THE OPTIMUM QUALITY INDEX FOR A SPATIAL REDUNDANT 4-8 IN-PARALLEL MANIPULATOR

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Abstract. The quality index for a redundant 4-4[1] in-parallel manipulator with a square platform and a square base was obtained previously. In this paper the optimum quality index for a redundant 4-8 manipulator is determined. The device has a square platform and an octagonal base connected by eight legs. The quality index is a dimensionless ratio which takes a maximum value of 1 at a central symmetrical configuration that is shown to correspond to the maximum value of the square root of the determinant of the product of the manipulator Jacobian by its transpose. The Jacobian matrix is none other than the normalized coordinates of the eight leg lines. It is shown that the quality index can be used as a constructive measure of not only acceptable and optimum design proportions but also an acceptable operating workspace.

1. Introduction

The quality index for redundant manipulators has been defined by Zhang and Duffy (1998) for a redundant 4-4 in-parallel manipulator (see Fig. 1) by the dimensionless ratio

$$\lambda = \sqrt{\frac{\det \mathbf{J}\mathbf{J}^{\mathrm{T}}}{\det \mathbf{J}_{\mathrm{m}}\mathbf{J}_{\mathrm{m}}^{\mathrm{T}}}} \tag{1}$$

where **J** is the six-by-eight Jacobian matrix of the normalized coordinates of the eight leg lines. It takes a maximum value of $\lambda = 1$ at a central symmetrical configuration that is shown to correspond to the maximum value of the square root of the determinant of the product of the manipulator Jacobian by its transpose (i.e. $(\det \mathbf{J}\mathbf{J}^{\mathrm{T}})^{1/2} = (\det \mathbf{J}_{\mathrm{m}}\mathbf{J}_{\mathrm{m}}^{\mathrm{T}})^{1/2}$). When the manipulator is actuated so that the moving platform departs from its central configuration, the determinant always diminishes and it

[1] These numbers indicate the number of connecting points in the top and base platform respectively.

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becomes zero when a special configuration is reached (The platform then gains one or more uncontrollable freedoms).

The quality index was defined initially for a planar 3-3 in-parallel device by the dimensionless ratio (see Lee, Duffy, and Keler (1996)).

$$\lambda = \frac{|\det \mathbf{J}|}{|\det \mathbf{J}_{\mathrm{m}}|} \tag{2}$$

where $\mathbf{J}$ is the three-by-three Jacobian matrix of the normalized coordinates of three leg lines. After that it was defined for an octahedral in-parallel manipulator by Lee, Duffy, and Hunt (1998), 3-6, and 6-6 in-parallel devices by Lee and Duffy (1999). For these cases $\mathbf{J}$ is the six-by-six matrix of the normalized coordinates of the six leg lines.

It has been shown by Lee, Duffy, and Hunt (1998) that by using the Grassmann-Cayley algebra (see White and Whiteley (1983)), for a general octahedron, when the leg lengths are not normalized, det$\mathbf{J}$ has dimension of $(\text{volume})^3$ and it is directly related to the products of volumes of tetrahedra which form the octahedron. In this way $\det \mathbf{J}$ has geometrical meaning. Similarly, $(\det \mathbf{J}\mathbf{J}^{\mathrm{T}})^{1/2}$ makes complete sense because by the Cauchy-Binet theorem, $\det \mathbf{J}\mathbf{J}^{\mathrm{T}} = \Delta_1^2 + \Delta_2^2 + \cdots + \Delta_m^2$, has geometrical meaning. Here, each $\Delta_i (1 \le i \le m = C_n^6)$ is simply the determinant of the 6×6 submatrices of $\mathbf{J}$ which is a $6 \times n$ matrix. Clearly when $n = 6$, (1) reduces to (2).

