Available online at www.sciencedirect.com
ScienceDirect
Mechanism and Machine Theory 43 (2008) 271–284
Mechanism and Machine Theory
www.elsevier.com/locate/mechmt

Stiffness mapping of compliant parallel mechanisms in a serial arrangement

Hyun K. Jung [a], Carl D. Crane III [a,*], Rodney G. Roberts [b]

[a] *Department of Mechanical and Aerospace Engineering, University of Florida, Gainesville, Florida 32611-6300, United States*
[b] *Florida State University, FAMU-FSU College of Engineering, Department of Electrical and Computer Engineering, United States*

Received 6 February 2006; accepted 9 April 2007
Available online 21 June 2007

Abstract

This paper presents a stiffness mapping of a mechanism having two compliant parallel mechanisms in a serial arrangement. A derivative of the wrench connecting two moving bodies is derived and applied to obtain the stiffness matrix of the mechanism. It is shown that the resultant compliance of two compliant parallel mechanisms that are serially arranged is not the summation of the compliances of the constituent mechanisms unless the external wrench applied to the mechanism is zero. A numerical example is presented to support the result.

Keywords: Compliance; Stiffness; Parallel mechanisms

1. Introduction

Compliant mechanisms can be considered as spatial springs having multiple degrees of freedom rather than one as line springs do. For many robot applications which involve contact between the robot and its environment, compliant mechanisms can be successfully implemented to compensate for the positional errors of the robot system or to provide force control measures [1–3].

The concepts of *twist* and *wrench* from screw theory, which was introduced by Ball [4] are employed throughout this paper to describe a small (or instantaneous) displacement of a rigid body and a force/torque applied to a body [5]. A small twist applied to a compliant mechanism generates a small change of the wrench which the compliant mechanism exerts on the environment and the stiffness matrix of the mechanism describes this relation.

Parallel mechanisms contain positive features compared to serial mechanisms such as high stiffness, compactness, and small positional errors at the expense of small work space and complexity of analysis. Thanks to its favorable properties, parallel mechanism-based compliant mechanisms have been studied by many

* Corresponding author. Tel.: +1 352 392 9461; fax: +1 352 392 1071.
E-mail addresses: hyunkwon.jung@gmail.com (H.K. Jung), ccrane@ufl.edu (C.D. Crane III), rroberts@eng.fsu.edu (R.G. Roberts).

0094-114X/$ - see front matter
doi:10.1016/j.mechmachtheory.2007.04.006

researchers: Dimentberg [6] studied properties of an elastically suspended body using screw theory. Griffis [3] obtained a global stiffness mapping model for the compliant mechanisms. Huang and Schimmels [7], Ciblak and Lipkin [8], and Roberts [9] studied synthesis of stiffness matrix. Simaan and Shoham [10] presented a line-based analytical formulation for the stiffness matrix of parallel robots.

2. Problem statement

Fig. 1 depicts the compliant mechanism whose stiffness matrix will be obtained in this paper. Body A is connected to ground by six compliant couplings and body B is connected to body A in the same way. Each compliant coupling has a spherical joint at each end and a prismatic joint with a spring in the middle. It is assumed that an external wrench $\underline{\mathbf{w}}_{\mathrm{ext}}$ is applied to body B and that both body B and body A are in static equilibrium. The poses of body A and body B and the spring constants and free lengths of all compliant couplings are known.

The stiffness matrix $[K]$ which maps a small twist of the moving body B in terms of the ground, ${}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}}$ (written in axis coordinates), into the corresponding wrench variation, $\delta\underline{\mathbf{w}}_{\mathrm{ext}}$ (written in ray coordinates), is desired to be derived and this relationship can be written as

$$\delta\underline{\mathbf{w}}_{\mathrm{ext}} = [K]^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}}. \tag{1}$$

The stiffness matrix will be derived by taking a derivative of the static equilibrium equations of body A and body B as

$$\underline{\mathbf{w}}_{\mathrm{ext}} = \sum_{i=1}^{6}\underline{\mathbf{w}}_i = \sum_{i=7}^{12}\underline{\mathbf{w}}_i, \tag{2}$$

$$\delta\underline{\mathbf{w}}_{\mathrm{ext}} = \sum_{i=1}^{6}\delta\underline{\mathbf{w}}_i = \sum_{i=7}^{12}\delta\underline{\mathbf{w}}_i, \tag{3}$$

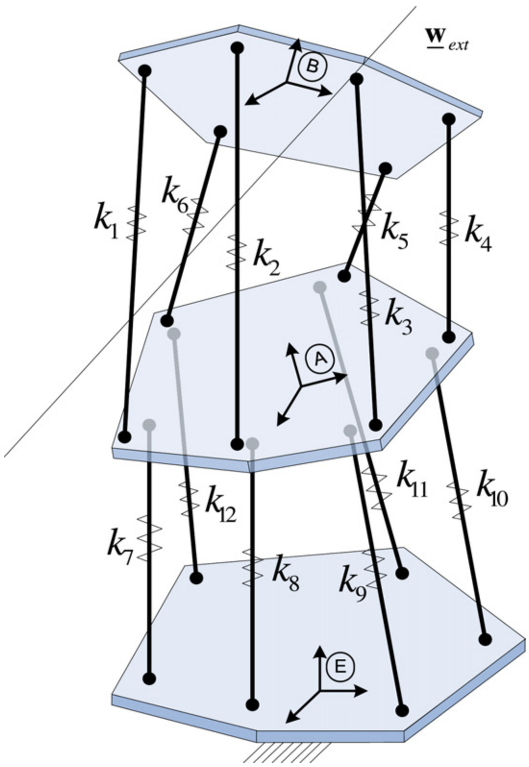

Fig. 1. Mechanism having two compliant parallel mechanisms in series.

where $\underline{\mathbf{w}}_i$ are the wrenches from the compliant couplings. A derivative of the spring wrench connecting body A and ground, which corresponds to $\delta\underline{\mathbf{w}}_i$ for $i = 7 \ldots 12$, is derived using a polar coordinate system in Section 3. In Section 4, a derivative of spring wrench connecting two moving bodies, which corresponds to $\delta\underline{\mathbf{w}}_i$ for $i = 1 \ldots 6$, is derived. The stiffness matrix of the compliant mechanism is then presented in Section 5. A numerical example and conclusions follow in Sections 6 and 7, respectively.

3. A derivative of spatial spring wrench joining a moving body and ground

Fig. 2 depicts a rigid body and a compliant coupling connecting the body and the ground. Body A can translate and rotate in a spatial space. The wrench which the spring exerts on body A can be written as

$$\underline{\mathbf{w}} = k(l - l_0)\underline{\mathbf{\$}}, \tag{4}$$

where k, l, and l_0 are respectively the spring constant, current spring length, and spring free length of the compliant coupling. Further, $\underline{\mathbf{\$}}$ represents the unitized Plücker coordinates of the line along the compliant coupling which may be written by

$$\underline{\mathbf{\$}} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{E}}_{P0} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times \underline{\mathbf{S}} \end{bmatrix}, \tag{5}$$

where $\underline{\mathbf{S}}$ is the unit vector along the compliant coupling and ${}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{E}}_{P0}$ and ${}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1}$ are the position of the pivot point P0 in the ground body and that of P1 in body A, respectively, measured with respect to a reference coordinate system attached to ground.

