STATIC ANALYSIS OF TENSEGRITY STRUCTURES

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ABSTRACT

In this paper the mathematical model to perform the static analysis of an antiprism tensegrity structure subjected to a wide variety of external loads is addressed. The virtual work approach is used to deduce the equilibrium equations and a method based on the Newton's Third Law is used to verify the numerical results. Two numerical examples are provided to demonstrate the use of the mathematical model, as well as the verification method.

Keywords: tensegrity, compliance, self-deployment.

INTRODUCTION

Tensegrity structures are spatial structures formed by a combination of rigid elements (the struts) and elastic elements (the ties). No pair of struts touch and the end of each strut is connected to three non-coplanar ties [1]. The struts are always in compression and the ties in tension. The entire configuration stands by itself and maintains its form solely because of the internal arrangement of the ties and the struts [2]. Tensegrity is an abbreviation of tension and integrity.

The development of tensegrity structures is relatively new and the works related have only existed for the 25 years. Kenner [3] established the relation between the rotation of the top and bottom ties. Tobie [2] presented procedures for the generation of tensile structures by physical and graphical means. Yin [1] obtained Kenner's results using energy considerations and found the equilibrium position for the unloaded tensegrity prisms. Stern [4] developed generic design equations to find the lengths of the struts and elastic ties needed to create a desired geometry. Since no external forces are considered his results are referred to the unloaded position of the structure. Knight [5] addressed the problem of stability of tensegrity structures for the design of a deployable antenna.

In this paper the problem of the determination of the equilibrium position of a tensegrity structure when external forces and external moments act on the structure is addressed.

NOMENCLATURE

Figure 1a shows a tensegrity structure formed by n struts each one of length L_S. In every structure it is possible to identify the top ties, the bottom ties and the lateral or connecting ties which are denoted as *T, B* and *L* respectively. Figure 1b shows the same structure. The bottom end of each strut is labeled consecutively as $E_1, E_2, \cdots, E_j, \cdots, E_n$ where 1 identifies the first strut and *n* stands for the last strut. Similarly the top ends of the struts are labeled as $A_1, A_2, \cdots, A_j, \cdots, A_n$. The selection of the first strut is arbitrary.

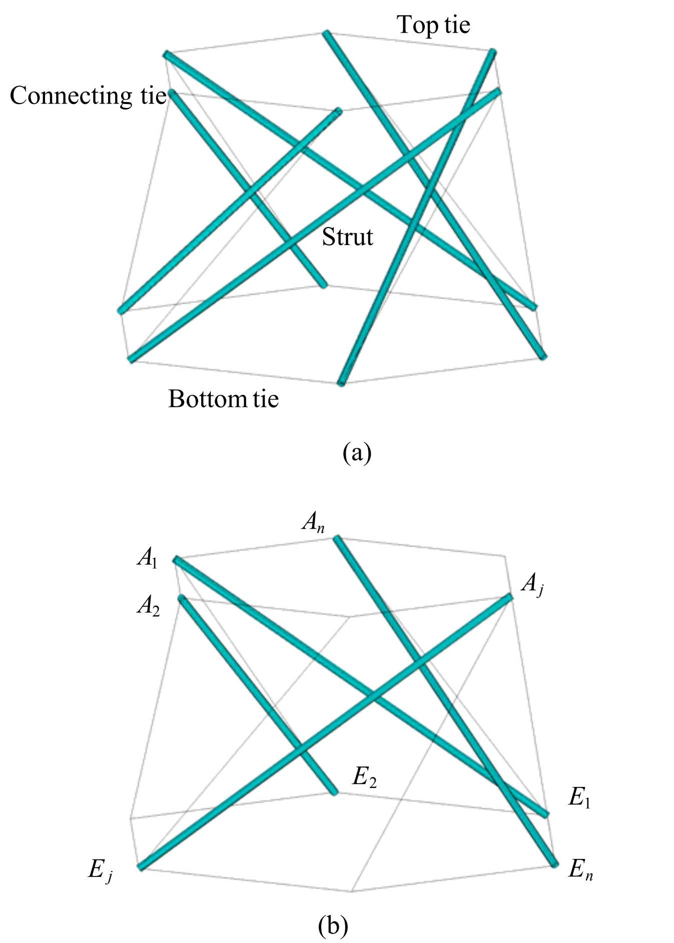

Figure 1. Nomenclature for tensegrity structures. a) Components; b) Strut ends.

GENERALIZED COORDINATES AND TRANSFORMATIONS MATRICES

Figure 2a shows an arbitrary point *P* located on a strut of length L_S. In a reference system *D* whose z axis is along the axis of the strut and with its origin located at the lower end of the strut, the coordinates of ${}^{D}\underline{P}$ are simply (0,0,*l*). However frequently is more convenient for purposes of analysis to express the location of *P* in the global reference system *A*.

If the lower end of one strut is constrained to move on the horizontal plane and also the rotation about its longitudinal axis is constrained, the strut can be modeled by a universal joint. In this way the joint provides the 4 degrees of freedom associated with the strut. The total system has *4*n* degrees of freedom which means there are *4*n* generalized coordinates.

For each strut the generalized coordinates are the horizontal displacements a_j, b_j of the lower end of the strut together with two rotations about the axes of the universal joint, ε_j and β_j. ε_j corresponds to the rotation of the strut about ${}^{B}\underline{x}$ axis and β_j corresponds to the rotation about ${}^{C}\underline{y}$ axis, see Fig. 2b.

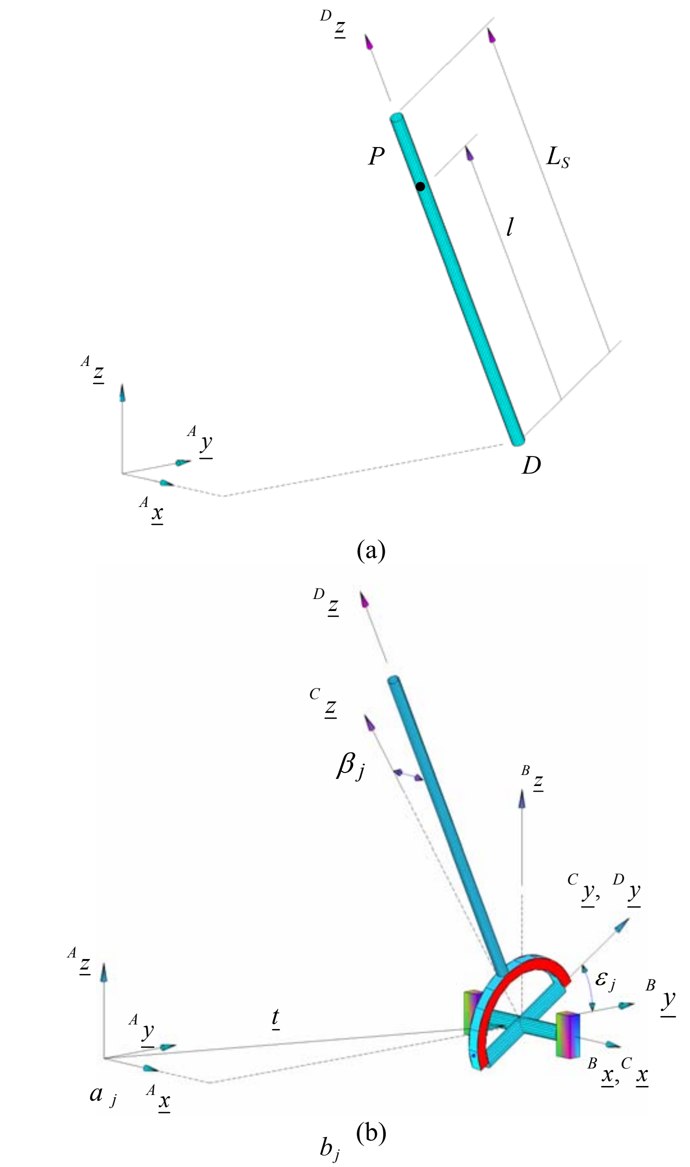

Figure 2. Degrees of freedom associated with one of the struts of a tensegrity structure. a) Arbitrary point on the strut; b) Strut modeled as a universal joint.

