strategies to select the coefficient column matrix $\underline{\mathbf{C}}$ which leads to a specific solution. For instance, if the norm of $\underline{\mathbf{X}}_{sol}$ is desired to be minimized, then by using a projection matrix $\left[A_{Null-P}\right]$ (see Strang [12]), the solution can be obtained as

$$\underline{\mathbf{X}}_{Min.sol} = \underline{\mathbf{X}}_p + \left[A_{Null-P}\right]\left(-\underline{\mathbf{X}}_p\right) \tag{28}$$

where

$$\left[A_{Null-P}\right]=\left[A_{Null}\right]\left(\left[A_{Null}\right]^T\left[A_{Null}\right]\right)^{-1}\left[A_{Null}\right]^T . \tag{29}$$

For another case, it may be desired that the solution is closest to a desired solution $\underline{\mathbf{X}}_d$ which may be constructed from operation ranges of adjustable springs, for instance the minimum and maximum stiffness coefficients and free lengths. Then the solution can be obtained as

$$\underline{\mathbf{X}}_{d.sol}=\underline{\mathbf{X}}_p+\left[A_{Null-P}\right]\left(\underline{\mathbf{X}}_d-\underline{\mathbf{X}}_p\right). \tag{30}$$

Unfortunately these methods involve mixed unit problems and do not guarantee a solution consisting of only positive stiffness coefficients and free lengths. A necessary and sufficient condition for there to always be a solution $\underline{\mathbf{X}}$ with positive components for any given $\underline{\mathbf{B}}$ is that there is a vector in the left null space of $[A]$ whose components are all positive (see Roberts et al. [15]).

A numerical example is presented. An external wrench $\underline{\mathbf{w}}_{ext}$ and a desired stiffness matrix $[K]$ are given as

$$\underline{\mathbf{w}}_{ext}=\left[-1.8832\ N \quad -2.8805\ N \quad 3.2851\ Ncm\right]$$

$$[K]=\begin{bmatrix} 0.0216\ \ N/cm & 2.2483\ \ N/cm & -2.2750\ \ N \\ 2.2483\ \ N/cm & 25.3914\ \ N/cm & 60.9800\ \ N \\ -5.1555\ \ N & 62.8632\ \ N & 270.4409\ \ Ncm \end{bmatrix}.$$

The geometry information of the mechanism shown in Figure 1 is given in Tables 1 and 2 where the mechanism is assumed to have five compliant couplings.

Table 1. Positions of pivot points in body E

Pivot points	E1	E2	E3	E4	E5
X	0.0000	0.6000	2.5000	3.9000	5.3000
Y	0.0000	0.8000	0.3000	0.9000	0.0000

(Unit: cm)

Table 2. Positions of pivot points in body A

Pivot points	A1	A2	A3	A4	A5
X	0.6000	1.4055	2.6736	3.3368	4.7284
Y	4.5000	2.7447	3.3209	3.9614	4.1442

(Unit: cm)

The spring parameters which have the minimum norm and satisfy the given conditions can be obtained by using Eq. (28) and these values are shown in Table 3.

Table 3. Spring parameters with minimum norm

Spring No.	1	2	3	4	5

Stiffness constant k	4.6674	7.2485	3.5188	5.0243	6.3280
Free length l_o	4.1678	2.1490	6.3995	1.9322	3.9104

(Unit: N/cm for k, cm for l_o)

The spring parameters which are closest to the given spring parameters as shown in Table 4 can be obtained by applying Eq. (30) and it is shown in Table 5.

Table 4. Given optimal spring parameters

Spring No.	1	2	3	4	5
Stiffness constant k	5.0	5.0	5.0	5.0	5.0
Free length l_o	3.0	3.0	3.0	3.0	3.0

(Unit: N/cm for k, cm for l_o)

Table 5. Spring parameters closest to given spring parameters

Spring No.	1	2	3	4	5
Stiffness constant k	4.8664	6.8783	3.8968	4.8990	6.2974
Free length l_o	4.3386	2.3374	5.0230	2.1667	4.0492

(Unit: N/cm for k, cm for l_o)

Two sets of the spring parameters, one in Table 3 and the other in Table 5, implement the given wrench and stiffness matrix.

(4) Compliance control without constraint on the pose of the mechanism

In this case the pose of body A is not constrained as fixed. A change of the pose of body A, which is considered to be in contact with the environment, may be compensated by attaching body E to the end of a robot system and by controlling the position of the robot end effector in a similar manner as described in the Theory of Kinestatic Control proposed by Griffis [2]. As presented in the previous section there are nine values to be fulfilled: six from the stiffness matrix and three from the wrench equations. A typical planar parallel mechanism which has three couplings is investigated since the mechanism has nine control input variables which is same with that of values to be fulfilled: six from the adjustable compliant couplings and three from the planar displacement of body A. The target variables may be expressed in matrix form as $\underline{\mathbf{B}}$ in Eq. (24) and control input variables $\underline{\mathbf{U}}$ may be written in matrix form as

$$\underline{\mathbf{U}} = \left[k_1, k_2, k_3, l_{o1}, l_{o2}, l_{o3}, x_o, y_o, \phi\right]^T \tag{31}$$

where the k_i 's and l_{oi} 's are the stiffness coefficient and free length of the ith compliant coupling, respectively. In addition, x_o and y_o are the position of point O in body A, which is coincident with the origin of the inertial frame E, and ϕ is the rotation angle of body A with respect to ground.