A polar coordinates system can be used to express the unit vector $\underline{\mathbf{S}}$ (see Fig. 3) as

$$\underline{\mathbf{S}} = \begin{bmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{bmatrix}. \tag{6}$$

It is obvious from Eqs. (5) and (6) that $\underline{\mathbf{\$}}$ is a function of α and β since ${}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{E}}_{P0}$ is fixed on ground. Hence a derivative of the spring wrench can be written as

$$\delta\underline{\mathbf{w}} = k\delta l\underline{\mathbf{\$}} + k(l - l_0)\delta\underline{\mathbf{\$}} = k\delta l\underline{\mathbf{\$}} + k\left(1 - \frac{l_0}{l}\right)\left(\frac{\partial\underline{\mathbf{\$}}}{\partial\alpha} l\,\delta\alpha + \frac{\partial\underline{\mathbf{\$}}}{\partial\beta} l\,\delta\beta\right), \tag{7}$$

where

$$\frac{\partial\underline{\mathbf{\$}}}{\partial\alpha} = \begin{bmatrix} \frac{\partial\underline{\mathbf{S}}}{\partial\alpha} \\ {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{E}}_{P0} \times \frac{\partial\underline{\mathbf{S}}}{\partial\alpha} \end{bmatrix}, \tag{8}$$

$$\frac{\partial\underline{\mathbf{\$}}}{\partial\beta} = \begin{bmatrix} \frac{\partial\underline{\mathbf{S}}}{\partial\beta} \\ {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{E}}_{P0} \times \frac{\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}. \tag{9}$$

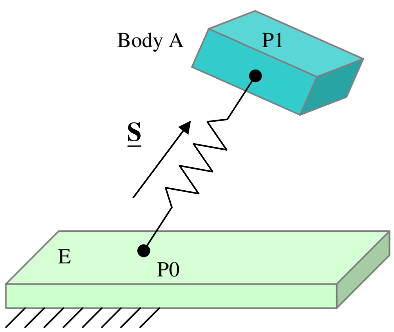

Fig. 2. Spatial compliant coupling joining body A and the ground.

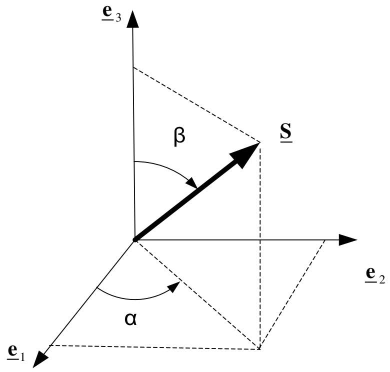

Fig. 3. Unit vector expressed in a polar coordinates system.

By taking a derivative of Eq. (6), $\frac{\partial \underline{\mathbf{S}}}{\partial \alpha}$ and $\frac{\partial \underline{\mathbf{S}}}{\partial \beta}$ can be explicitly written by

$$\frac{\partial \underline{\mathbf{S}}}{\partial \alpha} = \begin{bmatrix} -\sin\beta\sin\alpha \\ \sin\beta\cos\alpha \\ 0 \end{bmatrix}, \tag{10}$$

$$\frac{\partial \underline{\mathbf{S}}}{\partial \beta} = \begin{bmatrix} \cos\beta\cos\alpha \\ \cos\beta\sin\alpha \\ -\sin\beta \end{bmatrix}. \tag{11}$$

Since $\frac{\partial \underline{\mathbf{S}}}{\partial \alpha}$ is not a unit vector, a unit vector $\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha}$ is introduced as

$$\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha} = \begin{bmatrix} -\sin\alpha \\ \cos\alpha \\ 0 \end{bmatrix}, \tag{12}$$

$$\frac{\partial \underline{\mathbf{S}}}{\partial \alpha} = \sin\beta \frac{\partial \underline{\mathbf{S}}'}{\partial \alpha}. \tag{13}$$

Hence Eq. (7) can be rewritten as

$$\delta \underline{\mathbf{w}} = k\,\delta l\,\underline{\$} + k\left(1 - \frac{l_0}{l}\right)\left(\frac{\partial \underline{\$}'}{\partial \alpha} l \sin\beta\,\delta\alpha + \frac{\partial \underline{\$}}{\partial \beta} l\,\delta\beta\right), \tag{14}$$

where

$$\frac{\partial \underline{\$}'}{\partial \alpha} = \begin{bmatrix} \frac{\partial \underline{\mathbf{S}}'}{\partial \alpha} \\ {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{E}}_{P0} \times \frac{\partial \underline{\mathbf{S}}'}{\partial \alpha} \end{bmatrix}. \tag{15}$$

It is important to note that $\frac{\partial \underline{\$}'}{\partial \alpha}$ and $\frac{\partial \underline{\$}}{\partial \beta}$ are the unitized Plücker coordinates of the lines perpendicular to $\underline{\mathbf{S}}$ and go through the pivot point P0.

In Eq. (14) δl, $l\sin\beta\,\delta\alpha$, and $l\delta\beta$ can be considered as the change of the spring length and the changes of the direction of the spring (see Fig. 4). These values correspond to the projections of the variation of position P1, ${}^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1}$, into $\underline{\mathbf{S}}$, $\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha}$, and $\frac{\partial \underline{\mathbf{S}}}{\partial \beta}$, respectively. Thus

$${}^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1} = ({}^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1}\cdot\underline{\mathbf{S}})\underline{\mathbf{S}} + \left({}^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1}\cdot\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha}\right)\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha} + \left({}^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1}\cdot\frac{\partial \underline{\mathbf{S}}}{\partial \beta}\right)\frac{\partial \underline{\mathbf{S}}}{\partial \beta} = \delta l\,\underline{\mathbf{S}} + l\sin\beta\,\delta\alpha\frac{\partial \underline{\mathbf{S}}'}{\partial \alpha} + l\,\delta\beta\frac{\partial \underline{\mathbf{S}}}{\partial \beta}. \tag{16}$$

From the twist equation, the variation of position P1 can be written as

$${}^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1} = {}^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{0} + {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1}, \tag{17}$$

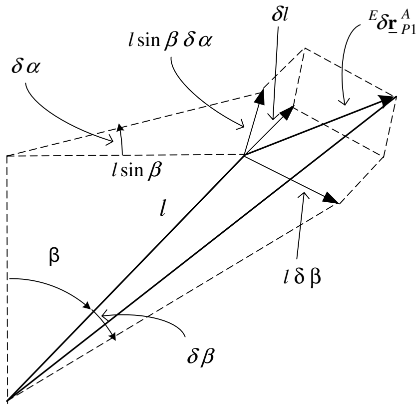

Fig. 4. Small change of position of P1 due to a small twist of body A.

where $^{E}\delta\underline{\mathbf{r}}_0^{A}$ is the differential of the position of point O in body A which is coincident with the origin of the inertial frame E measured with respect to the inertial frame. $^{E}\delta\underline{\boldsymbol{\varphi}}^{A}$ is the differential of the angle of body A with respect to the inertial frame.

