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Optimization of Ring Trusses for Antenna Structures Using Line Geometry

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Two types of trusses were optimized using the simple, yet powerful, mathematical technique of employing line geometry as defined by the mathematicians Plucker and Grassmann. The key benefit to this design methodology is simplicity. Multiple-node designs were analyzed, with particular focus placed on the triangular truss. The methodology is based on a geometric stability criterion called the quality index. This approach allowed for a simple equation to be created that identifies the optimal geometry based on the exterior angles and the number of sides. Trusses with single and double connecting points are considered using the quality index. An example of a double connection truss is the triangular truss. In this case, each node (either on the top or the base) has two structural elements connecting to the other nodes. It is shown that there are optimal design criteria for the upper, lower, and separation dimensions. These optimal geometric designs yield conical trusses. Further consideration of the 3–3 truss produced a closed-form solution. This equation holds significant potential for truss designs, providing a clear comparison between single and double connections.

Introduction

THE objective of this study was to create a design methodology for novel space structures that employs line geometry concepts developed in the late 19th and early 20th centuries[1] as developed by the mathematicians Plucker and Grassmann. The motivation was to create new, lightweight space structures, using as a starting point optimal geometric/static stability criteria derived from line geometry.[2,3] In this paper, the optimization process is developed by simple construction.

Two examples of existing trusses, which secure antennas, are shown in Figs. 1 and 2. The upper circular truss ring is connected to a lower ring by a redundant structure of connecting struts. It is the design of this truss ring that is the major object of this study.

Figures 3a–3c show regular planar polygons. A circle is drawn through the vertices of each polygon. These circles can be used to make the circular truss rings that carry the antennas. Each vertex or node of a polygon can be considered to be a single connecting point (one strut), a double connecting point (a pair of struts), or a triple connecting point (three struts), as shown in Fig. 3.

Background

The vector equation for a point can be expressed in terms of the Cartesian coordinates as

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} \tag{1}$$

These coordinates can be written $x = X/W$, $y = Y/W$, and $z = Z/W$ (Ref. 4), which expresses the point in terms of the homogeneous coordinates W, X, Y, and Z. A point is completely specified by the three independent ratios, X/W, Y/W, and Z/W, and therefore, there are ∞^3 points in three space.[5]

Similarly, the equation for a plane can be expressed in the form

$$D + Ax + By + Cz = 0 \tag{2}$$

or in terms of the homogeneous point coordinates by

$$DW + AX + BY + CZ = 0 \tag{3}$$

The homogeneous coordinates for a plane are D, A, B, and C, and a plane is completely specified by three independent ratios A/D, B/D, and C/D. Therefore, there are ∞^3 planes in three space. It is well known that in three-space the plane and the point are dual.

When Grassmann's (see Ref. 6) determinate principles are used, the six homogeneous coordinates for a line, which is the join of two points (x_1, y_1, z_1) and (x_2, y_2, z_2), can be obtained from the 2×4 array

$$\begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \end{bmatrix} \tag{4}$$

by calculating all of the 2×2 determinants of the 2×4 array as

$$L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, \quad M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}, \quad N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix}$$

$$P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \quad Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \quad L = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \tag{5}$$

The six homogeneous coordinates, L, M, N, P, Q, and R, or S, and $\bar{S}_0$ are superabundant by two because they must satisfy the following relationships:

$$\bar{S} \cdot \bar{S} = L^2 + M^2 + N^2 = d^2 \tag{6}$$

where d is the distance between the two points and

$$\bar{S} \cdot \bar{S}_0 = LP + MQ + NR = 0 \tag{7}$$

which is the orthogonality condition. Briefly, as mentioned, the vector equation for a line is given by $\boldsymbol{r} \times \bar{S} = \bar{S}_0$. Clearly, $\bar{S}$ and $\bar{S}_0$ are orthogonal because $\bar{S} \cdot \bar{S}_0 = \bar{S} \cdot \boldsymbol{r} \times \bar{S} = 0$. A line is completely specified by four independent ratios. Therefore, these are ∞^4 lines in three space.

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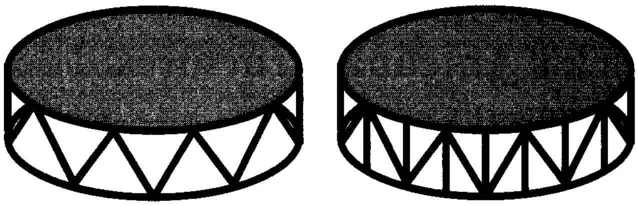


Fig. 1 Warren and TRW AstroMesh® truss.