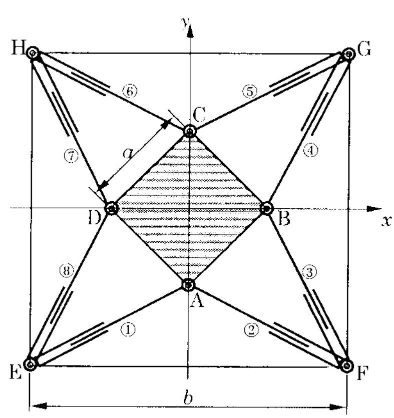


Figure 1. Plan view of a redundant 4-4 in-parallel manipulator

There are several advantages of having redundancy in parallel manipulators as proposed by Merlet (1996), Dasgupta and Mruthyunjaya (1998). Redundancy in actuation can be used to increase the reliability of parallel manipulators, eliminate certain type of singularities, and determine the unique position of the platform.

Many papers have been published on the optimal design of parallel manipulations (see for example Gosselin and Angeles (1988 and 1989), Zanganeh and Angeles (1997)). Unlike the case of a mechanism which is often designed for a specific task, the tasks to be performed by a manipulator are varied. Hence, there should not be any preferred general orientation for which the manipulator would have better properties.

In this paper a redundant 4-8 parallel manipulator is studied. The device has a square platform and an octagonal base connected by eight legs as shown in Fig. 2. The octagonal base is formed by separating from each vertex of a square by a small distance. Here we find that for the quality index of the redundant 4-8 parallel manipulator with a platform of side a to be a maximum, the base of the manipulator will originate from a square with side $\sqrt{2}a$ and the perpendicular distance between the platform and the base is $a/\sqrt{2}$. By using quality index, variable motions are investigated for which a moving platform rotates about a central axis or moves parallel to the base.

2. The determination of $\sqrt{\det \mathbf{J}_\mathrm{m}\mathbf{J}_\mathrm{m}^\mathrm{T}}$ for a redundant 4-8 in-parallel manipulator

The plan view of a redundant 4-8 parallel manipulator is shown in Fig. 2, which is derived simply by separating the double ball-and-socket joints in the base of a 4-4 manipulator. The device has a square platform of side a and an octagonal base formed by 4 pairs of joints E_D and E_A, F_A and F_B, G_B and G_C, H_C and H_D. Each pair of joints is separated from a vertex of a square of side b by a distance βb for which $0 \le \beta < 1/2$. Clearly the platform is degenerated when $\beta = 1/2$. The moving platform is parallel to the base with a height h. This manipulator is said to be redundant since the platform and the base are connected by eight actuated legs.

Firstly, it is necessary to determine the Plücker line coordinates of the eight legs of the platform. The Plücker coordinates for the line joining the points with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) were elegantly expressed by Grassmann by the six 2×2 determinants of the array

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \end{bmatrix} \tag{3}$$

where the direction ratios of the line are

$$L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, \qquad M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}, \qquad N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix} \tag{4}$$

and the moments of the line segment about the three coordinate axes are

$$P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \qquad Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \qquad R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}. \tag{5}$$

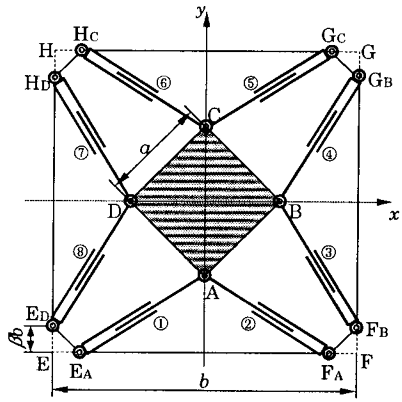


Figure 2. Plan view of a redundant 4-8 in-parallel manipulator

A fixed coordinate system is placed at the center of the octagonal base as shown in Fig. 2. Then the coordinates of the points A, B, C, and D on the platform are

$$A\begin{pmatrix}0 & -\frac{\sqrt{2}a}{2} & h\end{pmatrix},\quad B\begin{pmatrix}\frac{\sqrt{2}a}{2} & 0 & h\end{pmatrix},\quad C\begin{pmatrix}0 & \frac{\sqrt{2}a}{2} & h\end{pmatrix},\quad D\begin{pmatrix}-\frac{\sqrt{2}a}{2} & 0 & h\end{pmatrix}. \tag{6}$$