The stiffness matrix equations and wrench equations are highly nonlinear in terms of the displacement of the bodies as shown in Eqs. (15)-(22). In this section a derivative of the target variables $\underline{\mathbf{B}}$ with respect to input variables $\underline{\mathbf{U}}$ is derived and the derivative is used to obtain the small change of input variables for the desired small change of target values and it may be written as

$$\delta\underline{\mathbf{B}} = \frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}\delta\underline{\mathbf{U}} \tag{32}$$

$$\delta\underline{\mathbf{U}} = \left(\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}\right)^{-1}\delta\underline{\mathbf{B}} \tag{33}$$

where

$$\delta\underline{\mathbf{B}} = [\delta K_{11}, \delta K_{12}, \delta K_{13}, \delta K_{22}, \delta K_{32}, \delta K_{33}, \delta f_x, \delta f_y, \delta m_z]^T \tag{34}$$

$$\delta\underline{\mathbf{U}} = [\delta k_1, \delta k_2, \delta k_3, \delta l_{o1}, \delta l_{o2}, \delta l_{o3}, \delta x_o, \delta y_o, \delta\phi]^T \tag{35}$$

$$\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}} = \begin{bmatrix} \frac{\partial B_1}{\partial U_1} & \cdots & \frac{\partial B_1}{\partial U_9} \\ \vdots & \ddots & \vdots \\ \frac{\partial B_9}{\partial U_1} & \cdots & \frac{\partial B_9}{\partial U_9} \end{bmatrix}. \tag{36}$$

For instance, δB_1 can be written as

$$\begin{aligned} \delta B_1 &= \frac{d\,B_1}{d\,\underline{\mathbf{U}}}\delta\underline{\mathbf{U}} \\ &= \frac{\partial K_{11}}{\partial k_1}\delta k_1 + \frac{\partial K_{11}}{\partial k_2}\delta k_2 + \frac{\partial K_{11}}{\partial k_3}\delta k_3 + \frac{\partial K_{11}}{\partial l_{o1}}\delta l_{o1} + \frac{\partial K_{11}}{\partial l_{o2}}\delta l_{o2} + \frac{\partial K_{11}}{\partial l_{o3}}\delta l_{o3}\,. \\ &\quad + \frac{\partial K_{11}}{\partial x_o}\delta x_o + \frac{\partial K_{11}}{\partial y_o}\delta y_o + \frac{\partial K_{11}}{\partial \phi}\delta\phi \end{aligned} \tag{37}$$

In Eqs. (15)-(22) all elements of $\underline{\mathbf{B}}$ were presented as functions of not $\underline{\mathbf{U}}$ but $\underline{\mathbf{U}}_p$ which is defined as

$$\underline{\mathbf{U}}_p = \left[k_1, k_2, k_3, l_{o1}, l_{o2}, l_{o3}, l_1, l_2, l_3, \theta_1, \theta_2, \theta_3\right]^T. \tag{38}$$

Hence among Eq. (37) $\frac{\partial K_{11}}{\partial x_o}$, $\frac{\partial K_{11}}{\partial y_o}$, and $\frac{\partial K_{11}}{\partial \phi}$ are not obtained from simple differentiation.

Since $\underline{\mathbf{B}}$ is a function of $\underline{\mathbf{U}}_p$, δB_1 can also be written as

$$\delta B_1 = \frac{d\,B_1}{d\,\underline{\mathbf{U}}_p}\delta\underline{\mathbf{U}}_p$$
$$= \frac{\partial K_{11}}{\partial k_1}\delta k_1 + \frac{\partial K_{11}}{\partial k_2}\delta k_2 + \frac{\partial K_{11}}{\partial k_3}\delta k_3 + \frac{\partial K_{11}}{\partial l_{o1}}\delta l_{o1} + \frac{\partial K_{11}}{\partial l_{o2}}\delta l_{o2} + \frac{\partial K_{11}}{\partial l_{o3}}\delta l_{o3} \,. \qquad (39)$$
$$+ \frac{\partial K_{11}}{\partial l_1}\delta l_1 + \frac{\partial K_{11}}{\partial l_2}\delta l_2 + \frac{\partial K_{11}}{\partial l_3}\delta l_3 + \frac{\partial K_{11}}{\partial \theta_1}\delta \theta_1 + \frac{\partial K_{11}}{\partial \theta_2}\delta \theta_2 + \frac{\partial K_{11}}{\partial \theta_3}\delta \theta_3$$

In addition, the $\delta\, l_i$'s and $\delta\,\theta_i$'s in Eq. (39) can be substituted using Eqs. (4) and (5).