The alignment of the *z* axis on the fixed system with the axis of the rod can be accomplished using the following three consecutive transformations, [7]: translation, $\underline{t}=(a_j,\ b_j,\ 0)$, rotation ε about the current *x* axis (${}^{B}\underline{x}$) and rotation β about the current *y* axis (${}^{C}\underline{y}$).

The coordinates of *P* expressed in the global reference system are

$$ {}^{A}\underline{P} = {}^{A}_{B}T_{a,b,0} \; {}^{B}_{C}T_{\varepsilon} \; {}^{C}_{D}T_{\beta} \; {}^{D}\underline{P} \tag{1} $$

where

$$ {}^{A}_{B}T_{a,b,0} = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2} $$

$$ {}^{B}_{C}T_{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\varepsilon & -\sin\varepsilon & 0 \\ 0 & \sin\varepsilon & \cos\varepsilon & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{3} $$

$$ {}^{C}_{D}T_{\beta} = \begin{bmatrix} \cos\beta & 0 & \sin\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\beta & 0 & \cos\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{4} $$

$$ {}^{D}\underline{P} = [0 \;\; 0 \;\; l \;\; 1]^{T} . \tag{5} $$

Substituting the above three expressions into (1) yields

$$ {}^{A}\underline{P} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} l\sin\beta + a \\ -l\sin\varepsilon\cos\beta + b \\ l\cos\varepsilon\cos\beta \\ 1 \end{bmatrix} \tag{6} $$

In addition to the constraint imposed that the lower ends are to remain in the horizontal plane and for each strut to avoid the rotation about its longitudinal axis the following assumptions are made without loss of generality:

The external moments are applied along the axes of the universal joints.

- The struts are massless.
- All the struts have the same length.
- Only one external force is applied per strut.
- There are no dissipative forces acting on the system.
- All the ties are in tension at the equilibrium position, i.e., the initial lengths of the ties are greater than their respective free lengths.
- The free lengths of the top ties are equal.
- The free lengths of the bottom ties are equal.
- The free lengths of the connecting ties are equal.
- The stiffness of all the top ties is the same.
- The stiffness of all the bottom ties is the same.
- The stiffness of all the connecting ties is the same.

COORDINATES OF THE ENDS OF THE STRUTS

The Cartesian coordinates of the lower ends $\underline{E}^{j}$, expressed in the global reference system *A*, are obtained in terms of the generalized coordinates substituting l in (6) by 0, and replacing *a* and *b* by *aj, bj*

$$ {}^{A}\underline{E}_{j} = \begin{bmatrix} a_j \\ b_j \\ 0 \end{bmatrix} . \tag{7} $$

Similarly the coordinates of the upper end of the struts A_j are evaluated by replacing *l* by the length of the struts L_s in (6)

$$ {}^{A}\underline{A}_{j} = \begin{bmatrix} L_s\sin\beta_j + a_j \\ -L_s\sin\varepsilon_j\cos\beta_j + b_j \\ L_s\cos\varepsilon_j\cos\beta_j \end{bmatrix} . \tag{8} $$

Equations (7) and (8) permit one to obtain expressions for the lengths of the top, bottom and lateral ties in terms of the generalized coordinates as follows

$$ T_j = \left(\left(A_{j+1,x} - A_{j,x}\right)^2 + \left(A_{j+1,y} - A_{j,y}\right)^2 + \left(A_{j+1,z} - A_{j,z}\right)^2 \right)^{1/2} \tag{9} $$

$$ B_j = \left(\left(E_{j+1,x} - E_{j,x}\right)^2 + \left(E_{j+1,y} - E_{j,y}\right)^2 + \left(E_{j+1,z} - E_{j,z}\right)^2 \right)^{1/2} \tag{10} $$

$$ L_j = \left(\left(A_{j,x} - E_{j+1,x}\right)^2 + \left(A_{j,y} - E_{j+1,y}\right)^2 + \left(A_{j,z} - E_{j+1,z}\right)^2 \right)^{1/2} \tag{11} $$

where if $j = n$ then $j+1 = 1$

THE PRINCIPLE OF VIRTUAL WORK FOR TENSEGRITY STRUCTURES

The virtual work for systems able to store potential energy can be stated from [6] by

$$ \delta W = \delta W_{nc} + \delta W_{c} \tag{12} $$

where δW is the total virtual work, δW_{nc} is the virtual work performed for non-conservative forces and moments and δW_c is the virtual work performed by conservative forces. δW_{nc} can be represented as

$$ \delta W_{nc} = \delta W_{F} + \delta W_{M} \tag{13} $$

where δW_F is the total virtual work performed by non-conservative forces and δW_M is the total virtual work performed by non-conservative moments.

The virtual work performed by the conservative force *j*, δW_{cj} is $\delta W_{cj} = -\delta V_j$ where δV_j is the potential energy associated with the conservative force j, therefore the total contribution of the conservatives forces δW_c is

$$ \delta W_{c} = -\delta V \tag{14} $$

where δV is the summation over all the δV_j present in the structure.

Substituting (13) and (14) into (12) yields

$$ \delta W = \delta W_{F} + \delta W_{M} - \delta V . \tag{15} $$

In equilibrium the virtual work described by (15) must be zero [6], then the equilibrium conditions can be deduced from

$$ \delta W_{F} + \delta W_{M} - \delta V = 0 . \tag{16} $$

THE VIRTUAL WORK DUE TO THE EXTERNAL FORCES

As it is assumed that there is only one external force acting on each strut, the virtual work δW_F performed by all the external forces is given by

$$\delta W_F = \sum_{j=1}^{n} \underline{F}_j \cdot \delta \underline{r}_j \tag{17}$$

where $\underline{F}_j$ is the external force acting on the strut j and $\delta \underline{r}_j$ is the virtual displacement of $\underline{r}_j$ which is the vector that goes from the origin of the global reference system to the point of application of the external force. If the distance between the point of application of the force and the lower end of the strut is L_{Fj}, see Fig. 3, then an expression for $\underline{r}_j$ in the global system can be obtained from (6) replacing l by L_{Fj} and its rectangular coordinates are

$${}^{A}\underline{r}_j = \begin{bmatrix} {}^{A}r_{j\ x} \\ {}^{A}r_{j\ y} \\ {}^{A}r_{j\ z} \end{bmatrix} = \begin{bmatrix} L_{Fj} \sin\beta_j + a_j \\ -L_{Fj} \sin\varepsilon_j \cos\beta_j + b_j \\ L_{Fj} \cos\varepsilon_j \cos\beta_j \end{bmatrix} . \tag{18}$$

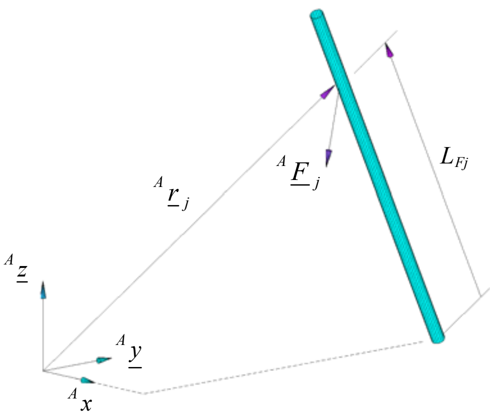

Figure 3. Location of the external force acting on the strut j.