From Eqs. (16) and (17), δl, $l\sin\beta\,\delta\alpha$, and $l\delta\beta$ can be expressed as

$$\delta l = {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}} = {}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}} = {}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}} = \underline{\mathbf{\$}}^{T\,E}\delta\underline{\mathbf{D}}^{A}, \tag{18}$$

$$l\sin\beta\,\delta\alpha = {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} = {}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} = {}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} = \frac{\partial\underline{\mathbf{\$}}''^{T}}{\partial\alpha}\,{}^{E}\delta\underline{\mathbf{D}}^{A}, \tag{19}$$

$$l\,\delta\beta = {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} = {}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} = {}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{\partial\underline{\mathbf{S}}}{\partial\beta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot{}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{\partial\underline{\mathbf{S}}}{\partial\beta} = \frac{\partial\underline{\mathbf{\$}}''^{T}}{\partial\beta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}, \tag{20}$$

where

$${}^{E}\delta\underline{\mathbf{D}}^{A} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{r}}_{0}^{A} \\ {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \end{bmatrix}, \tag{21}$$

$$\frac{\partial\underline{\mathbf{\$}}''}{\partial\alpha} = \begin{bmatrix} \frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix}, \tag{22}$$

$$\frac{\partial\underline{\mathbf{\$}}''}{\partial\beta} = \begin{bmatrix} \frac{\partial\underline{\mathbf{S}}}{\partial\beta} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}. \tag{23}$$

It is important to note that $\frac{\partial\underline{\mathbf{\$}}''}{\partial\alpha}$ and $\frac{\partial\underline{\mathbf{\$}}''}{\partial\beta}$ are the unitized Plücker coordinates of lines perpendicular to $\underline{\mathbf{S}}$ which pass through the pivot point P1 in body A and $^{E}\delta\underline{\mathbf{D}}^{A}$ is a small twist of body A with respect to ground. Substituting Eqs. (18)–(20) for δl, $l\sin\beta\,\delta\alpha$, and $l\delta\beta$ in Eq. (14) yields

$$\begin{aligned}\delta\underline{\mathbf{w}} &= k\,\delta l\,\underline{\mathbf{\$}} + k\left(1-\frac{l_0}{l}\right)\left(\frac{\partial\underline{\mathbf{\$}}'}{\partial\alpha}\,l\,\sin\beta\,\delta\alpha + \frac{\partial\underline{\mathbf{\$}}}{\partial\beta}\,l\,\delta\beta\right)\\ &= k\underline{\mathbf{\$}}\underline{\mathbf{\$}}^{T\,E}\delta\underline{\mathbf{D}}^{A} + k\left(1-\frac{l_0}{l}\right)\left(\frac{\partial\underline{\mathbf{\$}}'}{\partial\alpha}\frac{\partial\underline{\mathbf{\$}}''^{T}}{\partial\alpha} + \frac{\partial\underline{\mathbf{\$}}}{\partial\beta}\frac{\partial\underline{\mathbf{\$}}''^{T}}{\partial\beta}\right){}^{E}\delta\underline{\mathbf{D}}^{A} = [K_F]\,{}^{E}\delta\underline{\mathbf{D}}^{A},\end{aligned} \tag{24}$$

where

$$[K_F] = k\underline{\$}\underline{\$}^T + k\left(1 - \frac{l_0}{l}\right)\left(\frac{\partial\underline{\$}'}{\partial\alpha}\frac{\partial\underline{\$}''^T}{\partial\alpha} + \frac{\partial\underline{\$}}{\partial\beta}\frac{\partial\underline{\$}''^T}{\partial\beta}\right). \tag{25}$$

$[K_F]$ is the stiffness matrix of a spatial compliant coupling and maps a small twist of body A into the corresponding variation of the wrench. The first term of Eq. (25) is always symmetric and the second term is not. When the spring deviates from its equilibrium position due to an external wrench, the second term of Eq. (25) does not vanish and it makes the stiffness matrix asymmetric. This result agrees with the works of Griffis [3].

4. A derivative of spring wrench joining two moving bodies

Fig. 5 depicts two rigid bodies connected to each other by a compliant coupling with a spring constant k, a free length l_0, and a current length l. Body A can move in a spatial space and the compliant coupling exerts a wrench $\underline{\mathbf{w}}$ to body B which is in equilibrium. The spring wrench may be written by

$$\underline{\mathbf{w}} = k(l - l_0)\underline{\$}, \tag{26}$$

where

$$\underline{\$} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{\mathrm{E}}\underline{\mathbf{r}}_{P1}^{\mathrm{A}} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{\mathrm{E}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \times \underline{\mathbf{S}} \end{bmatrix} \tag{27}$$

and where $\underline{\mathbf{S}}$ is a unit vector along the compliant coupling and ${}^{\mathrm{E}}\underline{\mathbf{r}}_{P1}^{\mathrm{A}}$ and ${}^{\mathrm{E}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}}$ are the position vectors of the point $P1$ in body A and that of point $P2$ in body B, respectively, measured with respect to the reference system embedded in ground (body E). It is desired to express a derivative of the spring wrench in terms of the twist of body B ${}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}}$ and that of body A ${}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}$. The twist ${}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}}$ may be expressed as

$${}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}} = {}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}} + {}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}}, \tag{28}$$

where

$${}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}} = \begin{bmatrix} {}^{\mathrm{E}}\delta\underline{\mathbf{r}}_0{}^{\mathrm{B}} \\ {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{B}} \end{bmatrix}, \tag{29}$$

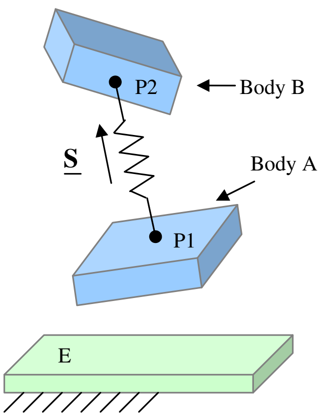

Fig. 5. Spatial compliant coupling joining two moving bodies.

$$ {}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}} = \begin{bmatrix} {}^{\mathrm{E}}\delta\underline{\mathbf{r}}_0{}^{\mathrm{A}} \\ {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \end{bmatrix}, \tag{30} $$

$$ {}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}} = \begin{bmatrix} {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_0{}^{\mathrm{B}} \\ {}^{\mathrm{A}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{B}} \end{bmatrix} \tag{31} $$

and where ${}^{\mathrm{E}}\delta\underline{\mathbf{r}}_0{}^{\mathrm{B}}$ is the differential of point O, which is in body B and coincident with the origin of the inertial frame, measured with respect to the inertial frame and ${}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{B}}$ is the differential of angle of body B with respect to the inertial frame. ${}^{\mathrm{E}}\delta\underline{\mathbf{r}}_0{}^{\mathrm{A}}$, ${}^{\mathrm{A}}\delta\underline{\mathbf{r}}_0{}^{\mathrm{B}}$, ${}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}}$, and ${}^{\mathrm{A}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{B}}$ are defined in the same way.