The coordinates of the points E_A, F_A, F_B, G_B, G_C, H_C, H_D, and E_D on the base are

$$\begin{aligned}&E_A\begin{pmatrix}-d_1 & -\frac{b}{2} & 0\end{pmatrix},\quad F_A\begin{pmatrix}d_1 & -\frac{b}{2} & 0\end{pmatrix},\quad F_B\begin{pmatrix}\frac{b}{2} & -d_1 & 0\end{pmatrix},\quad G_B\begin{pmatrix}\frac{b}{2} & d_1 & 0\end{pmatrix},\\ &G_C\begin{pmatrix}d_1 & \frac{b}{2} & 0\end{pmatrix},\quad H_C\begin{pmatrix}-d_1 & \frac{b}{2} & 0\end{pmatrix},\quad H_D\begin{pmatrix}-\frac{b}{2} & d_1 & 0\end{pmatrix},\quad E_D\begin{pmatrix}-\frac{b}{2} & -d_1 & 0\end{pmatrix}\end{aligned} \tag{7}$$

where $d_1 = (1-2\beta)b/2$.

Counting the 2×2 determinants of the various arrays of the joins of the pairs of points AE_A, AF_A, …, DE_D yields the normalized Jacobian matrix of the eight lines now all reduced to unit length which can be expressed in the form

$$\mathbf{J}=\frac{1}{l}\begin{bmatrix} d_1 & -d_1 & d_2 & d_2 & -d_1 & d_1 & -d_2 & -d_2 \\ -d_2 & -d_2 & d_1 & -d_1 & d_2 & d_2 & -d_1 & d_1 \\ h & h & h & h & h & h & h & h \\ -\frac{bh}{2} & -\frac{bh}{2} & -d_1h & d_1h & \frac{bh}{2} & \frac{bh}{2} & d_1h & -d_1h \\ d_1h & -d_1h & -\frac{bh}{2} & -\frac{bh}{2} & -d_1h & d_1h & \frac{bh}{2} & \frac{bh}{2} \\ \frac{\sqrt{2}ad_1}{2} & -\frac{\sqrt{2}ad_1}{2} & \frac{\sqrt{2}ad_1}{2} & -\frac{\sqrt{2}ad_1}{2} & \frac{\sqrt{2}ad_1}{2} & -\frac{\sqrt{2}ad_1}{2} & \frac{\sqrt{2}ad_1}{2} & -\frac{\sqrt{2}ad_1}{2} \end{bmatrix} \tag{8}$$

where $d_2 = (\sqrt{2}a - b)/2$.

Here, the device is in a symmetrical position so that the normalization divisor is the same for each leg, namely the leg length $l = AE_A = AF_A = BF_B = BG_B = CG_C = CH_C = DH_D = DE_D$, and for every leg

$$l = \sqrt{L^2 + M^2 + N^2} = \sqrt{\frac{1}{2}\left[a^2 - \sqrt{2}ab + (2\beta^2 - 2\beta + 1)b^2 + 2h^2\right]}. \tag{9}$$

Using Eq. (8), the determinant of the product $\mathbf{J}\,\mathbf{J}^{\mathrm{T}}$ can be expressed in the form

$$\det \mathbf{JJ}^{\mathrm{T}} = \frac{1}{l^{12}}\begin{vmatrix} 4(d_1^2+d_2^2) & 0 & 0 & 0 & 2(2d_1^2 - bd_2)h & 0 \\ 0 & 4(d_1^2+d_2^2) & 0 & 2(bd_2 - 2d_1^2)h & 0 & 0 \\ 0 & 0 & 8h^2 & 0 & 0 & 0 \\ 0 & 2(bd_2 - 2d_1^2)h & 0 & (b^2+4d_1^2)h^2 & 0 & 0 \\ 2(2d_1^2 - bd_2)h & 0 & 0 & 0 & (b^2+4d_1^2)h^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4a^2d_1^2 \end{vmatrix}. \tag{10}$$

Expanding (10) and using (9), then extracting the square root yields

$$\sqrt{\det \mathbf{JJ}^{\mathrm{T}}} = \frac{32\sqrt{2}(1-2\beta)^3 a^3 b^3 h^3}{\left[a^2 - \sqrt{2}ab + (2\beta^2 - 2\beta + 1)b^2 + 2h^2\right]^3}. \tag{11}$$