The virtual displacements can be deduced from (18) where the derivatives are taken with respect to the generalized coordinates ε_j, β_j, a_j and b_j as

$$\delta\, {}^{A}\underline{r}_j = \begin{bmatrix} {}^{A}\delta r_{j\ x} \\ {}^{A}\delta r_{j\ y} \\ {}^{A}\delta r_{j\ z} \end{bmatrix}$$

$$= \begin{bmatrix} \delta a_j + L_{Fj} \cos\beta_j \delta\beta_j \\ \delta b_j - L_{Fj} \cos\varepsilon_j \cos\beta_j \delta\varepsilon_j + L_{Fj} \sin\varepsilon_j \sin\beta_j \delta\beta_j \\ -L_{Fj} \sin\varepsilon_j \cos\beta_j \delta\varepsilon_j - L_{Fj} \cos\varepsilon_j \sin\beta_j \delta\beta_j \end{bmatrix} \tag{19}$$

Substituting (19) into (17), the general expression for the virtual work performed by external forces is given by

$$\delta W_F = \sum_{j=1}^{n} \Big(L_{Fj} \big[-{}^{A}F_{j\ y} \cos\varepsilon_j \cos\beta_j - {}^{A}F_{j\ z} \sin\varepsilon_j \cos\beta_j \big] \delta\varepsilon_j$$
$$+ L_{Fj} \big[{}^{A}F_{j\ x} \cos\beta_j + {}^{A}F_{j\ y} \sin\varepsilon_j \sin\beta_j - {}^{A}F_{j\ z} \cos\varepsilon_j \sin\beta_j \big] \delta\beta_j$$
$$+ {}^{A}F_{j\ x} \delta a_j + {}^{A}F_{j\ y} \delta b_j \Big) . \tag{20}$$

THE VIRTUAL WORK DUE TO THE EXTERNAL MOMENTS

The virtual work performed by the external moments is given by

$$\delta W_M = \sum_{j=1}^{n} \underline{M}_{\varepsilon\ j} \cdot \delta \underline{\varepsilon}_j + \underline{M}_{\beta\ j} \cdot \delta \underline{\beta}_j . \tag{21}$$

Provided that in this model of the tensegrity structure the external moments can be exerted only along the axis of the universal joint, $\underline{M}_{\varepsilon j}$, which is collinear with $\delta\underline{\varepsilon}_j$, and $\underline{M}_{\beta j}$, which is collinear with $\delta\underline{\beta}_j$ (see Fig. 4), and therefore Eq.(21) is simplified to

$$\delta W_M = \sum_{j=1}^{n} M_{\varepsilon\ j} \delta\varepsilon_j + M_{\beta\ j} \delta\beta_j . \tag{22}$$

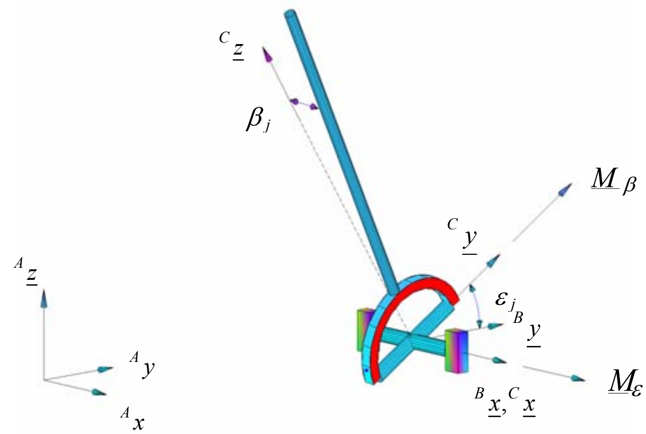

Figure 4. External moments applied to one of the struts of a tensegrity structure.

THE POTENTIAL ENERGY

Since the struts are considered massless the term related to the potential energy in the principle of virtual work is the resultant of the elastic potential energy contributions given by the ties. The potential elastic energy for a general tie j is given by, [6]

$$V_j = \frac{1}{2} k \left(w_j - w_{j0} \right)^2 \tag{23}$$

where V_j is the elastic potential energy for tie j, k the tie stiffness, w_j the current length of the tie j and w_{jo} the free length of the tie j. Therefore the differential of the potential energy for tie j is

$$\delta V_j = k \left(w_j - w_{j0} \right) \delta w_j . \tag{24}$$

The differential of the potential energy for all the tensegrity structure, δV, is the resultant of the contributions of the top ties, the bottom ties and the lateral ties and can be expressed as

$$\delta V = \sum_{j=1}^{n} k_T\left(T_j - T_o\right)\delta T_j + \sum_{j=1}^{n} k_B\left(B_j - B_o\right)\delta B_j + \sum_{j=1}^{n} k_L\left(L_j - L_o\right)\delta L_j \quad (25)$$

where k_T, k_B, k_L are the stiffness of the top, bottom and connecting ties respectively, T_0, B_0 and L_0 are the free lengths of the top, bottom and connecting ties respectively and T_j, B_j and L_j are given by (9), (10) and (11) and are functions of some of the generalized coordinates.

THE GENERAL EQUATIONS

Now that each one of the terms contributing to the virtual work has been evaluated, the equilibrium condition for the general tensegrity structure can be established. Substituting (20), (22) and (25) into (16) and re-grouping yields

$$f_1\delta a_1 + f_2\delta a_2 + \ldots.. f_n\delta a_n + f_{n+1}\delta b_1 + f_{n+2}\delta b_2 + \ldots.. f_{2n}\delta b_n + f_{2n+1}\delta\varepsilon_1 + f_{2n+2}\delta\varepsilon_2 + \ldots.. f_{3n}\delta\varepsilon_n + f_{3n+1}\delta\beta_1 + f_{3n+2}\delta\beta_2 + \ldots.. f_{4n}\delta\beta_n = 0 \quad (26)$$

where

$$f_i = {}^{A}F_{i\ x} - \sum_{j=1}^{n} k_T\left(T_j - T_o\right)\frac{\partial T_j}{\partial a_i} - \sum_{j=1}^{n} k_B\left(B_j - B_o\right)\frac{\partial B_j}{\partial a_i} - \sum_{j=1}^{n} k_L\left(L_j - L_o\right)\frac{\partial L_j}{\partial a_i} \quad (27)$$

$$f_{n+i} = {}^{A}F_{i\ y} - \sum_{j=1}^{n} k_T\left(T_j - T_o\right)\frac{\partial T_j}{\partial b_i} - \sum_{j=1}^{n} k_B\left(B_j - B_o\right)\frac{\partial B_j}{\partial b_i} - \sum_{j=1}^{n} k_L\left(L_j - L_o\right)\frac{\partial L_j}{\partial b_i} \quad (28)$$