The derivative of the spring wrench in Eq. (26) can be written by

$$ {}^{\mathrm{E}}\delta\underline{\mathbf{w}} = k\,\delta l\,\underline{\$} + k(l - l_0)\,{}^{\mathrm{E}}\delta\underline{\$} \tag{32} $$

and it is required to express δl and ${}^{\mathrm{E}}\delta\underline{\$}$ in Eq. (32) in terms of the twists of the bodies.

From the twist equation, the variation of position of point P2 in body B with respect to body A can be expressed as

$$ {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}{}^{\mathrm{B}} = {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_0^{\mathrm{B}} + {}^{\mathrm{A}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{B}} \times {}^{\mathrm{A}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}}, \tag{33} $$

where ${}^{\mathrm{A}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}}$ is the position of $P2$, which is embedded in body B, measured with respect to a coordinate system embedded in body A which at this instant is coincident and aligned with the reference system attached to ground. It can also be decomposed into three perpendicular vectors along $\underline{\mathbf{S}}$, $\frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}$ which are defined in a similar way as Eqs. (6), (12), and (11). These three vectors correspond to the change of the spring length δl and the directional changes of the spring such as $l\sin\beta\,\delta\alpha$ and $l\,\delta\beta$ in terms of body A in a way that is analogous to that shown in Fig. 4. Thus the variation of position of point P2 in body B in terms of body A can be written as

$$ \begin{aligned} {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}^{\mathrm{B}} &= ({}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \cdot \underline{\mathbf{S}})\underline{\mathbf{S}} + \left({}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \cdot \frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha}\right)\frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} + \left({}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \cdot \frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\right)\frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} \\ &= \delta l\,\underline{\mathbf{S}} + l\,\sin\beta\,\delta\alpha\frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} + l\,\delta\beta\frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}. \end{aligned} \tag{34} $$

From Eqs. (33) and (34), δl in Eq. (32) can be obtained as

$$ \delta l = {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \cdot \underline{\mathbf{S}} = {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_0^{\mathrm{B}} \cdot \underline{\mathbf{S}} + {}^{\mathrm{A}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{B}} \times {}^{\mathrm{A}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \cdot \underline{\mathbf{S}} = {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_0^{\mathrm{B}} \cdot \underline{\mathbf{S}} + {}^{\mathrm{A}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{B}} \cdot {}^{\mathrm{A}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \times \underline{\mathbf{S}} = \underline{\$}^{T\,\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}}. \tag{35} $$

In the same way, $l\sin\beta\,\delta\alpha$ and $l\,\delta\beta$ can be expressed as

$$ l\,\sin\beta\,\delta\alpha = {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \cdot \frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} = \frac{{}^{\mathrm{A}}\partial\underline{\$}''^{\mathrm{T}}}{\partial\alpha}\,{}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}}, \tag{36} $$

$$ l\,\delta\beta = {}^{\mathrm{A}}\delta\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \cdot \frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} = \frac{{}^{\mathrm{A}}\partial\underline{\$}''^{\mathrm{T}}}{\partial\beta}\,{}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}}, \tag{37} $$

where

$$ \frac{{}^{\mathrm{A}}\partial\underline{\$}''}{\partial\alpha} = \begin{bmatrix} \frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ {}^{\mathrm{A}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \times {}^{\mathrm{A}}\frac{\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix}, \tag{38} $$

$$ \frac{{}^{\mathrm{A}}\partial\underline{\$}''}{\partial\beta} = \begin{bmatrix} \frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} \\ {}^{\mathrm{A}}\underline{\mathbf{r}}_{P2}^{\mathrm{B}} \times \frac{{}^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}. \tag{39} $$

Now in Eq. (32), only ${}^{\mathrm{E}}\delta\underline{\$}$ is yet to be obtained. It is a derivative of the unit screw along the spring in terms of the inertial frame and may be written as

$$ {}^{\mathrm{E}}\delta\underline{\$} = \begin{bmatrix} {}^{\mathrm{E}}\delta\underline{\mathbf{S}} \\ {}^{\mathrm{E}}\delta\underline{\mathbf{r}}_{P1}^{\mathrm{A}} \times \underline{\mathbf{S}} + {}^{\mathrm{E}}\underline{\mathbf{r}}_{P1}^{\mathrm{A}} \times {}^{\mathrm{E}}\delta\underline{\mathbf{S}} \end{bmatrix}. \tag{40} $$

Using an intermediate frame attached to body A, a derivative of the unit vector $\underline{\mathbf{S}}$ can be written by

$$^{\mathrm{E}}\delta\underline{\mathbf{S}} = {}^{\mathrm{A}}\delta\underline{\mathbf{S}} + {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}. \tag{41}$$

Thus $^{\mathrm{E}}\delta\underline{\$}$ may be decomposed into three screws as

$$\begin{aligned}
^{\mathrm{E}}\delta\underline{\$} &= \begin{bmatrix} ^{\mathrm{E}}\delta\underline{\mathbf{S}} \\ ^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times \underline{\mathbf{S}} + {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times {}^{\mathrm{E}}\delta\underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} ^{\mathrm{A}}\delta\underline{\mathbf{S}} + {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}} \\ ^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times \underline{\mathbf{S}} + {}^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times ({}^{\mathrm{A}}\delta\underline{\mathbf{S}} + {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \end{bmatrix} \\
&= \begin{bmatrix} ^{\mathrm{A}}\delta\underline{\mathbf{S}} \\ ^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times {}^{\mathrm{A}}\delta\underline{\mathbf{S}} \end{bmatrix} + \begin{bmatrix} ^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}} \\ ^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times ({}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} \\ ^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times \underline{\mathbf{S}} \end{bmatrix}.
\end{aligned} \tag{42}$$