By doing the above substitutions in Eq. (39) and collecting the coefficients according to each term of $\delta\underline{\mathbf{U}}$, $\delta\, B_1$ can be written as

$$\delta B_1 = \begin{bmatrix} \frac{\partial K_{11}}{\partial k_1} & \frac{\partial K_{11}}{\partial k_2} & \frac{\partial K_{11}}{\partial k_3} & \frac{\partial K_{11}}{\partial l_{o1}} & \frac{\partial K_{11}}{\partial l_{o2}} & \frac{\partial K_{11}}{\partial l_{o3}} & \frac{\partial K_{11}}{\partial x_o} & \frac{\partial K_{11}}{\partial y_o} & \frac{\partial K_{11}}{\partial \phi} \end{bmatrix} \begin{bmatrix} \delta k_1 \\ \delta k_2 \\ \delta k_3 \\ \delta l_{o1} \\ \delta l_{o2} \\ \delta l_{o3} \\ \delta x_o \\ \delta y_o \\ \delta\phi \end{bmatrix} . \qquad (40)$$
$$= \frac{d\,B_1}{d\,\underline{\mathbf{U}}}\delta\underline{\mathbf{U}}$$

All terms of $\delta\underline{\mathbf{B}}$ may be obtained in the same way in which $\delta\, B_1$ was obtained and $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ can be derived by combining all $\frac{d\,B_i}{d\,\underline{\mathbf{U}}}$'s.

A numerical example is presented. The mechanism shown in Figure 1 is in static equilibrium under the external wrench $\underline{\mathbf{w}}_{ext}$ and the geometry information and the spring parameters are given below. The mechanism is assumed to have three compliant couplings.

Table 6. Positions of pivot points

Pivot points	E1	E2	E3	A1	A2	A3
X	0.0000	0.6000	2.5000	0.6000	1.4055	2.6736
Y	0.0000	0.8000	0.2000	4.5000	2.7447	3.3209

(Unit: cm)

Table 7. Initial spring parameters

Spring No.	1	2	3
Stiffness constant k	5.5	5.7	5.1
Free length l_o	4.8	3.1	2.0

(Unit: N/cm for k, cm for l_o)

The external wrench $\underline{\mathbf{w}}_{ext}$ and the initial stiffness matrix $[K]_I$ are calculated from the geometry of the mechanism and the spring parameters;

$$\underline{\mathbf{w}}_{ext} = [-2.0409\ N \quad -0.9263\ N \quad 12.8594\ Ncm]$$

$$[K]_I = \begin{bmatrix} 0.1679 \quad N/cm & 3.9107 \quad N/cm & 3.9623 \quad N \\ 3.9107 \quad N/cm & 14.9590 \quad N/cm & 10.9558 \quad N \\ 3.0360 \quad N & 12.9966 \quad N & 25.9764 \quad Ncm \end{bmatrix}.$$

The desired stiffness matrix $[K]_D$ is given below.

$$[K]_D = \begin{bmatrix} 0.6679 \quad N/cm & 4.3107 \quad N/cm & 4.1823 \quad N \\ 4.3107 \quad N/cm & 15.4290 \quad N/cm & 10.8358 \quad N \\ 3.2560 \quad N & 12.8766 \quad N & 26.3764 \quad Ncm \end{bmatrix}.$$

Since the difference between the desired stiffness matrix and the initial stiffness matrix is not at all close to infinitesimal, the difference is divided into a number of small $\delta\underline{\mathbf{B}}$'s and Eq. (33) is applied repeatedly to obtain the spring parameters and the displacement of body B which implement the desired stiffness matrix and the given wrench.

The calculated spring parameters and the pose of body A are shown in Table 8 and Table 9 and the initial and final pose of the mechanism is shown in Figure 4.

Table 8. Calculated spring parameters

Spring No.	1	2	3
Stiffness constant k	6.2563	5.5311	5.1492
Free length l_o	5.4810	4.3584	3.1954

(Unit: N/cm for k, cm for l_o)

Table 9. Positions of pivot points in body A

Pivot points	A1	A2	A3
X	0.8201	1.9165	3.0661
Y	5.2909	3.7010	4.4874

(Unit: cm)

4. COMPLIANCE CONTROL OF MECHANISM HAVING TWO PLANAR COMPLIANT PARALLEL MECHANISMS IN SERIES

In this section a mechanism having two planar compliant parallel mechanisms that are serially arranged as shown in Figure 2 is investigated. The mechanism is assumed to be in static equilibrium under the external wrench $\underline{\mathbf{w}}_{ext}$.