$$f_{2n+i} = L_{Fi}\left[-{}^{A}F_{i\ y}\cos\varepsilon_i\cos\beta_i - {}^{A}F_{i\ z}\sin\varepsilon_i\cos\beta_i\right] + M_{\varepsilon\ i} - \sum_{j=1}^{n} k_T\left(T_j - T_o\right)\frac{\partial T_j}{\partial\varepsilon_i} - \sum_{j=1}^{n} k_B\left(B_j - B_o\right)\frac{\partial B_j}{\partial\varepsilon_i} - \sum_{j=1}^{n} k_L\left(L_j - L_o\right)\frac{\partial L_j}{\partial\varepsilon_i} \quad (29)$$

$$f_{3n+i} = L_{Fi}\left[{}^{A}F_{i\ x}\cos\beta_i + {}^{A}F_{i\ y}\sin\varepsilon_i\sin\beta_i - {}^{A}F_{i\ z}\cos\varepsilon_i\sin\beta_i\right] + M_{\beta\ i} - \sum_{j=1}^{n} k_T\left(T_j - T_o\right)\frac{\partial T_j}{\partial\beta_i} - \sum_{j=1}^{n} k_B\left(B_j - B_o\right)\frac{\partial B_j}{\partial\beta_i} - \sum_{j=1}^{n} k_L\left(L_j - L_o\right)\frac{\partial L_j}{\partial\beta_i} \quad (30)$$

$i = 1, 2, \ . \ . \ . \ , n$.

Equation (26) must be satisfied for all the values of the virtual displacements which in general are different from zero, then

$$f_1 = 0, \quad f_2 = 0, \quad \vdots \quad f_{4n} = 0 \quad (31)$$

where f_i is given by Eq. (27) to (30). Equations (31) represent a strongly coupled system of *4*n* equations depending only on the *4*n* generalized coordinates. The equilibrium position for a general tensegrity structure is obtained by solving numerically the set (31) for a_1, b_1, ε_1, β_1, ... , a_n, b_n, ε_n, β_n. After that Eq. (7) and (8) yield explicitly expressions for the coordinates of the ends of the struts in the global coordinate system.

INITIAL CONDITIONS

To be able to solve (31) iteratively it is necessary to first find a proper set of values for the generalized coordinates in the unloaded position. This is accomplished using Yin's method [1], which is presented here without proof. Yin derived the following expressions:

$$k_L\left(1 - \frac{L_o}{L}\right) R_B - 2k_T\left(R_T - R_{To}\right)\sin\frac{\gamma}{2} = 0, \quad (32)$$

$$k_L\left(1 - \frac{L_o}{L}\right) R_T - 2k_B\left(R_B - R_{Bo}\right)\sin\frac{\gamma}{2} = 0, \quad (33)$$

$$L - \sqrt{L_s^2 + 2R_B R_T\left[\cos(\alpha+\gamma) - \cos\alpha\right]} = 0 \quad (34)$$

where

$$R_{T0} = \frac{T_o}{2\sin\frac{\gamma}{2}} \text{ and } R_{B0} = \frac{B_o}{2\sin\frac{\gamma}{2}} . \quad (35)$$

The angles γ and α are given by

$$\gamma = \frac{2\pi}{n} \text{ and } \alpha = \frac{\pi}{2} - \frac{\pi}{n} \quad (36)$$

where n is the number of struts.

The three unknowns R_B, R_T and the length of the connecting ties L are solved using Eq. (32) through (34). These values are used to solve the following set of generalized coordinates for the unloaded position.

$$a_{j,0} = R_B\cos(\ (j-1)\ \gamma) , \ j = 1, 2, \ldots , n , \quad (37)$$

$$b_{j,0} = R_B\sin(\ (j-1)\ \gamma) , \ j = 1, 2, \ldots , n , \quad (38)$$

$$\tan\varepsilon_{j,0} = \frac{b_{j,0} - R_T\sin(\ (j-1)\ \gamma+\alpha)}{H} , \quad (39)$$

$$\tan\beta_{j,0} = \frac{R_T\cos(\ (j-1)\ \gamma+\alpha) - a_{j,0}}{\left(\dfrac{b_{j,0} - R_T\sin(\ (j-1)\ \gamma+\alpha)}{\sin\varepsilon_{j,0}}\right)} , \quad (40)$$

$$\text{where } H = \sqrt{L_s^2 - R_B^2 - R_T^2 - 2R_B R_T\sin\frac{\gamma}{2}} \quad (41)$$

and if $j = 1$ then $j - 1 = n$.

VERIFICATION OF THE NUMERICAL RESULTS

Because of the complexity of the equilibrium equations it is essential to verify the answers obtained. An independent validation of the results can be accomplished using Newton's Third Law. If there are no external moments acting on an isolated strut, it is sufficient to perform the summation of moments with respect to the lower end of the strut. If there are external moments the verification process involves additional steps.

The unitized Plücker coordinates of a line joining two finite points *(x1, y1, z1)* and *(x2, y2, z2)*, as is the case of the forces in the ties, can be written in the global reference system as, [8]

$$^{A}\hat{\$} = \frac{1}{\sqrt{L^2 + M^2 + N^2}} \begin{bmatrix} L & M & N & P & Q & R \end{bmatrix}^T \tag{42}$$

where

$$L = \begin{vmatrix} 1 & x_1 \\ 1 & x_2 \end{vmatrix}, \; M = \begin{vmatrix} 1 & y_1 \\ 1 & y_2 \end{vmatrix}, \; N = \begin{vmatrix} 1 & z_1 \\ 1 & z_2 \end{vmatrix} \tag{43}$$

and

$$P = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}, \; Q = \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix}, \; R = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \tag{44}$$

The subindex *1* in (43) and (44) identifies the end of the tie attached to the current strut and the subindex *2* is for the remaining end of the tie. Further the coordinates of the ends of the ties can be evaluated using (7) and (8).

If *L, M* and *N* are simultaneously equal to zero (42) must be modified to

$$^{A}\hat{\$} = \frac{1}{\sqrt{P^2 + Q^2 + R^2}} \begin{bmatrix} 0 & 0 & 0 & P & Q & R \end{bmatrix}^T \tag{45}$$

When an external force $^{A}\underline{F}_j$ and its point of application are known, the Plücker coordinates are obtained by

$$^{A}\underline{\$}_F = \begin{bmatrix} ^{A}\underline{F}_j \\ ^{A}\underline{r}_j \times {}^{A}\underline{F}_j \end{bmatrix} \tag{46}$$

where $^{A}\underline{F}_j$ corresponds to the external force expressed in the global reference system and $^{A}\underline{r}_j$ is given by (18).

The Plücker coordinates can be expressed in a new system that is translated and rotated with respect to the global reference system. If the new system is the *C* system, this is the system defined for the axes of the universal joint, the expression that relates the Plücker coordinates in the *A* system and the *C* system is, [8]

$$^{C}\underline{\$} = e^{-1}\; {}^{A}\underline{\$} \tag{47}$$

where

$$e^{-1} = \begin{bmatrix} {}^{A}_{C}R^T & \underline{O}_3 \\ {}^{A}_{C}R^T A_3^T & {}^{A}_{C}R^T \end{bmatrix} \tag{48}$$

$$A_3 = \begin{bmatrix} 0 & 0 & b_j \\ 0 & 0 & -a_j \\ -b_j & a_j & 0 \end{bmatrix} \tag{49}$$

$${}^{A}_{C}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varepsilon_j & -\sin\varepsilon_j \\ 0 & \sin\varepsilon_j & \cos\varepsilon_j \end{bmatrix} \tag{50}$$

and $\underline{0}3$ is a 3 by 3 zeroes matrix.