Since $\underline{\mathbf{S}}$ is a function of α and β from the vantage of body A and $l\sin\beta\,\delta\alpha$ and $l\delta\beta$ were already described in Eqs. (36) and (37), the first screw in Eq. (42) can be written as

$$\begin{aligned}
\begin{bmatrix} ^{\mathrm{A}}\delta\underline{\mathbf{S}} \\ ^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times {}^{\mathrm{A}}\delta\underline{\mathbf{S}} \end{bmatrix} &= \begin{bmatrix} \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha + \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta \\ ^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times \left(\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha + \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta\right) \end{bmatrix} = \begin{bmatrix} \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha \\ ^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\alpha}\delta\alpha \end{bmatrix} + \begin{bmatrix} \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta \\ ^{\mathrm{E}}\underline{\mathbf{r}}^{\mathrm{A}}_{P1} \times \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\delta\beta \end{bmatrix} \\
&= \frac{^{\mathrm{A}}\partial\underline{\$}'}{\partial\alpha}\frac{1}{l}l\sin\beta\,\delta\alpha + \frac{^{\mathrm{A}}\partial\underline{\$}}{\partial\beta}\frac{1}{l}l\,\delta\beta = \frac{1}{l}\left(\frac{^{\mathrm{A}}\partial\underline{\$}'}{\partial\alpha}\frac{^{\mathrm{A}}\partial\underline{\$}''^{T}}{\partial\alpha} + \frac{^{\mathrm{A}}\partial\underline{\$}}{\partial\beta}\frac{^{\mathrm{A}}\partial\underline{\$}''^{T}}{\partial\beta}\right){}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}}.
\end{aligned} \tag{43}$$

As to the second screw in Eq. (42), $^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}$ can be decomposed into three perpendicular vectors along $\underline{\mathbf{S}}$, $\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}$, respectively, as

$$^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}} = \left\{({}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \cdot \underline{\mathbf{S}}\right\}\underline{\mathbf{S}} + \left\{({}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \cdot \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha}\right\}\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} + \left\{({}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \cdot \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\right\}\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}. \tag{44}$$

From the fact that $\underline{\mathbf{S}}$, $\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}$ are unit vectors and perpendicular to each other (see Fig. 4), each dot product of Eq. (44) can be expressed as

$$({}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \cdot \underline{\mathbf{S}} = 0, \tag{45}$$

$$\begin{aligned}
({}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \cdot \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} &= {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \cdot \left(\underline{\mathbf{S}} \times \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha}\right) = -{}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \cdot \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} \\
&= -\begin{bmatrix} \underline{\mathbf{0}} \\ \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}^{T} \begin{bmatrix} ^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{0} \\ ^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \end{bmatrix} = -\begin{bmatrix} \underline{\mathbf{0}} \\ \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}^{T} {}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}},
\end{aligned} \tag{46}$$

$$\begin{aligned}
({}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}) \cdot \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} &= {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \cdot \left(\underline{\mathbf{S}} \times \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\right) = {}^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \cdot \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\
&= \begin{bmatrix} \underline{\mathbf{0}} \\ \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix}^{T} \begin{bmatrix} ^{\mathrm{E}}\delta\underline{\mathbf{r}}^{\mathrm{A}}_{0} \\ ^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{0}} \\ \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix}^{T} {}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}},
\end{aligned} \tag{47}$$

where $\underline{\mathbf{0}} = [0 \quad 0 \quad 0]^{T}$.

Hence, $^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}}$ can be rewritten as

$$^{\mathrm{E}}\delta\underline{\boldsymbol{\varphi}}^{\mathrm{A}} \times \underline{\mathbf{S}} = -\frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha}\begin{bmatrix} \underline{\mathbf{0}} \\ \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta} \end{bmatrix}^{T} {}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}} + \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}}{\partial\beta}\begin{bmatrix} \underline{\mathbf{0}} \\ \frac{^{\mathrm{A}}\partial\underline{\mathbf{S}}'}{\partial\alpha} \end{bmatrix}^{T} {}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}} \tag{48}$$

and the second screw in Eq. (42) can be expressed as

$$\begin{bmatrix} {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{S}}) \end{bmatrix} = \begin{bmatrix} -\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}+\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\left\{-\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}+\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}\right\} \end{bmatrix}$$

$$= -\begin{bmatrix} \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}+\begin{bmatrix} \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A}\times\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}{}^{E}\delta\underline{\mathbf{D}}^{A}$$

$$= \left(-\frac{{}^{A}\partial\underline{\underline{\$}}''}{\partial\alpha}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\end{bmatrix}^{T}+\frac{{}^{A}\partial\underline{\underline{\$}}''}{\partial\beta}\begin{bmatrix}\mathbf{0}\\ \frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\end{bmatrix}^{T}\right){}^{E}\delta\underline{\mathbf{D}}^{A}. \tag{49}$$

As to the third screw in Eq. (42), ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$ can be decomposed into three perpendicular vectors along $\underline{\mathbf{S}}$, $\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}$, respectively, as

$${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}={}^{E}\delta\underline{\mathbf{r}}_{0}^{A}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times{}^{E}\underline{\mathbf{r}}_{P1}^{A}=({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}})\underline{\mathbf{S}}+\left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}+\left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}. \tag{50}$$

The first dot product in Eq. (50) can be expressed as

$${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}}={}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\underline{\mathbf{S}}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{r}}_{P1}^{A}\cdot\underline{\mathbf{S}}={}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\underline{\mathbf{S}}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}}=\underline{\underline{\$}}^{T\,E}\delta\underline{\mathbf{D}}^{A}. \tag{51}$$

In the same way, the second and third dot products in Eq. (50) can be written as

$$\begin{aligned}{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}&={}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\\&={}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\underline{\mathbf{r}}_{P1}^{A}\times\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\\&=\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha}{}^{E}\delta\underline{\mathbf{D}}^{A},\end{aligned} \tag{52}$$

$$\begin{aligned}{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}&={}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\times\underline{\mathbf{r}}_{P1}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\\&={}^{E}\delta\underline{\mathbf{r}}_{0}^{A}\cdot\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}+{}^{E}\delta\underline{\boldsymbol{\varphi}}^{A}\cdot\underline{\mathbf{r}}_{P1}^{A}\times\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\\&=\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}{}^{E}\delta\underline{\mathbf{D}}^{A}.\end{aligned} \tag{53}$$

Finally, ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}}$ of the third screw in Eq. (42) can be expressed as

$$\begin{aligned}{}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}}&=\left\{(\underline{\underline{\$}}^{T\,E}\delta\underline{\mathbf{D}}^{A})\underline{\mathbf{S}}+\left(\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha}{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}+\left(\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\right\}\times\underline{\mathbf{S}}\\&=\left(\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha}{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}\times\underline{\mathbf{S}}+\left(\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}\times\underline{\mathbf{S}}\\&=\left(\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha}{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}-\left(\frac{{}^{A}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha},\end{aligned} \tag{54}$$

since $\underline{\mathbf{S}}$, $\frac{{}^{A}\partial\underline{\mathbf{S}}'}{\partial\alpha}$, and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\beta}$ are unit vectors and perpendicular to each other (see Fig. 4).