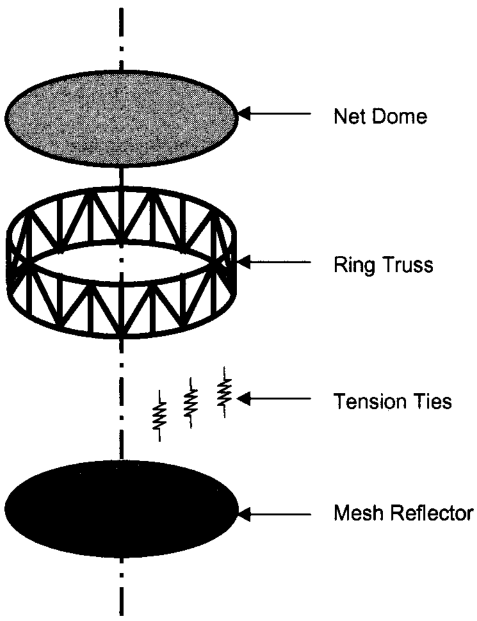


Fig. 2 Exploded tension truss design.

The quality index, or geometric stability index, for a parallel-legged structure is defined as a number between 0 and 1, derived from the determinant of the Jacobian matrix comprising the lines between the upper and lower surfaces.[7] In the case of a 3–3 device (where there are three nodes on the upper surface and three nodes on the lower surface), the Jacobian is a square matrix (6 × 6) that can be written as

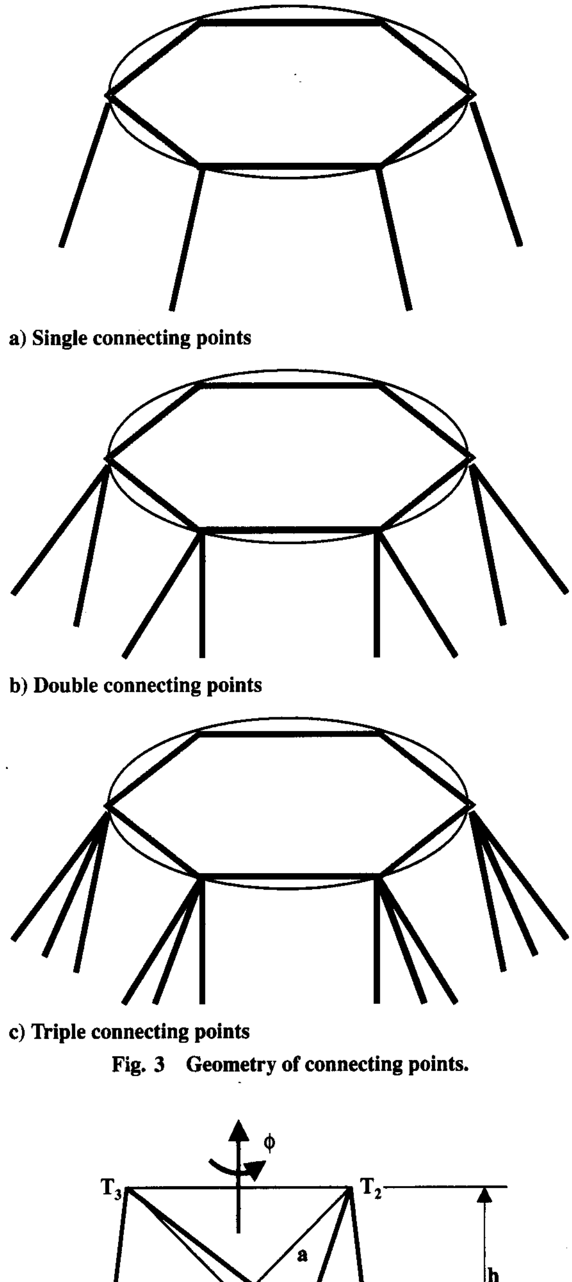
$$J = \begin{bmatrix} \bar{S}_1 & \bar{S}_2 & \bar{S}_3 & \bar{S}_4 & \bar{S}_5 & \bar{S}_6 \\ \bar{S}_1^0 & \bar{S}_2^0 & \bar{S}_3^0 & \bar{S}_4^0 & \bar{S}_5^0 & \bar{S}_6^0 \end{bmatrix} \tag{8}$$

The determinant for this square matrix is a function based on the geometric variables describing the structure (Fig. 4). These variables are a for the side length of the upper surface, b for the side length of the lower surface, and h for the distance between the surfaces. Obviously, for equilibrium, there must be at least six connecting struts for a three-dimensional structure (with six degrees of freedom). It is well established that five or fewer connecting struts will not fully constrain a pair of rigid bodies.