(1) Stiffness matrix

The stiffness matrix of the mechanism was derived by Jung et al. [10] [13] and is restated as

$$[K] = [K_F]_{R,L} \left([K_F]_{R,L} + [K_F]_{R,U} + (\underline{\mathbf{w}}_{ext} \times) \right)^{-1} [K_F]_{R,U} \quad (41)$$

where

$$[K_F]_{R,L} = \sum_{i=4}^{6} [K_F]_i \quad (42)$$

$$[K_F]_{R,U} = \sum_{i=1}^{3} [K_F]_i \ . \quad (43)$$

$[K_F]_i$ was defined in Eq. (8) and $\underline{\mathbf{w}}_{ext} \times$ is the external wrench expressed as a spatial cross product operator (see Featherstone [14]) which may be written in the planar case as

$$[K_M]_{R,U} = \begin{bmatrix} 0 & 0 & -f_y \\ 0 & 0 & f_x \\ f_y & -f_x & 0 \end{bmatrix} . \quad (44)$$

(2) Constraints on stiffness matrix

Applying the constraint presented by Ciblak and Lipkin [11], $[K_F]_{R,L}$ and $[K_F]_{R,U}$ may be written as

$$[K_F]_{R,L} = \begin{bmatrix} K_{11}^L & K_{12}^L & K_{13}^L \\ K_{12}^L & K_{22}^L & K_{32}^L + f_x \\ K_{13}^L + f_y & K_{32}^L & K_{33}^L \end{bmatrix} \quad (45)$$

$$[K_F]_{R,U} = \begin{bmatrix} K_{11}^U & K_{12}^U & K_{13}^U \\ K_{12}^U & K_{22}^U & K_{32}^U + f_x \\ K_{13}^U + f_y & K_{32}^U & K_{33}^U \end{bmatrix} \quad (46)$$

where $\underline{\mathbf{w}}_{ext} = [\, f_x,\ f_y,\ m_z \,]^T$ is the external wrench.

Substituting Eqs. (44)-(46) in Eq. (41) and carrying out a symbolic operation using Maple® software lead to

$$[K]-[K]^T = \begin{bmatrix} 0 & 0 & -f_y \\ 0 & 0 & f_x \\ f_y & -f_x & 0 \end{bmatrix}$$

which is identical to the statement by Ciblak and Lipkin [11] for planar compliant parallel mechanisms. This result indicates that mechanisms having two planar compliant parallel mechanisms in a serial arrangement also contain only six independent variables.

(3) Compliance control

Since the stiffness matrix of the mechanism shown in Figure 2 is complicated and nonlinear in terms of the spring parameters and the displacement of the constituent rigid bodies, a derivative of the stiffness matrix and the static equilibrium equation is derived and applied for compliance control of the mechanism.

The independent stiffness matrix elements and the external wrench may be written in matrix form as

$$\underline{\mathbf{B}} = \left[K_{11}, K_{12}, K_{13}, K_{22}, K_{32}, K_{33}, f_x^{\ A}, f_y^{\ A}, m_z^{\ A}, f_x^{\ B}, f_y^{\ B}, m_z^{\ B} \right]^T \tag{47}$$

where $\underline{\mathbf{w}}^A = \left[f_x^{\ A}, f_y^{\ A}, m_z^{\ A} \right]^T$ and $\underline{\mathbf{w}}^B = \left[f_x^{\ B}, f_x^{\ B}, m_z^{\ B} \right]^T$ are the wrenches from the compliant couplings connecting body A to ground and from the couplings connecting body B to body A, respectively.

The spring parameters and the displacements of the rigid bodies may be written as

$$\underline{\mathbf{U}} = [k_1, k_2, k_3, k_4, k_5, k_6, l_{o1}, l_{o2}, l_{o3}, l_{o4}, l_{o5}, l_{o6}, x_o^{\ A}, y_o^{\ A}, \phi^A, x_o^{\ B}, y_o^{\ B}, \phi^B]^T \tag{48}$$

where k_i 's and l_{oi} 's are the stiffness coefficient and free length of ith compliant coupling, respectively. In addition, $x_o^{\ A}$ and $y_o^{\ A}$ are the position of point O in body A which is coincident with the origin of the inertial frame E and ϕ^A is the rotation angle of body A with respect to ground. $x_o^{\ B}$, $y_o^{\ B}$, and ϕ^B are defined in the same way in terms of the inertial frame E.

The stiffness matrix and the wrench can also be expressed as functions of $\underline{\mathbf{U}}_p$ which is defined as

$$\begin{aligned} \underline{\mathbf{U}}_p = [&k_1, k_2, k_3, k_4, k_5, k_6, l_{o1}, l_{o2}, l_{o3}, l_{o4}, l_{o5}, l_{o6}, l_1, l_2, l_3, l_4, l_5, l_6, \\ &\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, r_{x,1}, r_{x,2}, r_{x,3}, r_{y,1}, r_{y,2}, r_{y,3}]^T \end{aligned} \tag{49}$$

where the l_i 's and θ_i 's are the current spring length and rising angle of ith compliant coupling. In addition $r_{x,i}$ and $r_{y,i}$ are the pivot positions of ith compliant coupling in body A.