Now taking the partial deviate of (11) with respect to h and b respectively and equating to zero yield

$$\frac{96\sqrt{2}(1-2\beta)^3 a^3 b^3 h^2\left[a^2 - \sqrt{2}ab + (2\beta^2 - 2\beta + 1)b^2 - 2h^2\right]}{\left[a^2 - \sqrt{2}ab + (2\beta^2 - 2\beta + 1)b^2 + 2h^2\right]^4} = 0 \tag{12}$$

and

$$\frac{96\sqrt{2}(-1+2\beta)^3 a^3 b^2 h^3\left[-a^2 + (2\beta^2 - 2\beta + 1)b^2 - 2h^2\right]}{\left[a^2 - \sqrt{2}ab + (2\beta^2 - 2\beta + 1)b^2 + 2h^2\right]^4} = 0. \tag{13}$$

When a, b, and h are not equal to zero (and $\beta \neq 1/2$), Eqs. (12) and (13) yield

$$a^2 - \sqrt{2}ab + (2\beta^2 - 2\beta + 1)b^2 - 2h^2 = 0, \tag{14}$$

$$-a^2 + (2\beta^2 - 2\beta + 1)b^2 - 2h^2 = 0. \tag{15}$$

Subtracting Eq. (15) from Eq. (14) gives

$$2a^2 - \sqrt{2}ab = 0. \tag{16}$$

Thus,

$$b = \sqrt{2}a\,. \tag{17}$$

Substituting (17) into Eq. (15) yields

$$(4\beta^2 - 4\beta + 1)a^2 - 2h^2 = 0\,. \tag{18}$$

There are two solutions for h in the above equation, here we only take the positive solution (the negative solution is simply a reflection through the base)

$$h = \frac{1}{\sqrt{2}}(1 - 2\beta)a\,. \tag{19}$$

Finally substituting (17) and (19) into (11) we obtain

$$\sqrt{\det \mathbf{J}_{\mathrm{m}}\mathbf{J}_{\mathrm{m}}^{\mathrm{T}}} = \left(\sqrt{\det \mathbf{J}\mathbf{J}^{\mathrm{T}}}\right)_{\max} = 4\sqrt{2}a^3 \tag{20}$$

where $\mathbf{J}_{\mathrm{m}}$ denotes the Jacobian matrix for the configuration at which the 4-8 manipulator has a maximum quality index as shown in Fig. 3. Clearly, this maximum value of $(\det \mathbf{J}\mathbf{J}^{\mathrm{T}})^{1/2}$ is independent to the value of β.

Fig. 4 illustrates the compatibility of these two results. It can be observed that as the distance between the pairs of separation points of the double ball-and-socket joints E, F, G and H of the original 4-4 manipulator are increased, the value of h decreases (see (19)) from $h = a/\sqrt{2}$ ($\beta = 0$, concentric ball-and-socket joints) to h=0 (β=1/2, platform has degenerated).

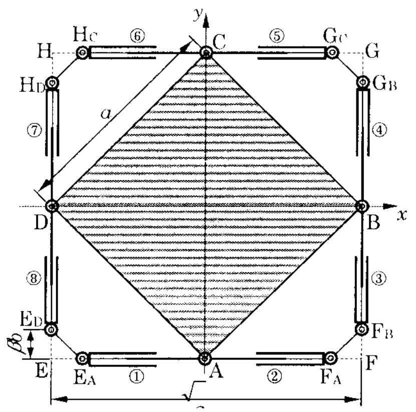


Figure 3. The manipulator in the central configuration with the highest quality index

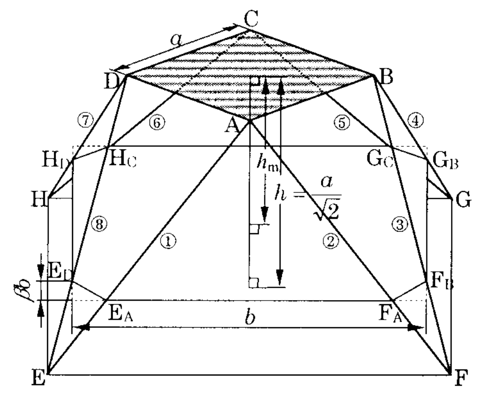


Figure 4. Compatibility between redundant 4-4 and 4-8 manipulators ($h_m = \frac{(1-2\beta)a}{\sqrt{2}}$)