Figure 5 shows the free body diagram for an arbitrary strut modeled with a universal joint. In addition to the forces in the ties there is an external force $\underline{F}j$ which is known, a reaction force $\underline{R}$ passing through the lower end and a reaction moment $\underline{RM}$ at the lower end. Newton's Third Law expressed in Plücker coordinates in the *C* system is

$$\begin{aligned} &F_{Aj\,Aj+1}\,{}^{C}\hat{\$}_{Aj\,Aj+1} + F_{Aj\,Aj-1}\,{}^{C}\hat{\$}_{Aj\,Aj-1} + F_{Aj\,Ej+1}\,{}^{C}\hat{\$}_{Aj\,Ej+1} \\ &+ F_{Ej\,Ej+1}\,{}^{C}\hat{\$}_{Ej\,Ej+1} + F_{Ej\,Ej-1}\,{}^{C}\hat{\$}_{Ej\,Ej-1} + F_{Ej\,Aj-1}\,{}^{C}\hat{\$}_{Ej\,Aj-1} \\ &+ {}^{C}\underline{\$}_F + M_\varepsilon\,{}^{C}\hat{\$}_{M\varepsilon} + M_\beta\,{}^{C}\hat{\$}_{M\beta} + {}^{C}\underline{\$}_R + {}^{C}\underline{\$}_{RM} = \underline{0} \end{aligned} \tag{51}$$

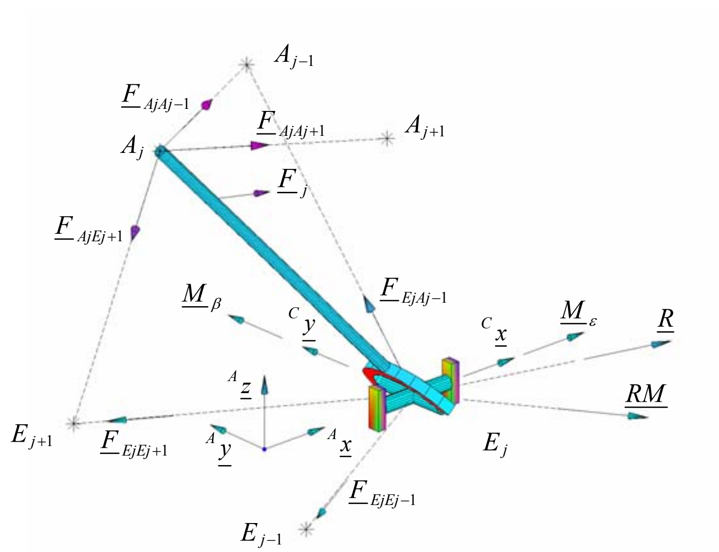

Figure 5. Free Body diagram for an arbitrary strut modeled with a universal joint.

The coefficients $F_{Aj\,Aj+1}$, $F_{Aj\,Aj-1}, \cdots$ represent the magnitudes of the forces in the ties $A_{j\,Aj+1}$, $A_{j\,Aj-1} \cdots$ and are given by *k*(w-wo)* where *k* is the stiffness, *w* the current length and *wo* the free length of the tie. The current lengths are given by (9) and (11) for the top ties and connecting ties respectively. It should be noted that the magnitude does not depend of the reference system which is used.

The unitized Plücker coordinates $^{C}\hat{\$}$ for each one of the ties can be calculated in the *A* system using (42) thorough (44) and then converted to the *C* system using (47) through (50).

The Plücker coordinates of the external force acting on the current strut $\underline{\$}_F$, can be evaluated in the *A* system using (46) and then converted to the *C* system with the aid of (47) through (50).

M_ε and M_β are the magnitudes of the external moments and their unitized Plücker coordinates in the C system are given by ${}^C\hat{\$}_{M\varepsilon} = [0\ 0\ 0\ 1\ 0\ 0]^T$ and ${}^C\hat{\$}_{M\beta} = [0\ 0\ 0\ 0\ 1\ 0]^T$.

Since the reaction force ${}^C\$_R$ expressed in the C system is a pure force and the reaction moment ${}^C\$_{RM}$ expressed in the C system is a pure moment they have the form $[{}^CR_x\ {}^CR_y\ {}^CR_z\ 0\ 0\ 0]^T$ and $[0\ 0\ 0\ {}^CRM_x\ {}^CRM_y\ {}^CRM_z]^T$ respectively. Further they are the only unknowns in (51).

After expanding (51), rows four and five represent the components in the x and y directions of the summation of moments about the lower end of the strut. A universal joint cannot exert moment along its own axes. If after a numerical evaluation, RMx and RMy are both zero, then the equilibrium of moments is maintained solely due to the forces in the ties and to the external loads (if any) and therefore the current position is an equilibrium position.

EXAMPLE 1: ANALYSIS OF A TENSEGRITY STRUCTURE WITH 3 STRUTS.

ANALYSIS FOR THE UNLOADED POSITION

It is required to evaluate the unloaded equilibrium position of a tensegrity structure with 3 struts and with the stiffness and free lengths shown in Table 1. Each of the struts has a length $L_s = 100mm$.

Table 1. Stiffness and free lengths for example 1.

	Stiffness (N/mm)	Free lengths (mm)
Top ties	$k_T = 0.5$	$T_0 = 35$
Bottom ties	$k_B = 0.3$	$B_0 = 52$
Connecting ties	$k_L = 1$	$L_0 = 80$

For this example $n = 3$ then (35) and (36) yield $\gamma = 120°$, $\alpha = 15°$, $R_{T0} = 20.207mm$ and $R_{B0} = 32.02mm$.

The solution of (32), (33) and (34) yields $R_B = 33.0568mm$, $R_T = 22.8422mm$ and from (41), $H = 84.1287mm$. The initial values of the generalized coordinates are obtained from (37) through (40) and are listed in Table 2.

Table 2. Initial values of the generalized coordinates for the structure of the example 1.

	Strut 1	Strut 2	Strut 3
a (mm)	33.0568	-16.5284	-16.5284
b (mm)	0	28.6280	-28.6280
ε (rad)	-0.1349	0.5491	-0.4443
β (rad)	-0.5567	0.1660	0.3716

ANALYSIS FOR THE LOADED POSITION

It is required to evaluate the final equilibrium position of the structure when the external forces listed in Table 3 are applied vertically at the upper end of the struts and there are no constraints acting on the struts.

Table 3. External forces and their application points acting on the structure of the example 1.

	Strut 1	Strut 2	Strut 3
F_x (N)	0	0	0
F_y (N)	0	0	0
F_z (N)	-10	-10	-10
L_F (mm)	100	100	100

Since the system has 3 struts and there are no constraints then there are 12 degrees of freedom and therefore 12 equations are required, one per each generalized coordinate. Equation (27) yields f_1, f_2 and f_3, Eq. (28) yields f_4, f_5 and f_6, Eq. (29) yields f_7, f_8 and f_9, and Eq. (30) yields f_{10}, f_{11} and f_{12}. Each f_i is equated to zero and then the system is solved numerically using the software developed and the initial values listed in Table 2. The values of the generalized coordinates for the structure when the external forces are applied are shown in Table 4.

Table 4. Generalized coordinates for the final position for the structure of the example 1.

	Strut 1	Strut 2	Strut 3
a (mm)	40.8573	-20.4241	-20.4332
b (mm)	-0.0053	35.3861	-35.3808
ε (rad)	-0.0269	0.6808	-0.6643
β (rad)	-0.7434	0.3271	0.3635

Using the values of Tables 2 and 4, Eq. (7) and (8) yield the coordinates of the ends of the struts for the initial and final position. The results are summarized in Table 5. Figure 6 shows the structure in its initial and final equilibrium position.