A similar approach taken in Section 3 is applied to obtain a derivative of the stiffness matrix and the wrench $\frac{d\underline{\mathbf{B}}}{d\underline{\mathbf{U}}}$: each element of $\underline{\mathbf{B}}$ is differentiated with respect to $\underline{\mathbf{U}}_p$ and the terms not belonging to $\delta\underline{\mathbf{U}}$ are substituted in the terms of $\delta\underline{\mathbf{U}}$. In other words the δl_i 's, $\delta\theta_i$'s, $\delta r_{x,i}$'s, and $\delta r_{y,i}$'s are expressed in terms of the twists of the bodies. The coefficient of each term of $\delta\underline{\mathbf{U}}$ corresponds to an element of the derivative matrix in an analogous way to Eq. (40).

Springs 4, 5, and 6 connect body A and ground and thus δl_i and $\delta\theta_i$ for i=4, 5, 6 can be written in a similar way to Eqs. (4) and (5) as

$$\delta l_i = \underline{\$}_i^T \, {}^E\delta\underline{\mathbf{D}}^A = \underline{\$}_i^T \begin{bmatrix} \delta x_o{}^A \\ \delta y_o{}^A \\ \delta \phi^A \end{bmatrix} \tag{50}$$

$$\delta\theta_i = \frac{1}{l_i}\frac{\partial \underline{\$}_i^{'}}{\partial\theta_i}^T \, {}^E\delta\underline{\mathbf{D}}^A = \frac{1}{l_i}\frac{\partial \underline{\$}_i^{'}}{\partial\theta_i}^T \begin{bmatrix} \delta x_o{}^A \\ \delta y_o{}^A \\ \delta \phi^A \end{bmatrix}. \tag{51}$$

Since springs 1, 2, and 3 join body B and body A, δl_i and $\delta\theta_i$ for i=1, 2, 3 may be expressed as

$$\delta l_i = \underline{\$}_i^T \, {}^A\delta\underline{\mathbf{D}}^B = \underline{\$}_i^T \begin{bmatrix} \delta x_o{}^B - \delta x_o{}^A \\ \delta y_o{}^B - \delta y_o{}^A \\ \delta \phi^B - \delta\phi^A \end{bmatrix} \tag{52}$$

$$\delta\theta_i = \frac{1}{l_i}\frac{\partial \underline{\$}_i^{'}}{\partial\theta_i}^T \, {}^A\delta\underline{\mathbf{D}}^B + \delta\phi^A = \frac{1}{l_i}\frac{\partial \underline{\$}_i^{'}}{\partial\theta_i}^T \begin{bmatrix} \delta x_o{}^B - \delta x_o{}^A \\ \delta y_o{}^B - \delta y_o{}^A \\ \delta \phi^B - \delta\phi^A \end{bmatrix} + \delta\phi^A. \tag{53}$$

Lastly the $\delta r_{x,i}$'s, and $\delta r_{y,i}$'s are the positions of the pivot point in body A and by using the twist equation it can be written as

$$\begin{bmatrix} \delta r_{x,i} \\ \delta r_{y,i} \\ 0 \end{bmatrix} = \begin{bmatrix} \delta x_o{}^A \\ \delta y_o{}^A \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \delta\phi^A \end{bmatrix} \times \begin{bmatrix} r_{x,i} \\ r_{y,i} \\ 0 \end{bmatrix}. \tag{54}$$

Now all terms in the differential of $\underline{\mathbf{B}}$ are expressed in terms of $\delta\underline{\mathbf{U}}$ and writing it in matrix form gives

$$\delta\underline{\mathbf{B}} = \frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}\delta\underline{\mathbf{U}} \tag{55}$$

where $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ is ["a"] 12 × 18 matrix.

It is required to obtain the small change of input values $\delta\underline{\mathbf{U}}$ corresponding to a small change of stiffness matrix and wrench $\delta\underline{\mathbf{B}}$. Since the number of columns of the matrix is greater than the number of rows it is a redundant system. There are in general an infinite number of solutions and a variety of constraints may be imposed on the system.

Since $\delta\underline{\mathbf{U}}$ is the change from the current values, minimizing the norm of $\delta\underline{\mathbf{U}}$ may be one reasonable option. Then in a similar way to Eq. (28) $\delta\underline{\mathbf{U}}_{\min}$ may be obtained as

$$\delta\underline{\mathbf{U}}_{\min} = \delta\underline{\mathbf{U}}_{p.sol} + \left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]\left(\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]^T\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]\right)^{-1}\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]^T\left(-\delta\underline{\mathbf{U}}_{p.sol}\right) \tag{56}$$

where $\delta\underline{\mathbf{U}}_{p.sol}$ is a particular solution of Eq. (55) and $\left[\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}_{Null}\right]$ is the null space of matrix $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ (see Strang [12]).