The quality index[7] is defined as

$$\lambda = \frac{|\det J|}{|\det J_m|}, \qquad 0 \leq \lambda \leq 1 \tag{9}$$

where J is the aforementioned 6 × 6 Jacobian matrix of the coordinates of the six connector lines and J_m is the matrix when the platform is in an optimum, most geometrically stable position. The quality index has two clear meanings so far. When $\lambda = 0$, the octahedron is in a stable singularity and will degenerate or collapse under a load. When $\lambda = 1$, it is in an optimal configuration to sustain loads. However, when λ is neither zero nor one, it is hard to say exactly how much one configuration is better than another. One cannot say that a configuration with $\lambda = 0.8$ is twice as good as a configuration with $\lambda = 0.4$ without further analyses. However, the quality index helps to design platforms by setting dimensions that give the best quality index, $\lambda = 1$. Also, it gives an idea of certain designs that must be avoided because they would lead to zero or very low-quality indices.



a) Single connecting points

b) Double connecting points

c) Triple connecting points

Fig. 3 Geometry of connecting points.

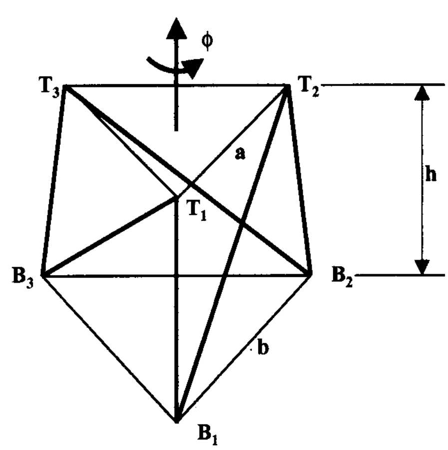


Fig. 4 Double connecting 3–3 truss.

Optimization Methodology

The optimization methodology presented here stems from the work in line geometry and the quality index. Specifically, these calculations provide a design-independent metric to describe geometric stability. Simply put, when the quality index reaches 0, the matrix is in a singularity and, therefore, unstable. Because stability is not a go/no-go function, it is useful to have a gradient scale value to describe it.

This paper expands the use of the quality index to include the number of connection points or nodes in the upper circular ring (UCR), which are joined to an equal number of connecting points in the lower circular ring (LCR) by a series of struts. For example, a 3–3 truss means there are three connecting points in the UCR and three connecting points in the LCR. As a further example, a 6–6 truss means there are six connecting points in the UCR and six connecting points in the LCR. Figure 3a, shows an arrangement of six single connecting points. Figures 3b and 3c show a construction of potential redundant structures containing 12 and 18 connecting struts.

In general, if there are 3, 4, 5, 6, 7, 8, ..., connecting points in the UCR and LCR, a design will be based on optimizing the geometry that is formed by connecting pairs of equilateral triangles, squares, and regular polygons (pentagons, hexagons, etc.). Previous work[7] has shown that symmetry is required for the upper and lower surfaces. Clearly, for numbers of connecting struts greater than six, the truss will be a redundant structure because six reactions are required for six degrees of freedom, but this does not present a problem in the optimization process.

It is well known that the sum of the exterior angles of a planar polygon is 360 deg. For a regular polygon for which each side is equal and each exterior angle β is equal, then $\beta = 360/n$. When this information is used, it is a relatively simple task to construct any regular polygon, and it was useful in the construction of a standard function to describe these structures.

Ring Truss Optimization with Double Connecting Points

Consider initially a 3–3 truss in which the UCR and LCR are equilateral triangles located in parallel planes and connected by six struts B_1T_1, B_1T_2, B_2T_2, B_2T_3, B_3T_3, and B_3T_1, as shown in Fig. 4. The upper and lower triangles have sides with lengths a and b, respectively. The corresponding pairs of sides (T_1T_2, B_1B_2), (T_2T_3, B_2B_3), and (T_3T_1, B_3B_1) are parallel, and the relative rotation angle ϕ between the triangles is zero (as shown).