Substituting Eq. (54) for ${}^{\mathrm{E}}\delta\underline{\mathbf{r}}_{P1}^{\mathrm{A}} \times \underline{\mathbf{S}}$ of the third screw in Eq. (42) yields

$$\begin{aligned}\begin{bmatrix}\underline{\mathbf{0}}\\ {}^{\mathrm{E}}\delta\underline{\mathbf{r}}_{P1}^{\mathrm{A}}\times\underline{\mathbf{S}}\end{bmatrix} &= \begin{bmatrix}\underline{\mathbf{0}}\\ \left(\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha}{}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}\right)\frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}}{\partial\beta} - \left(\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}{}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}\right)\frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}'}{\partial\alpha}\end{bmatrix}\\ &= \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}}{\partial\beta}\end{bmatrix}\left(\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha}{}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}\right) - \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}'}{\partial\alpha}\end{bmatrix}\left(\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}{}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}\right)\\ &= \left(\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}}{\partial\beta}\end{bmatrix}\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha} - \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}'}{\partial\alpha}\end{bmatrix}\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}\right){}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}.\end{aligned} \tag{55}$$

By replacing δl and ${}^{\mathrm{E}}\delta\underline{\$}$ in Eq. (32) with Eqs. (35), (43), (49), and (55) and sorting it into the twists, the derivative of the spring wrench can be rewritten as

$${}^{\mathrm{E}}\delta\underline{\mathbf{w}} = k\,\delta l\,\underline{\$} + k(l-l_0){}^{\mathrm{E}}\delta\underline{\$} = [K_F]^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}} + [K_M]^{\mathrm{E}}\,\delta\underline{\mathbf{D}}^{\mathrm{A}}, \tag{56}$$

where

$$[K_F] = k\underline{\$}\underline{\$}^{T} + k\left(1-\frac{l_0}{l}\right)\left(\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}'}{\partial\alpha}\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha} + \frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}}{\partial\beta}\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}\right), \tag{57}$$

$$[K_M] = k(l-l_0)\left(\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}}{\partial\beta}\end{bmatrix}\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\alpha} - \frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''}{\partial\alpha}\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}}{\partial\beta}\end{bmatrix}^{T} + \frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''}{\partial\beta}\begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}'}{\partial\alpha}\end{bmatrix}^{T} - \begin{bmatrix}\underline{\mathbf{0}}\\ \frac{{}^{\mathrm{A}}\partial\underline{\underline{\mathbf{S}}}'}{\partial\alpha}\end{bmatrix}\frac{{}^{\mathrm{A}}\partial\underline{\underline{\$}}''^{T}}{\partial\beta}\right). \tag{58}$$

It is important to note that $[K_M]$ is identical to the negative of the spring wrench expressed as a spatial cross-product operator (see [11,12]). To prove it, all terms in Eq. (58) are explicitly expressed in a polar coordinate system and ${}^{\mathrm{E}}\underline{\mathbf{r}}_{P1}^{\mathrm{A}} = \begin{bmatrix}p_x & p_y & p_z\end{bmatrix}^{T}$ to yield

$$[K_M] = \begin{bmatrix}[\mathbf{0}] & [\mathbf{K12}]\\ [\mathbf{K12}] & [\mathbf{K22}]\end{bmatrix}, \tag{59}$$

where

$$[\mathbf{K12}] = k(l-l_0)\begin{bmatrix}0 & c_\beta & -s_\beta s_\alpha\\ -c_\beta & 0 & s_\beta c_\alpha\\ s_\beta s_\alpha & -s_\beta c_\alpha & 0\end{bmatrix}, \tag{60}$$

$$[\mathbf{K22}] = k(l-l_0)\begin{bmatrix}0 & p_x s_\beta s_\alpha - p_y s_\beta c_\alpha & -p_z s_\beta c_\alpha + p_x c_\beta\\ -p_x s_\beta s_\alpha + p_y s_\beta c_\alpha & 0 & p_y c_\beta - p_z s_\beta s_\alpha\\ p_z s_\beta c_\alpha - p_x c_\beta & -p_y c_\beta + p_z s_\beta s_\alpha & 0\end{bmatrix} \tag{61}$$

and where $[\underline{\mathbf{0}}]$ is 3×3 zero matrix, $c_\alpha = \cos(\alpha)$, and $s_\alpha = \sin(\alpha)$, etc.

In the same way the spring wrench can be explicitly written as

$$\underline{\mathbf{w}} = k(l-l_0)\underline{\$} = k(l-l_0)\begin{bmatrix}\underline{\mathbf{S}}\\ {}^{\mathrm{E}}\underline{\mathbf{r}}_{P1}^{\mathrm{A}}\times\underline{\mathbf{S}}\end{bmatrix} = k(l-l_0)\begin{bmatrix}s_\beta c_\alpha\\ s_\beta s_\alpha\\ c_\beta\\ p_y c_\beta - p_z s_\beta s_\alpha\\ p_z s_\beta c_\alpha - p_x c_\beta\\ p_x s_\beta s_\alpha - p_y s_\beta c_\alpha\end{bmatrix} = \begin{bmatrix}f_x\\ f_y\\ f_z\\ m_x\\ m_y\\ m_z\end{bmatrix} = \begin{bmatrix}\underline{\mathbf{f}}\\ \underline{\mathbf{m}}\end{bmatrix}. \tag{62}$$

By comparing Eqs. (60) and (61) with Eq. (62) it is obvious that

$$[\mathbf{K12}] = k(l - l_0)\begin{bmatrix} 0 & f_z & -f_y \\ -f_z & 0 & f_x \\ f_y & -f_x & 0 \end{bmatrix} = -\underline{\mathbf{f}}\times, \tag{63}$$

$$[\mathbf{K22}] = \begin{bmatrix} 0 & m_z & -m_y \\ -m_z & 0 & m_x \\ m_y & -m_x & 0 \end{bmatrix} = -\underline{\mathbf{m}}\times, \tag{64}$$

where $\underline{\mathbf{f}}\times$ and $\underline{\mathbf{m}}\times$ are skew-symmetric matrices representing vector multiplication.

Then $[K_M]$ can be expressed as

$$[K_M] = \begin{bmatrix} [\underline{\mathbf{0}}] & -\underline{\mathbf{f}}\times \\ -\underline{\mathbf{f}}\times & -\underline{\mathbf{m}}\times \end{bmatrix} = -\underline{\mathbf{w}}\times, \tag{65}$$

where $\underline{\mathbf{w}}\times$ is the spring wrench expressed as a spatial cross-product operator (see [11]).