3. Implementation of the quality index

From (1) and (20), the quality index for the 4-8 manipulator will be

$$\lambda = \frac{\sqrt{\det \mathbf{J}\mathbf{J}^{\mathrm{T}}}}{4\sqrt{2}a^3} \ . \tag{21}$$

The variation of the quality index will now be investigated for a number of simple motions of the top platform. Here, a 4-8 parallel manipulator with platform side $a = 1$ and base side $b = \sqrt{2}$ is taken as an example.

First, consider a pure translation of the platform from the initial position along the z axis while remaining parallel to the base. For such movement, from (11) and (21), the quality index is given by

$$\lambda = \frac{8(1-2\beta)^3 b^3 h^3}{\left[a^2 - \sqrt{2}ab + (2\beta^2 - 2\beta + 1)b^2 + 2h^2\right]^3} \ . \tag{22}$$

With $a = 1$ and $b = \sqrt{2}$, this reduces to

$$\lambda = \frac{16\sqrt{2}(1-2\beta)^3 h^3}{\left[(1-2\beta)^2 + 2h^2\right]^3} \tag{23}$$

and is plotted in Fig. 5 as a function of h for different values of β. From the figure, we can see the height for the maximum quality index is reduced as β increases. Each value of β designates the distance between the separation points in the base and is a first design parameter. Clearly, β=0 is the best overall design.

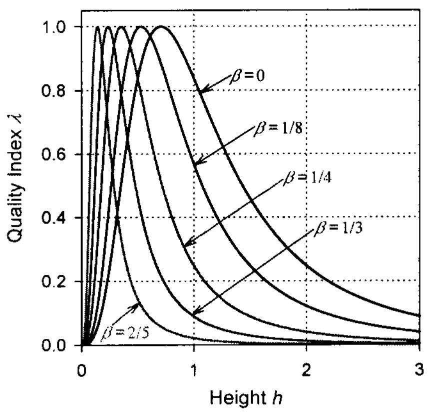
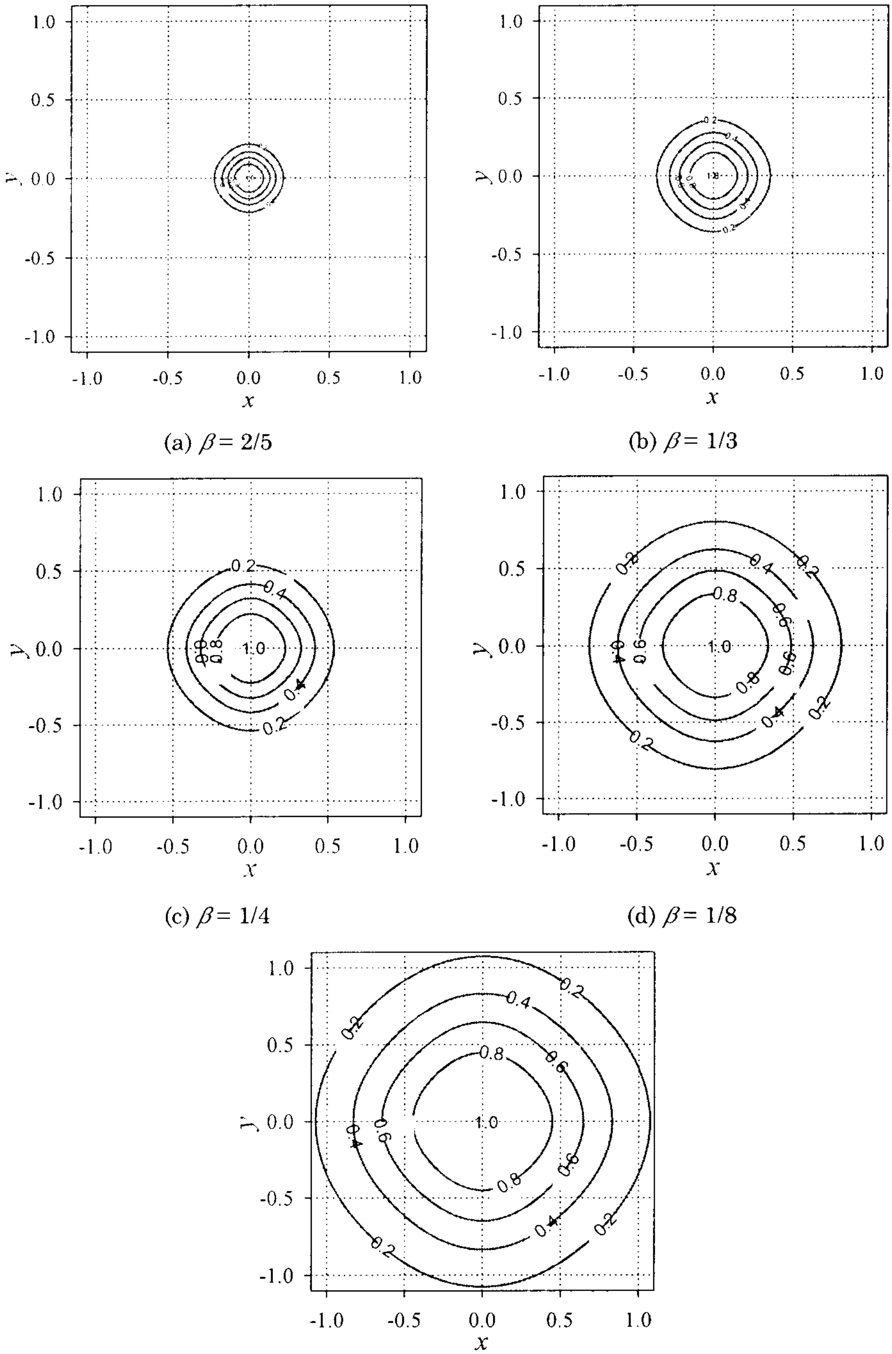