The quality index can be expressed as a function of ϕ, a, b, and h, and values for these parameters are determined to obtain the optimal quality index $\lambda = 1$. The optimum configuration, $\lambda = 1$, occurs at $\phi = 60$ deg, and when $a = b$, we obtain the Star of David as shown in Fig. 5. The distance h between the planes can be computed for any ratio b/a. Note that the plan view illustrates the true sides of the upper ($T_1T_2T_3$) and lower ($B_1B_2B_3$) triangles for which $T_1T_2 = T_2T_3 = T_3T_1 = a$ and $B_1B_2 = B_2B_3 = B_3B_1 = b$, where $a = b$. Furthermore, the connectors B_1T_1, B_2T_1, $B_2T_2, \ldots$, do not lie in the planes perpendicular to the triangles $T_1T_2T_3$ and $B_1B_2B_3$ so that the true leg lengths will not be revealed in the elevation view.

However, there is a global optimum for which $a = b/2$, $h = b/2$, and $h/\ell = 1/\sqrt{2}$, shown in Fig. 6a, where ℓ is the length of each connector. The six connectors lie pairwise in planes perpendicular to the triangles $T_1T_2T_3$ and $B_1B_2B_3$. The elevation view captures the true lengths ℓ of the pair of connectors B_3T_2 and B_2T_2.

Note that the connecting struts are all length $\ell = \sqrt{(2)}h = b/\sqrt{2}$ and that they are inclined at 45 deg to the base and top triangles.

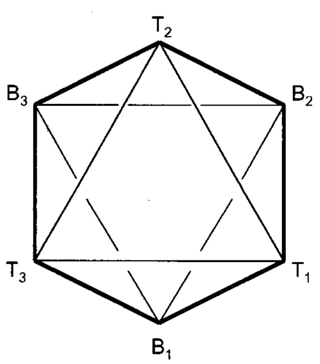


Fig. 5 Star of David plan view.

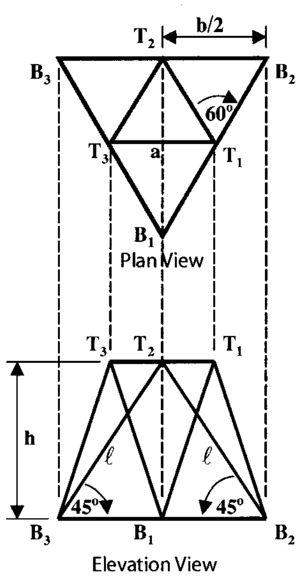


Fig. 6a Optimal Star of David design.

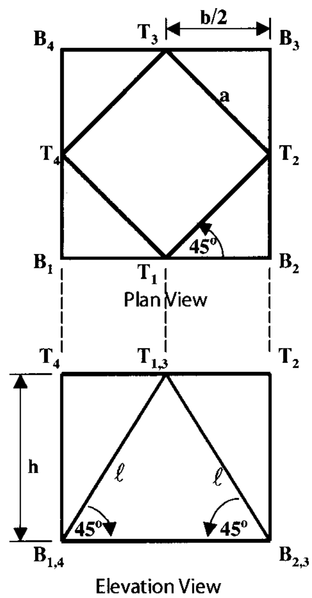


Fig. 6b Two views of 4–4 structure, $n = 4$.

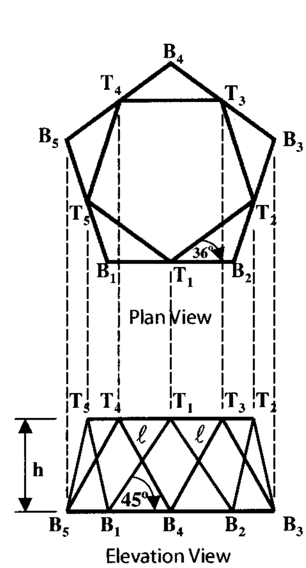


Fig. 6c Two views of 5–5 structure, $n = 5$.