Finally the derivative of the spring wrench can be written as

$$^{\mathrm{E}}\delta\underline{\mathbf{w}} = [K_F]^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}} - (\underline{\mathbf{w}}\times)^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}. \tag{66}$$

As shown in Eq. (66), the derivative of the spring force joining two rigid bodies depends not only on a relative twist between two bodies but also on the twist of the intermediate body, in this case body A, in terms of the inertial frame. $[K_F]$ which maps a small twist of body B in terms of body A into the corresponding change of wrench upon body B is identical to the stiffness matrix of the spring assuming the body A is stationary.

5. Stiffness matrix of the mechanism

The stiffness matrix $[K]$ which maps a small twist of body B in terms of the inertial frame into the corresponding change of the wrench on body B is derived from Eq. (3) (see Fig. 1). Since springs 1–6 join the two moving bodies and springs 7–12 connect body A to ground, the derivatives of the spring wrenches can be written as

$$\sum_{i=1}^{6}\delta\underline{\mathbf{w}}_i = \sum_{i=1}^{6}([K_F]_i{}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}} - (\underline{\mathbf{w}}_i\times)^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}) = [K_F]_{R,U}{}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}} - (\underline{\mathbf{w}}_{\mathrm{ext}}\times)^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}, \tag{67}$$

$$\sum_{i=7}^{12}\delta\underline{\mathbf{w}}_i = \sum_{i=7}^{12}[K_F]_i{}^{\mathrm{A}}\delta\underline{\mathbf{D}}_i^{\mathrm{B}} = [K_F]_{R,L}{}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}, \tag{68}$$

where

$$[K_F]_{R,L} = \sum_{i=7}^{12}[K_F]_i, \tag{69}$$

$$[K_F]_{R,U} = \sum_{i=1}^{6}[K_F]_i \tag{70}$$

and where $\underline{\mathbf{w}}_{\mathrm{ext}}\times$ is the external wrench expressed as a spatial cross-product operator.

From Eqs. (67) and (68) and the twist equation (28), twist $^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}$ can be written as Eq. (72).

$$\begin{aligned}[K_F]_{R,L}{}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}} &= [K_F]_{R,U}{}^{\mathrm{A}}\delta\underline{\mathbf{D}}^{\mathrm{B}} - (\underline{\mathbf{w}}_{\mathrm{ext}}\times)^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}} \\ &= [K_F]_{R,U}({}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}} - {}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}}) - (\underline{\mathbf{w}}_{\mathrm{ext}}\times)^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}},\end{aligned} \tag{71}$$

$$^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{A}} = ([K_F]_{R,L} + [K_F]_{R,U} + (\underline{\mathbf{w}}_{\mathrm{ext}}\times))^{-1}[K_F]_{R,U}{}^{\mathrm{E}}\delta\underline{\mathbf{D}}^{\mathrm{B}}. \tag{72}$$

Substituting Eq. (72) for ${}^{E}\delta\underline{\mathbf{D}}^{A}$ in Eq. (68) and comparing it with Eq. (1) yield the stiffness matrix as Eq. (74).

$$[K]^{E}\delta\underline{\mathbf{D}}^{B} = [K_F]_{R,L}{}^{E}\delta\underline{\mathbf{D}}^{A} = [K_F]_{R,L}([K_F]_{R,L} + [K_F]_{R,U} + (\underline{\mathbf{w}}_{\text{ext}}\times))^{-1}[K_F]_{R,U}{}^{E}\delta\underline{\mathbf{D}}^{B}, \tag{73}$$

$$[K] = [K_F]_{R,L}([K_F]_{R,L} + [K_F]_{R,U} + (\underline{\mathbf{w}}_{\text{ext}}\times))^{-1}[K_F]_{R,U}. \tag{74}$$

It was generally accepted that the resultant compliance, which is the inverse of the stiffness, of serially connected mechanisms is the summation of the compliances of all constituent mechanisms (see Griffis [3]). However, the stiffness matrix derived from this research shows a different result. Taking an inverse of the stiffness matrix equation (74) yields

$$[K]^{-1} = [K_F]_{R,L}^{-1} + [K_F]_{R,U}^{-1} + [K_F]_{R,U}^{-1}(\underline{\mathbf{w}}_{\text{ext}}\times)[K_F]_{R,L}^{-1}. \tag{75}$$

The third term in Eq. (75) is newly introduced in this research and it does not vanish unless the external wrench is zero.

6. Numerical example

The geometry information and spring properties of the mechanism shown in Fig. 1 are presented in Tables 1–5. The external wrench $\underline{\mathbf{w}}_{\text{ext}}$ is given as

$$\underline{\mathbf{w}}_{\text{ext}} = [-0.3 \quad 0.4 \quad 0.8 \quad -2.3 \quad -1.3 \quad 0.7]^{T} \quad (\text{unit}: \ [\text{N},\ \text{N},\ \text{N},\ \text{N cm},\ \text{N cm},\ \text{N cm}]).$$

Two stiffness matrices are calculated: $[K]_1$ from Eq. (74) and $[K]_2$ from the same equation but without the matrix $\underline{\mathbf{w}}\times$. The numerical results are

$$[K]_1 = \begin{bmatrix} 0.3429 & -0.0077 & -0.2661 & -0.7853 & 1.7378 & -0.4076 \\ -0.0077 & 0.5103 & 1.7122 & 1.2760 & 0.2157 & -0.2885 \\ -0.2661 & 1.7122 & 10.5103 & 20.0012 & 0.7518 & -0.2695 \\ -0.7853 & 2.0760 & 19.6012 & 54.3222 & 1.1348 & 1.2570 \\ 0.9378 & 0.2157 & 0.4518 & 0.4348 & 12.1329 & -3.8667 \\ -0.0076 & 0.0115 & -0.2695 & -0.0430 & -1.5667 & -0.0798 \end{bmatrix},$$

$$[K]_2 = \begin{bmatrix} 0.3039 & -0.0109 & -0.2641 & -0.7858 & 1.3134 & -0.2770 \\ -0.0504 & 0.4617 & 1.7122 & 1.9863 & -0.0875 & 0.0375 \\ -0.4364 & 1.6222 & 10.6633 & 21.7834 & -0.7144 & 0.5956 \\ -1.0788 & 1.9862 & 20.7574 & 59.4736 & -1.1874 & 2.5822 \\ 0.9754 & 0.0759 & -0.0901 & -0.1283 & 12.1852 & -3.0399 \\ -0.0319 & 0.0218 & -0.0576 & 0.6983 & -1.6440 & -0.1157 \end{bmatrix},$$

where the units of upper left 3×3 sub matrix is N/cm, that of lower right 3×3 sub matrix is N cm, and that of remainder is N.