Body B is considered to be in contact with the environment and it may be required to maintain the pose of body B. Fixing the pose of body B indicates that the twist of body B is equal to zero which may be written as

$$\delta x_o{}^B = 0\,,\ \delta y_o{}^B = 0\,,\ \delta\theta^B = 0\,.$$

This constraint can implement by removing the last three columns of $\frac{d\,\underline{\mathbf{B}}}{d\,\underline{\mathbf{U}}}$ and the last three rows of $\delta\underline{\mathbf{U}}$ and solving the problem in a similar way to that of the previous problem since the system is still redundant.

If both of the bodies are required to be stationary then the twists of the bodies should be zero and it may be written that

$$\delta x_o{}^A = 0\,,\ \delta y_o{}^A = 0\,,\ \delta\theta^A = 0\,,$$

$$\delta x_o^{\ B} = 0\ ,\ \delta y_o^{\ B} = 0\ ,\ \delta \theta^B = 0\ .$$

This can be implemented by removing the last six columns of $\frac{d\,\mathbf{B}}{d\,\underline{\mathbf{U}}}$ and the last six rows of $\delta \underline{\mathbf{U}}$ and solving the problem which is not redundant.

(4) Numerical example

The geometry information and spring parameters of the mechanism shown in Figure 2 and the external wrench $\underline{\mathbf{w}}_{ext}$ are given below.

$$\underline{\mathbf{w}}_{ext} = \left[-1.7\,N \quad 2.5\,N \quad 12.7\,N\right]$$

Table 10. Spring parameters of the compliant couplings

Spring No.	1	2	3	4	5	6
Stiffness constant k	5.0	5.0	5.0	5.0	5.0	5.0
Free length l_o	3.0614	0.6791	2.3608	2.8657	0.7258	1.2732

(Unit: N/cm for k, cm for l_o)

Table 11. Positions of pivot points

Pivot points	E1	E2	E3	B1	B2	B3
X	0.0000	1.0000	3.0000	-0.8000	0.4453	1.1965
Y	0.0000	0.8000	0.0000	4.5000	3.7726	4.6186

(Unit: cm)

Table 4-11. Continued.

A1	A2	A3	A4
0.2000	1.1261	2.1646	1.2760
2.3000	1.8179	1.9252	2.6038

The initial stiffness matrix $\left[K\right]_I$ is calculated from the geometry of the mechanism and the spring parameters as

$$[K]_I = \begin{bmatrix} 1.5992 \quad N/cm & -1.1571 \quad N/cm & -6.0650 \quad N \\ -1.1571 \quad N/cm & 5.7047 \quad N/cm & 7.7521 \quad N \\ -3.5650 \quad N & 9.4521 \quad N & 22.6794 \quad Ncm \end{bmatrix}.$$

The desired stiffness matrix $\left[K\right]_D$ is given as

$$[K]_D = \begin{bmatrix} 1.6442 \quad N/cm & -1.1921 \quad N/cm & -6.0230 \quad N \\ -1.1921 \quad N/cm & 5.7417 \quad N/cm & 7.7931 \quad N \\ -3.5230 \quad N & 9.4931 \quad N & 22.6384 \quad Ncm \end{bmatrix}.$$

Since the difference between the desired stiffness matrix and the initial stiffness matrix is not at all close to infinitesimal, the difference is divided into a number of small δB 's and the problem is solved repeatedly to obtain the spring parameters and the displacements of body B and body A which implement the desired stiffness matrix and the external wrench. Three sets of spring parameters are obtained: one with no constraint on the displacements of body A and body B, another with body A fixed, and the other with bodies A and B fixed. The calculated spring parameters are presented in Tables 12, 13, and 14, respectively. In addition, the initial and final poses of the mechanism are shown in Figures 5 and 6.

Table 12. Spring parameters with no constraint

Spring No.	1	2	3	4	5	6
Stiffness constant k	5.5786	4.8852	5.5513	5.1865	5.1506	5.1505
Free length l_o	2.5502	0.5084	1.9188	2.7939	0.8039	1.3746

(Unit: N/cm for k, cm for l_o)

Table 13. Spring parameters with body B fixed

Spring No.	1	2	3	4	5	6
Stiffness constant k	6.3556	4.4317	5.9605	5.3306	5.5194	4.4334
Free length l_o	3.0234	0.6588	2.6016	2.8192	0.6220	1.0270

(Unit: N/cm for k, cm for l_o)

Table 14. Spring parameters with body A and body B fixed

Spring No.	1	2	3	4	5	6
Stiffness constant k	12.3070	10.5719	6.9847	1.9411	4.9528	3.8741
Free length l_o	2.6786	1.0769	2.5033	3.7435	0.7229	1.0333

(Unit: N/cm for k, cm for l_o)

The results indicate that there are greater changes of the spring parameters with more constraints imposed on the bodies. These control methods all require the inverse of $\frac{d\underline{\mathbf{B}}}{d\underline{\mathbf{U}}}$ or $\left[\frac{d\underline{\mathbf{B}}}{d\underline{\mathbf{U}}_{Null}}\right]^T\left[\frac{d\underline{\mathbf{B}}}{d\underline{\mathbf{U}}_{Null}}\right]$ depending on the constraint and it may cause a singularity problem. With more constraints the mechanism is more vulnerable to the singularity problem.