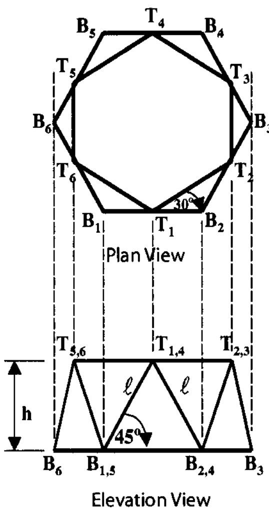


Fig. 6d Two views of 6–6 structure, $n = 6$.

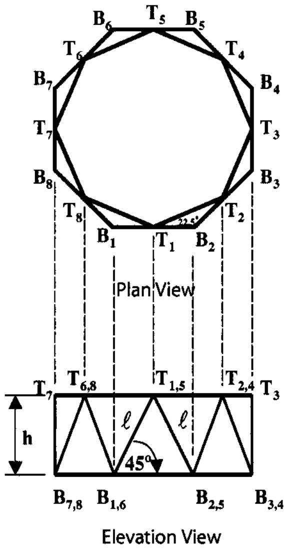


Fig. 6e Two views of 8–8 structure, $n = 8$.

A careful comparison of the results for the global optimum design of the 3–3 truss, the 4–4 truss,[5] and 6–6 truss[8] reveals that there is an extraordinarily simple global optimum design for any n–n truss ring with double connecting struts. The vertical distance h between the UCR and LCR is always given by $h = b/2$. The connecting strut lengths $\ell = \sqrt{(2)}h = b/\sqrt{2}$ are equal and are always inclined to the UCR and LCR equally at 45 deg. The variable a is easily obtained by direct measurement from the plan view or by calculation:

$$a = b \cdot \cos(\pi/n) \tag{10}$$

For the square (Fig. 6b):

$$n = 4, \qquad \phi = 45\,\text{deg}, \qquad a/b = \cos(45\,\text{deg}) = \sqrt{2}\big/2 \approx 0.707$$

For the pentagon (Fig. 6c):

$$n = 5, \qquad \phi = 36\,\text{deg}, \qquad a/b = \cos(36\,\text{deg}) \approx 0.809$$

For the hexagon (Fig. 6d):

$$n = 6, \qquad \phi = 30\,\text{deg}$$

$$a/b = \cos(30\,\text{deg}) = \sqrt{3}\big/2 \approx 0.866$$

For the octagon (Fig. 6e):

$$n = 8, \qquad \phi = 22.5\,\text{deg}, \qquad a/b = \cos(22.5\,\text{deg}) \approx 0.924$$

The preceding design procedure essentially determines the connecting polygon with side a that yields the optimum static stability.

The support truss obtained for $n = 6$ (6–6) with double connecting points is shown in Fig. 7. This was obtained using the construction illustrated by Fig. 6d. Note that M. M. Mikulas and G. Greschik originally proposed this as a conical truss type during in April 2000 at the 2nd Space Technology Alliance Inflatable Structures Working Group, sponsored by the U.S. Air Force Research Laboratory, Dayton, Ohio. Note that the designs based on the optimal static/geometric criteria, which stem from line geometry, immediately yield conical trusses.

Optimization of Ring Trusses with Single Connecting Points

For each statically optimized truss with double connecting points, a corresponding truss can be obtained with single connecting points simply by separating the pairs of double connecting points. This is shown in Fig. 8, where the points B_1, B_2, B_3 and T_1, T_2, T_3 of the 3–3 truss (Fig. 5) are each separated into corresponding pairs of points B_{11}, B_{21}, $B_{22}, \ldots,$ and T_{11}, T_{12}, $T_{22}, \ldots.$ Corresponding pairs of points are connected to form six connecting struts, $B_{11}T_{11}$, $B_{21}T_{12}$, $B_{22}T_{22}$, $B_{31}T_{23}$, $B_{32}T_{33}$, and $B_{13}T_{31}$.

The optimized solution is more complicated because a pair of new variables, the dimensionless ratios α and β, have been introduced, which essentially measure the separation (Fig. 8). Care must be taken in choosing α and β because it has been shown that the structure

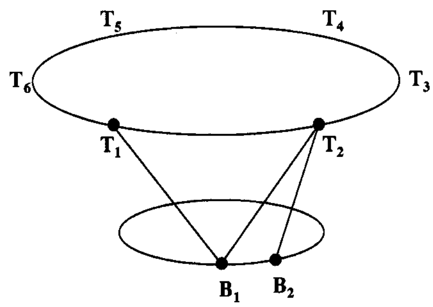


Fig. 7 Double connecting 6–6 antenna support structure.