Table 5. Lower and upper coordinates for the unloaded and final position for the structure of the example 1 (mm).

	Strut 1		Strut 2		Strut 3	
	Initial Pos.	Final Pos.	Initial Pos.	Final Pos.	Initial Pos.	Final Pos.
E_x	33.0568	40.8573	-16.528	-20.424	-16.528	-20.433
E_y	0	-0.0053	28.6280	35.3861	-28.628	-35.380
E_z	0	0	0	0	0	0
A_x	-19.781	-26.824	0	11.7016	19.7819	15.1224
A_y	11.4211	1.9750	-22.842	-24.217	11.4211	22.2429
A_z	84.1287	73.5888	84.1287	73.5888	84.1287	73.5888

the equilibrium of moments the result is $\underline{0}$ what means the current position is indeed an equilibrium position.

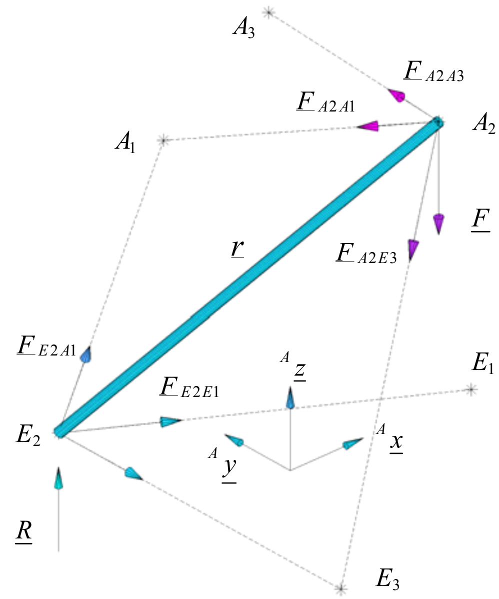

Figure 7. Free body diagram for the second strut of the structure of example 1 in the last position.

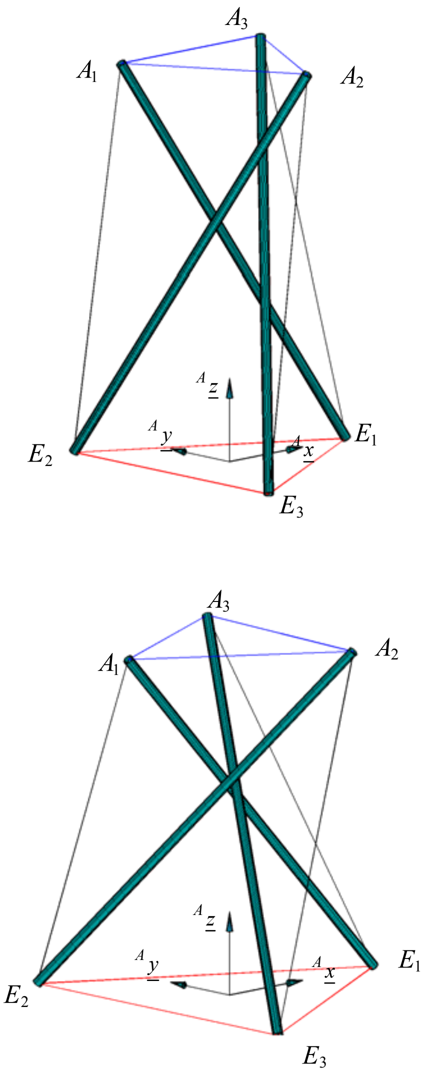

Figure 6. Unloaded and final equilibrium position for the structure of the example 1.

Figure 7 shows the free body diagram of strut 2 for the final position. The summation of moments with respect to the lower end E_2 is given by the following equation

$$\underline{r}\times\underline{F}+\underline{r}\times\underline{F}_{A2A1}+\underline{r}\times\underline{F}_{A2A3}+\underline{r}\times\underline{F}_{A2E3} \quad = \quad \underline{0}$$

where

$$\underline{r}=\underline{A}_2-\underline{E}_2 \; ,$$

$$\underline{F}_{A2A1}=k_T(|\underline{A}_1-\underline{A}_2|-T_0)\frac{\underline{A}_1-\underline{A}_2}{|\underline{A}_1-\underline{A}_2|} \; ,$$

$$\underline{F}_{A2A3}=k_T(|\underline{A}_3-\underline{A}_2|-T_0)\frac{\underline{A}_3-\underline{A}_2}{|\underline{A}_3-\underline{A}_2|} \; ,$$

$$\underline{F}_{A2E3}=k_L(|\underline{E}_3-\underline{A}_2|-L_0)\frac{\underline{E}_3-\underline{A}_2}{|\underline{E}_3-\underline{A}_2|} \; .$$

The numerical values are obtained from Tables 1 and 5. In addition from Table 2, F=[0 0 –10]^T. It can be verified that when the numerical values are replaced in the expression for the equilibrium of moments the result is $\underline{0}$ what means the current position is indeed an equilibrium position.

EXAMPLE 2: ANALYSIS OF A TENSEGRITY STRUCTURE WITH 4 STRUTS.

ANALYSIS FOR THE UNLOADED POSITION

It is required to evaluate the unloaded equilibrium position of a tensegrity structure with 4 struts with the stiffness and free lengths shown in Table 6. Each of its struts has a length $L_s = 100mm$.

Table 6. Stiffness and free lengths for example 2.

	Stiffness (N/mm)	Free lengths (mm)
Top ties	$k_T = 0.5$	$T_0 = 40$
Bottom ties	$k_B = 0.5$	$B_0 = 40$
Connecting ties	$k_L = 0.5$	$L_0 = 40$

For this example $n = 4$ then (35) and (36) yield $\gamma = 90°$, $\alpha = 22.55°$, $R_{T0} = 28.2843mm$ and $R_{B0} = 28.2843mm$.

The solution of (32), (33) and (34) yields $R_B = 41.2528mm$, $R_T = 41.2528mm$ and from (41), $H = 64.7280mm$. The parameters that define the location of the struts at the initial position are obtained from (37) through (40) and are listed in Table 7.

Table 7. Parameters for the location of the struts for the structure of example 2 in the unloaded position.

	Strut 1	Strut 2	Strut 3	Strut 4
a (mm)	41.2528	0	-41.2528	0
b (mm)	0	41.2528	0	-41.2528
ε (rad)	-0.4234	0.8275	0.4234	-0.8275
β (rad)	-0.7813	-0.2960	0.7813	0.2960

ANALYSIS FOR THE LOADED POSITION

It is required to evaluate the final equilibrium position of the structure when the external moments listed in Table 8 are applied along the axes of the universal joints that model the structure, see Fig. 8, and the lower ends of the struts are constrained in such a way that they cannot move in the horizontal plane.