5. CONCLUSIONS

Compliance control of planar mechanisms was investigated. The compliant mechanisms consist of rigid bodies connected by adjustable springs that are in static equilibrium under an external wrench. The initial external wrench was not ignored and this leads to a more practical model of real systems at the cost of increased complexity of the stiffness matrix of the mechanism. Planar compliant parallel mechanisms contain six independent variables in their stiffness matrices and it was shown that even a mechanism having two planar compliant parallel mechanisms in series has this same property.

Adjustable springs that are assumed to be able to change their stiffness coefficients and free lengths were employed to control the compliance of the mechanism and to regulate the external wrench at the same time. The stiffness matrix and the external wrench are highly nonlinear in terms of the constituent spring properties and the displacements of the constituent rigid bodies. Hence a derivative of the stiffness matrix and the external wrench were derived and applied in an iterative manner. Since the system is in general redundant, several strategies and constraints were applied to select a proper solution.

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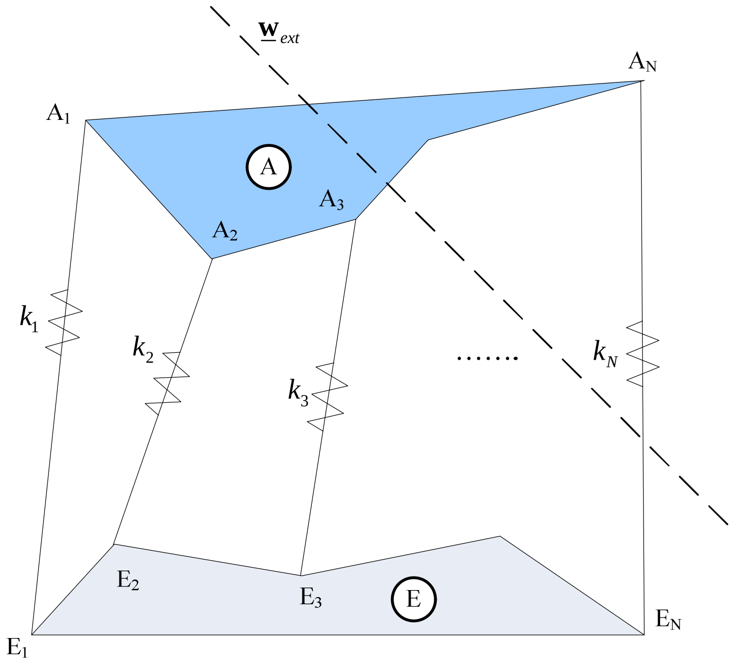