Figure 5. Quality index for platform vertical movement

In Fig. 6, the contours of quality index as the platform of the redundant 4-8 parallel manipulator is translated away from the central location while remaining parallel to the base when $h = h_m = a/\sqrt{2}$, are drawn for various values of β. Comparing these figures it is clear that the smaller β, the larger workspace area of the platform is with high quality index.



(a) $\beta = 2/5$

(b) $\beta = 1/3$

(c) $\beta = 1/4$

(d) $\beta = 1/8$

(e) $\beta = 0$

Figure 6. Quality index for platform parallel translation with different values of β

Fig. 7 illustrates the variation of the quality index for rotations about x' and y' axes located in the moving platform in its central configuration and which are parallel to the x and y axes located in the base platform respectively. The $x'y'$ and xy planes are parallel with a distance $h_m = a/\sqrt{2}$. Rotations about any line in the $x'y'$ plane passing through the intersection point of x' and y' axes are simply linear combination of rotations about the x' and y' axes.

Finally, Fig. 8 shows how the quality index varies as the platform is rotated about the vertical axis z through its center. The legs being adjusted in length to keep the platform parallel to the base at a distance h_m. It is shown in the figure that the manipulator has the highest quality index $\lambda = 1$ when $\theta = 0°$, and $\lambda = 0$ (singularity) when $\theta = \pm 90°$.

Again, from Figs. 7 and 8 we can clearly see that better designs are obtained as β reduces to zero. Hence the best 4-8 parallel manipulator design is obtained when the pair of base joints are as close as possible.