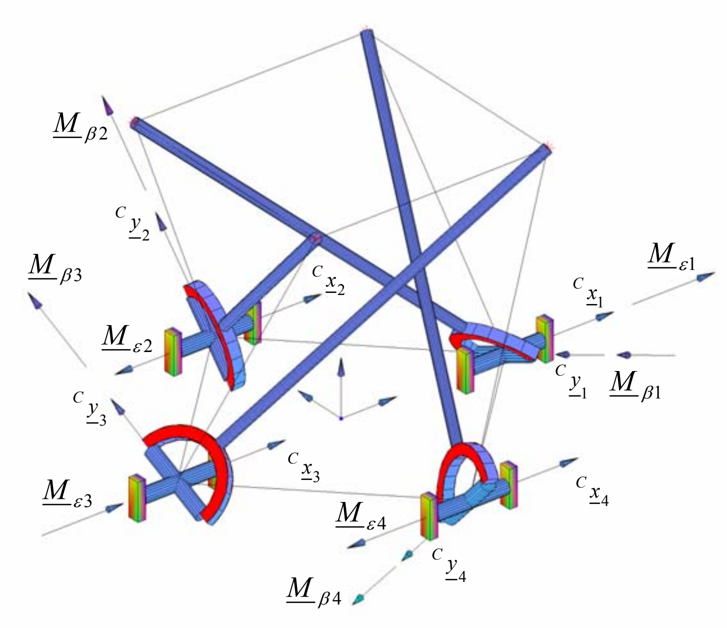

Figure 8. Directions of the external moments for the structure of example 2.

Table 8. External moments acting on the structure of example 2.

	Strut 1	Strut 2	Strut 3	Strut 4
M_ε (N.mm)	450	-900	450	-900
M_β (N.mm)	450	450	450	450

Because there are 2 constraints per strut there are 8 degrees of freedom for this system, and they are associated with the rotations of the struts. The generalized coordinates are ε_1, β_1, ε_2, β_2, ε_3, β_3 and ε_4, β_4, where the subscript indicates the number of the strut.

The required 8 equations are obtained as follows: Eq. (29) yields f_9, f_{10}, f_{11} and f_{12} and Eq. (30) yields f_{13}, f_{14}, f_{15} and f_{16}. The solution of this system yields the generalized coordinates for the final position listed in Table 9.

Using the values of Tables 7 and 9, Eq. (7) and (8) yield the coordinates of the ends of the struts for the initial and final position. The results are summarized in Table 10. Figure 9 shows the structure in its initial and final equilibrium position. It should be noted that the numerical values of *Ex,Ey* and *Ez* are the same for the unloaded and final position due to the constraints imposed to the lower ends.

Table 9. Generalized coordinates for the final position for the structure of example 2.

	Strut 1	Strut 2	Strut 3	Strut 4
ε (rad)	-0.468	0.6311	0.8140	-1.1716
β (rad)	-0.291	0.1598	1.2136	0.5948

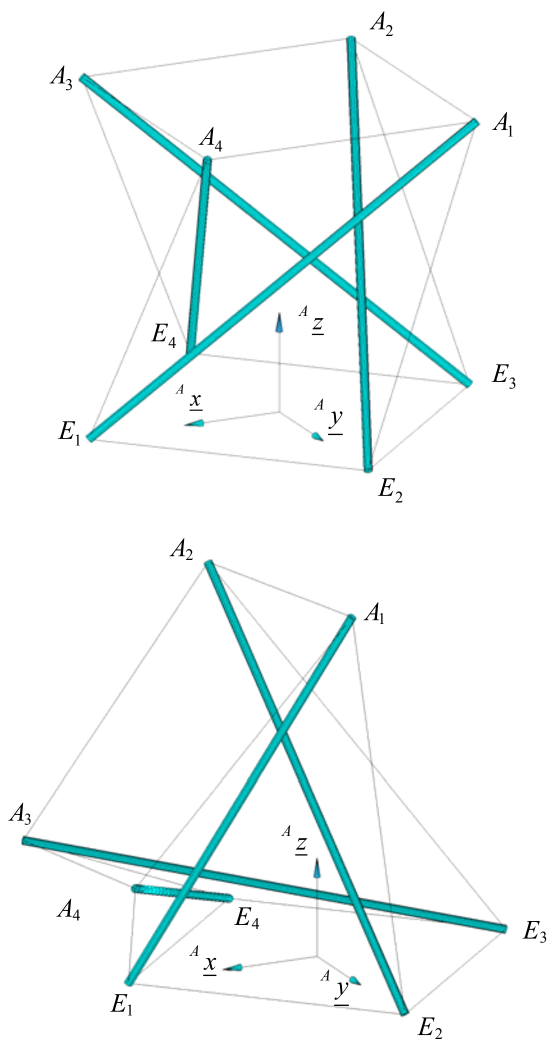

Figure 9. Unloaded and final equilibrium position for the structure of example 2.

Figure 10 shows the free body diagram for the fourth strut in its final position modeled with a universal joint. In addition to the forces in the ties and the external moments, all known, there are a reaction force $\underline{R}$ and a reaction moment $\underline{RM}$, these both of which are unknowns. The equilibrium equation expressed in Plücker coordinates in the C system, this is the system defined by the axes of the universal joint, see Eq.(51), is

$$\begin{aligned} & k_T(|\underline{A}_4-\underline{A}_1|-T_0)\,{}^C\hat{\$}_{A4A1} + k_T(|\underline{A}_4-\underline{A}_3|-T_0)\,{}^C\hat{\$}_{A4A3} \\ & + k_L(|\underline{A}_4-\underline{E}_1|-L_0)\,{}^C\hat{\$}_{A4E1} + k_B(|\underline{E}_4-\underline{E}_1|-B_0)\,{}^C\hat{\$}_{E4E1} \\ & + k_B(|\underline{E}_4-\underline{E}_3|-B_0)\,{}^C\hat{\$}_{E4E3} + k_L(|\underline{E}_4-\underline{A}_3|-L_0)\,{}^C\hat{\$}_{E4A3} \\ & + M_\varepsilon\,{}^C\hat{\$}_{M\varepsilon} + M_\beta\,{}^C\hat{\$}_{M\beta} + {}^C\underline{\$}_R + {}^C\underline{\$}_{RM} = \underline{0} \end{aligned}$$

Table 10. Lower and upper coordinates for the unloaded and final position for the example 2 (mm).

	Strut 1		Strut 2		Strut 3		Strut 4	
	Init. Pos.	Fin. Pos.	Init.Pos.	Fin. Pos.	Init.Pos.	Fin. Pos.	Init.Pos.	Fin. Pos.
E_x	41.2528	41.2528	0	0	-41.2528	-41.2528	0	0
E_y	0	0	41.2528	41.2528	0	0	-41.2528	-41.2528
E_z	0	0	0	0	0	0	0	0
A_x	-29.1701	12.5079	-29.1701	15.9151	29.1701	52.4370	29.1701	56.0322
A_y	29.1701	43.2859	-29.1701	-16.9992	-29.1701	-25.4176	29.1701	35.0631
A_z	64.7280	85.4404	64.7280	79.7083	64.7280	24.0035	64.7280	32.1911

${}^{C}\hat{\$}_{A4A1}$, ${}^{C}\hat{\$}_{A4A3}\cdots$ are the unitized Plücker coordinates of the lines joining the points A_4A_1, $A_4A_3\cdots$ They are calculated in the *A* system with the aid of (42) thorough (44) and using the data of Table 10 for the structure in its final position. Then they are converted to the *C* system using (47) through (50).