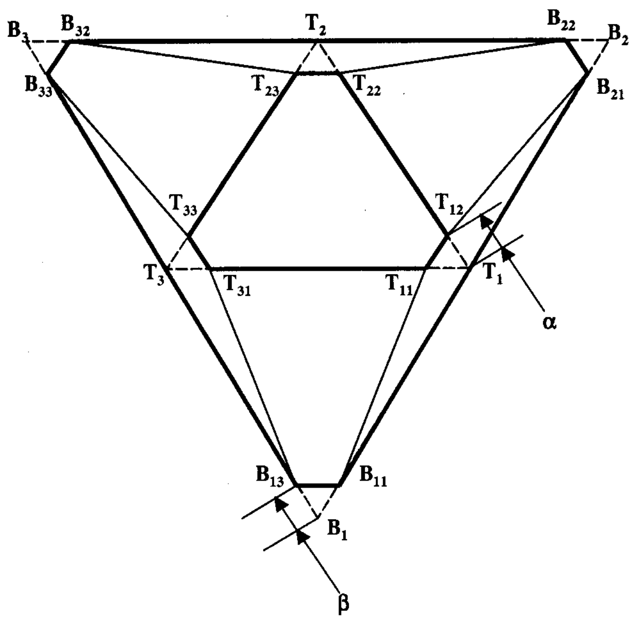


Fig. 8 Construction of a 6–6 truss from a 3–3 truss.

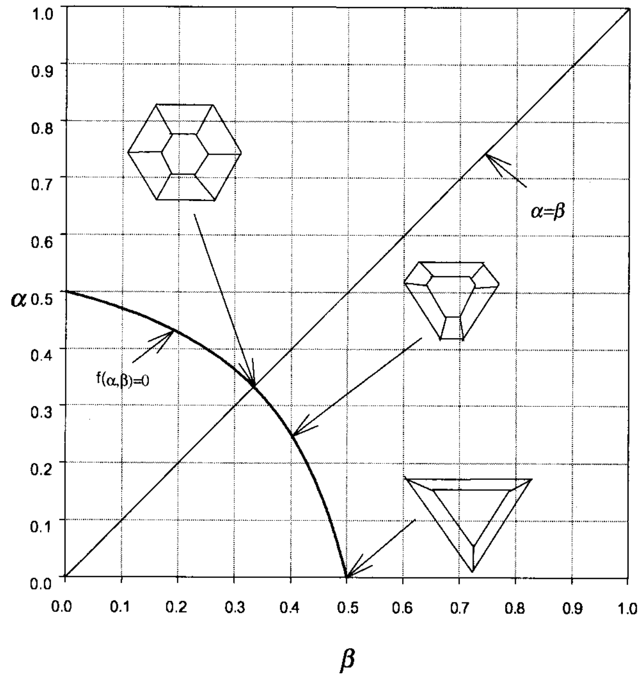


Fig. 9 Function $f(\alpha, \beta) = 3\alpha\beta - 2\alpha - 2\beta + 1 = 0$ configurations of $(\det J) = 0$.

is in an unstable critical point when the ratios α and β satisfy the condition

$$f(\alpha, \beta) = 3\alpha\beta - 2\alpha - 2\beta + 1 = 0 \tag{11}$$

The function $f(\alpha, \beta) = 0$ is shown in Fig. 9, which also illustrates two statically unstable configurations.[8]

It has been determined that given $b = 2a$ for values $\alpha = \frac{1}{4}$ and $\beta = \frac{1}{6}$ then $h = 0.3b$ and $\ell = \sqrt{(2)}h$ for optimum static stability. The six connecting struts are inclined at 45 deg to the top and base, and the distance between the planes is $h = 0.3b$, compared with the distance $h = 0.5b$ for the original 3–3 truss.

Conclusions

A new method for optimized ring trusses was presented. Ring trusses with three–eight sides were addressed. A closed-form solution for the triangular (octahedron) design was determined. Whereas single and double connecting points were the focus of this work, triple (or higher) connections are possible and can be optimized using the same procedures used herein. A conical truss type was discovered during the development of the 6–6 optimal static/geometric design. This can be traced directly from line geometry theory, as would be expected.

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