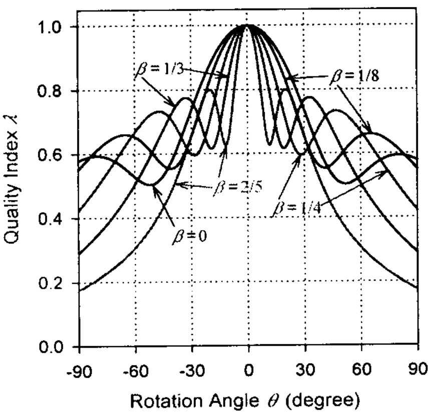


Figure 7. Quality index for platform rotations about x' axis or y' axis

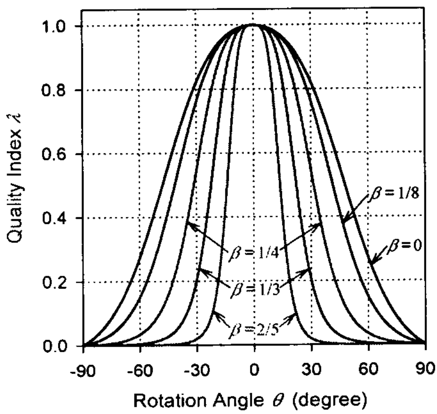


Figure 8. Quality index for platform rotation about z axis

4. Conclusions

In this paper, the maximum value of the square root of the determinant of the product of Jacobian matrix by its transpose, which is used for quality index for a redundant 4-8 manipulator, has been obtained. Similar to the 4-4 redundant manipulator, the maximum value is also $4\sqrt{2}a$ for a square moving platform with side a, regardless of the base size.

It is shown in this paper how the quality index varies for various motions of the platform. Comparing those quality index curves with different joint separations, we have obtained useful design information for the 4-8 redundant parallel manipulator, i.e., after avoiding the use of concentric ball-and-socket joints so as to overcome the interference problems, we should keep two base joints as close as possible. Such kind of design was also pointed by Lee and Duffy (1999) for non-redundant 6-6 parallel manipulator to avoid full cycle mobility of platform. Thus, the obvious choice for the design of parallel manipulators is an arrangement that entirely lacks one-to-one correspondences, e.g. the 4-4 redundant parallel manipulator in Fig. 1 in which all correspondences are two-to-one.

5. Acknowledgements

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References

Dasgupta, B. and Mruthyunjaya, T.S. (1998). Force redundancy in parallel manipulators: theoretical and practical issues. *Mechanism and Machine Theory*, Vol. 33, No. 6, pp. 727-742.

Gosselin, C. and Angeles, J. (1988). The optimum kinematic design of a planar three-degree-of-freedom parallel manipulator. *ASME Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 110, No. 3, pp. 35-41.

Gosselin, C. and Angeles, J. (1989). The optimum kinematic design of a spherical three-degree-of-freedom parallel manipulator. *ASME Journal of Mechanisms, Transmissions, and Automation in Design*, Vol. 111, No. 6, pp. 202-207.

Lee, J., Duffy, J., and Keler, M. (1996). The optimum quality index for the stability of in-parallel planar platform devices. *Proceedings of the ASME 24th Biennial Mechanisms Conference*, Irvine, Ca., USA, 96-DETC/MECH-1135.

Lee, J., Duffy, J., and Hunt, K.H. (1998). A practical quality index based on the octahedral manipulator. *The International Journal of Robotics Research*, Vol. 17, No. 10, pp. 1081-1090.

Lee, J. and Duffy. J. (1999). The optimum quality index for some spatial in-parallel devices. *1999 Florida Conference on Recent Advances in Robotics*, Gainesville, FL, USA.

Merlet, J.P. (1996). Redundant parallel manipulators. *Laboratory Robotics and Automation*, Vol. 8, No. 1, pp. 17-24.

White, N. and Whiteley, W. (1983). The algebraic geometry of stresses in frameworks. *S.I.A.M. Journal of Algebraic and Discrete Methods*, Vol. 4, No. 4, pp. 481-511.

Zanganeh, K.E. and Angeles, J. (1997). Kinematic isotropy and the optimum design of parallel manipulators. *The International Journal of Robotics Research*, Vol. 16, No. 2, pp. 185-197.

Zhang, Y. and Duffy, J. (1998). The optimum quality index for a redundant 4-4 in-parallel manipulator. *RoManSy 12: Theory and Practice of Robots and Manipulators, Proceedings of the Twelfth CISM-IFToMM Symposium*, Paris, pp. 289-296.