Figure 1. Planar compliant parallel mechanism with *N* number of couplings

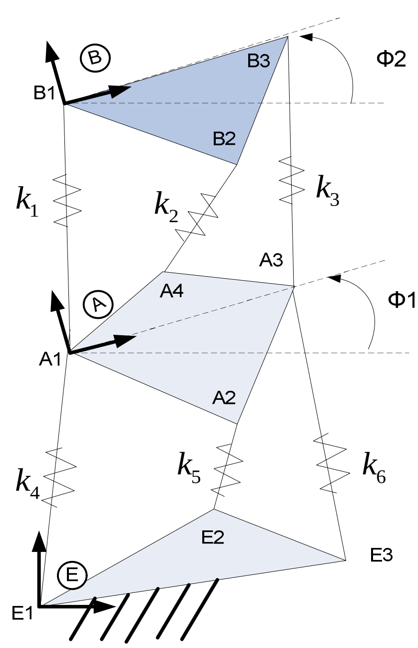

Figure 2. Mechanism having two planar compliant parallel mechanisms in series

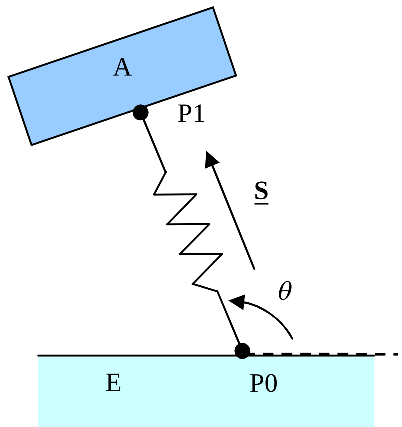

Figure 3. Planar compliant coupling connecting body A and the ground

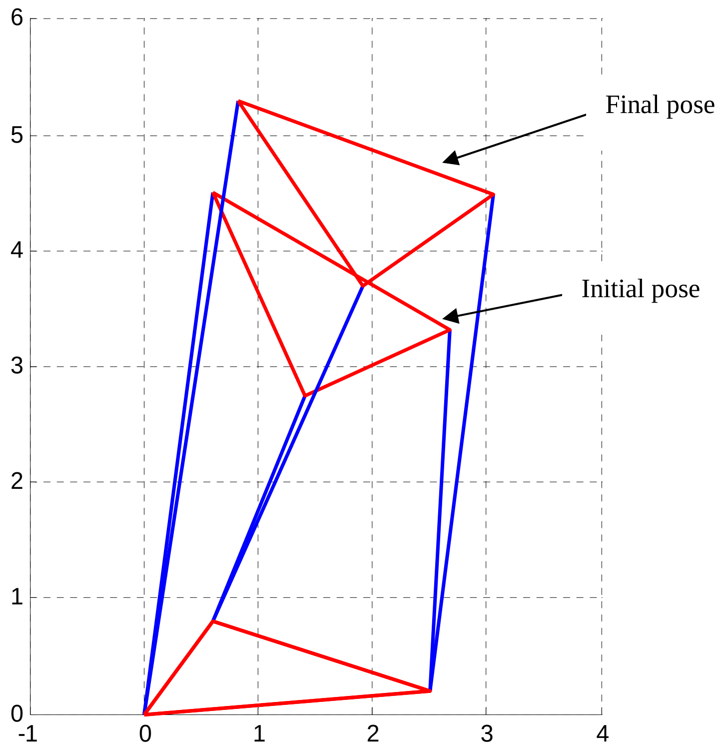

Figure 4. Poses of the compliant parallel mechanism for numerical example in section 4.4

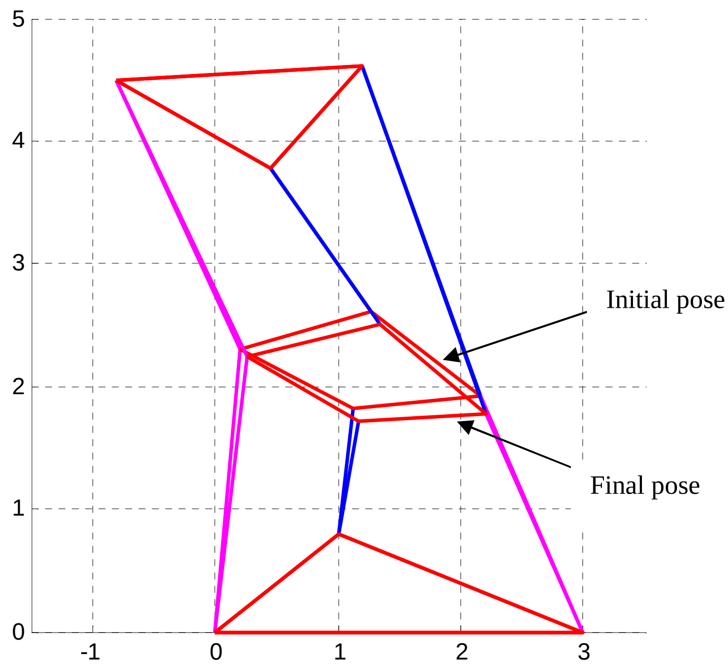

Figure 5. Poses of the compliant mechanism with body B fixed

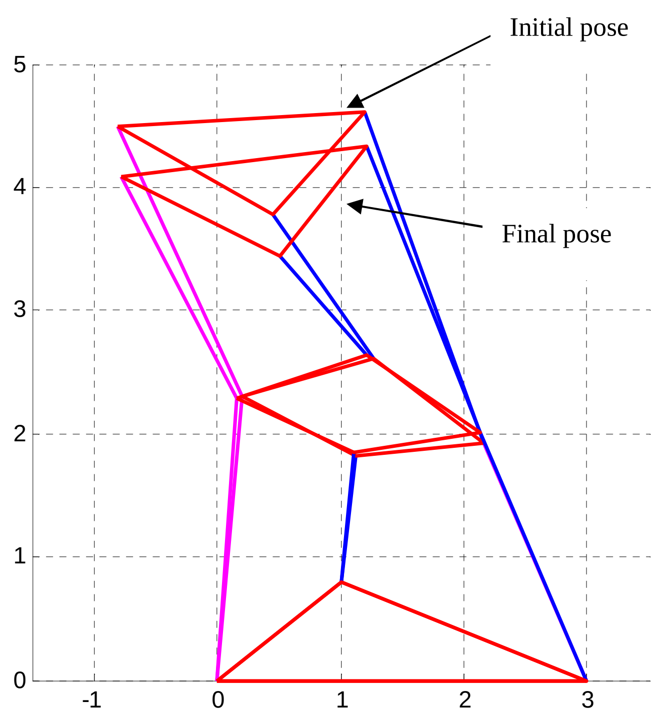

Figure 6. Poses of the compliant mechanism with no constraint