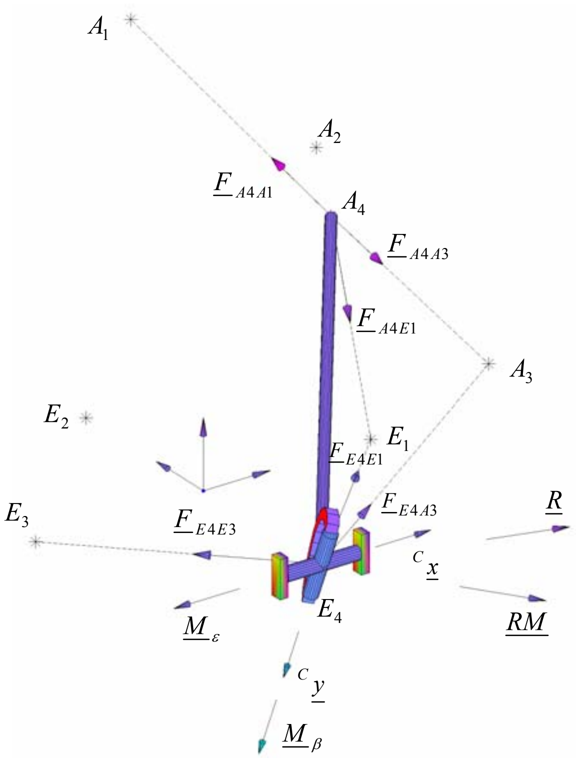

Figure 10. Free body diagram for the fourth strut of example 2 in the last position.

It should be noted that *aj* and *bj* in (49) correspond to the coordinates *x* and *y* for the lower end of the strut 4. They do not change for this example and their numerical values are listed in Table 7. The angle *εj* in (50) is one of the generalized coordinates associated with the rotations of strut 4 in the last position and is listed in Table 9.

$M\varepsilon\,{}^{C}\underline{\$}_{M\varepsilon}$ are the Plücker coordinates of the moment acting along axis ${}^{C}\underline{x}_4$ and according to the initial data is given by 900* $[0\ 0\ 0\ {-1}\ 0\ 0]^T$, see Table 8.

Similarly $M\beta\,{}^{C}\underline{\$}_{M\beta}$ are the Plücker coordinates of the moment acting along axis ${}^{C}\underline{y}_4$ and is given by 450* $[0\ 0\ 0\ 0\ 1\ 0]^T$, see Table 4.

${}^{C}\underline{\$}_{R}$ is the reaction force through E_4 which components are $[Rx\ Ry\ Rz\ 0\ 0\ 0]^T$ and ${}^{C}\underline{\$}_{RM}$ is the reaction moment along the axes of the *C* system which components are ^{C} $[0\ 0\ 0\ RMx\ RMy\ RMz]^T$.

The substitution of numerical values in the last expression yields

$$14.6319\begin{bmatrix}-0.6284\\-0.6622\\0.4082\\54.8493\\-74.9186\\-37.1052\end{bmatrix}+10.5691\begin{bmatrix}-0.0588\\-0.2611\\-0.9635\\21.6248\\49.1177\\-14.6290\end{bmatrix}+4.9205\begin{bmatrix}-0.2965\\0.3217\\-0.8992\\-26.6442\\25.8241\\18.0246\end{bmatrix}$$

$$+9.1701\begin{bmatrix}0.7071\\0.2748\\0.6515\\0\\0\\0\end{bmatrix}+9.1701\begin{bmatrix}-0.7071\\0.2748\\0.6515\\0\\0\\0\end{bmatrix}+9.9022\begin{bmatrix}0.8768\\-0.2669\\0.4\\0\\0\\0\end{bmatrix}$$

$$-900\begin{bmatrix}0\\0\\0\\1\\0\\0\end{bmatrix}+450\begin{bmatrix}0\\0\\0\\0\\1\\0\end{bmatrix}+\begin{bmatrix}{}^{C}R_x\\{}^{C}R_y\\{}^{C}R_z\\0\\0\\0\end{bmatrix}+\begin{bmatrix}0\\0\\0\\{}^{C}RM_x\\{}^{C}RM_y\\{}^{C}RM_z\end{bmatrix}=\underline{0}$$

The expansion of rows four and five yield *RMx=0* and *RMy=0*, this is the component of the reaction moment along the axes of the universal joint is zero. Since the universal joint cannot exert any reaction moment along its axes, the foregoing results confirm that the current position is an equilibrium position. The same procedure is executed for the software for all the struts and all the positions of the structure.

CONCLUSIONS

The model allows one to analyze a general anti-prism tensegrity structure subjected to a wide variety of external loads and the software developed is able to solve the system of equations generated for the model. The results are presented both numerically and in a three dimension graphical representation, which permits one to visualize the behavior of the structure.

The model is developed using the virtual work approach and all the results are checked using Newton's Third Law. This verification assures that the answers produced by the numerical method accurately correspond to equilibrium positions.

Mathematical models for variations of the basic configuration of tensegrity structures such as the reinforced tensegrity prisms might be developed following the same procedure presented in this research.

The mathematical model always assumes that the ties are in tension. If under the action of the external load the distance between two strut ends which are connected by a tie is less than the free length for that tie, the model is no longer a valid representation of the structure and as result no convergence is found and the software cannot yield a solution for that particular situation.

Also when two struts or a tie and a strut intersect, the Jacobian for the structure vanishes, Lee et al. [9], and corresponds to a singular configuration that the software cannot solve.

In certain situations even though there are no singularities and all the ties are in tension, a small increase in the external load can make it impossible for the software to converge to a solution, i.e., it is not possible to find a new equilibrium position. The system suffers a sudden change and it jumps from one equilibrium position to another for a smooth transition force. This is known as a catastrophe, Hines [10], and Arnold's [11]. Catastrophe Theory is a well developed classical method. It describes sudden changes caused by a gradually changing input. It offers a better understanding of the phenomena reported here which is beyond the scope of this work.

ACKNOWLEGEMENTS

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DEDICATION

This paper is dedicated to the memory of Dr. Joseph Duffy whom the other two authors had the pleasure of working with in this endeavor. Dr. Duffy was a mentor and colleague and an inspiration to a generation of kinematicians. His contributions to the field are truly significant and he will be dearly missed both professionally and personally.

REFERENCES

[1] Yin, J.P., 2000, "An Analysis for the Design of Self-Deployable Tensegrity and Reinforced Tensegrity Prisms with Elastic Ties," Report for the Center for Intelligent Machines and Robotics, University of Florida, Gainesville, FL.

[2] Tobie, R.S., 1976, "A Report on an Inquiry into The Existence, Formation and Representation of Tensile Structures," Master of Industrial Design thesis, Pratt Institute, New York.

[3] Kenner, H., 1976, *Geodesic Math and How to Use It*, University of California Press, Berkeley and Los Angeles, CA.

[4] Stern, I.P., 1999, "Development of Design Equations for Self-Deployable N-Strut Tensegrity Systems," Master of Science thesis, University of Florida, Gainesville, FL.

[5] Knight, B.F., 2000, "Deployable Antenna Kinematics using Tensegrity Structure Design," Ph.D. thesis, University of Florida, Gainesville, FL.

[6] Doughty, S., 1998, *Mechanics of Machines*, John Wiley & Sons, New York.

[7] Crane, C. and Duffy, J., 1998, *Kinematic Analysis of Robot Manipulators*, Cambridge University Press, New York.

[8] Kunt, K.H., 1978, *Kinematic Geometry of Mechanisms*, Oxford, New York.

[9] Lee, J., Duffy, J., and Hunt, K., 1998, "A Practical Quality Index Based on the Octahedral Manipulator," The International Journal of Robotics Research, **17**, pp. 1081-1090.

[10] Hines, R., 1995, "Catastrophe Analysis of Compliant Mechanisms," Master of Science thesis, University of Florida, Gainesville, FL.

[11] Arnold's, V.I., 1992 *Catastrophe Theory*, Springer-Verlag, Berlin.