Table 1
Spring properties (unit: N/cm for k, cm for l_0)

Spring no.	1	2	3	4	5	6
Stiffness coefficient, k_i	4.6	4.7	4.5	4.4	5.3	5.5
Free length, l_{0i}	1.6305	1.0276	4.0098	1.8592	1.7591	3.8364
Spring no.	7	8	9	10	11	12
Stiffness coefficient, k_i	4.4	4.9	4.7	4.5	5.1	4.8
Free length, l_{0i}	4.4718	1.2760	5.2149	2.6780	2.2712	3.4244

Table 2
Positions of pivots in ground (unit: cm)

No.	1	2	3	4	5	6
X	0.0000	1.3000	0.6000	−0.7000	−1.1000	−0.5000
Y	0.0000	1.1000	2.7000	2.6000	1.8000	0.4000
Z	0.0000	0.2000	0.1000	−0.1000	0.3000	0.1000

Table 3
Positions of pivots in bottom side of body A in terms of the inertial frame (unit: cm)

No.	1	2	3	4	5	6
X	0.2000	1.1833	0.4616	−0.6575	−1.1452	−0.2189
Y	1.2000	2.1235	3.5111	3.3783	2.5652	1.6879
Z	3.2000	3.1843	3.3010	3.1013	3.0704	3.1196

Table 4
Positions of pivots in top side of body A in terms of the inertial frame (unit: cm)

No.	1	2	3	4	5	6
X	0.2086	1.4860	0.7553	−0.5501	−0.9278	−0.2945
Y	1.2033	2.3329	3.9187	3.7867	2.9797	1.5942
Z	3.2996	3.2514	3.3121	3.2792	3.3590	3.3804

Table 5
Positions of pivots in body B in terms of the inertial frame (unit: cm)

No.	1	2	3	4	5	6
X	−0.3000	0.9216	0.2183	−0.8385	−1.2525	−0.5589
Y	1.6000	2.7822	3.8980	3.9919	2.8972	2.0875
Z	5.5000	5.5000	5.4782	5.8447	5.8317	5.7745

The result is evaluated in the following way:

1. A small wrench $\delta\underline{\mathbf{w}}_T$ is applied in addition to $\underline{\mathbf{w}}_{\text{ext}}$ to body B and twists ${}^{\text{E}}\delta\underline{\mathbf{D}}_1^{\text{B}}$ and ${}^{\text{E}}\delta\underline{\mathbf{D}}_2^{\text{B}}$ are obtained by multiplying the inverse matrices of the stiffness matrices, $[K]_1$ and $[K]_2$, respectively, by $\delta\underline{\mathbf{w}}_T$ as of Eq. (1). Corresponding positions for body B are then determined, based on the calculated twists ${}^{\text{E}}\delta\underline{\mathbf{D}}_1^{\text{B}}$ and ${}^{\text{E}}\delta\underline{\mathbf{D}}_2^{\text{B}}$.
2. ${}^{\text{E}}\delta\underline{\mathbf{D}}^{\text{A}}$ is calculated by multiplying the inverse matrix of $[K_F]_{R,L}$ by $\delta\underline{\mathbf{w}}_T$ as of Eq. (68). The position of body A is then determined from this twist.
3. The wrench between body B and body A is calculated for the two cases based on knowledge of the positions of bodies A and B and the spring parameters. The change in wrench for the two cases is determined as the difference between the new equilibrium wrench and the original. The changes in the wrenches are named $\delta\underline{\mathbf{w}}_{\text{AB},1}$ and $\delta\underline{\mathbf{w}}_{\text{AB},2}$ which correspond to the matrices $[K]_1$ and $[K]_2$.
4. The given change in wrench $\delta\underline{\mathbf{w}}_T$ is compared to $\delta\underline{\mathbf{w}}_{\text{AB},1}$ and $\delta\underline{\mathbf{w}}_{\text{AB},2}$.

The given wrench $\delta\underline{\mathbf{w}}_T$ and the numerical results are presented as below.

$$\delta\underline{\mathbf{w}}_T = 10^{-4}\times[0.5 \quad -0.2 \quad 0.4 \quad 0.3 \quad -0.8 \quad 0.4]^{\text{T}},$$
$${}^{\text{E}}\delta\underline{\mathbf{D}}_1^{\text{B}} = 10^{-3}\times[0.3522 \quad -0.3081 \quad 0.0912 \quad -0.0137 \quad -0.0429 \quad -0.0367]^{\text{T}},$$
$${}^{\text{E}}\delta\underline{\mathbf{D}}_2^{\text{B}} = 10^{-3}\times[0.3354 \quad -0.2682 \quad 0.0845 \quad -0.0132 \quad -0.0404 \quad -0.0365]^{\text{T}},$$
$${}^{\text{E}}\delta\underline{\mathbf{D}}^{\text{A}} = 10^{-3}\times[0.1113 \quad -0.0100 \quad -0.0067 \quad 0.0081 \quad -0.0267 \quad -0.0650]^{\text{T}},$$
$$\delta\underline{\mathbf{w}}_{E\text{A}} = 10^{-4}\times[0.5000 \quad -0.1995 \quad 0.4017 \quad 0.3035 \quad -0.8000 \quad 0.4000]^{\text{T}},$$

$$\delta\underline{\mathbf{w}}_{\mathrm{AB},1} = 10^{-4} \times [0.4997 \quad -0.1998 \quad 0.4020 \quad 0.3041 \quad -0.8010 \quad 0.4011]^{\mathrm{T}},$$
$$\delta\underline{\mathbf{w}}_{\mathrm{AB},2} = 10^{-4} \times [0.5462 \quad -0.0693 \quad 0.3534 \quad -0.7831 \quad -0.2319 \quad 0.0967]^{\mathrm{T}},$$

where $\delta\underline{\mathbf{w}}_{E\mathrm{A}}$ is the wrench between body A and ground. The difference between $\delta\underline{\mathbf{w}}_{E\mathrm{A}}$ and $\delta\underline{\mathbf{w}}_{T}$ is small and is due to the fact that the twist was not infinitesimal. The difference between $\delta\underline{\mathbf{w}}_{\mathrm{AB},1}$ and $\delta\underline{\mathbf{w}}_{T}$ is also small and is most likely attributed to the same fact. However, the difference between $\delta\underline{\mathbf{w}}_{\mathrm{AB},2}$ and $\delta\underline{\mathbf{w}}_{T}$ is not negligible. This indicates that the stiffness matrix formula derived in this paper produces the proper result and that the term $\underline{\mathbf{w}}_{\mathrm{ext}}\times$ cannot be neglected in Eq. (74).

7. Conclusions

In this paper, a derivative of the spring wrench connecting two moving bodies was derived by using screw theory and an intermediate frame and applied to obtain the stiffness matrix of a mechanism having two compliant parallel mechanisms in a serial arrangement. It was shown that a derivative of the spring wrench connecting two moving bodies depends not only on a relative twist between the two bodies but also on the twist of the intermediate body in terms of the inertial frame.

The derived stiffness matrix showed that the resultant compliance of two serially arranged parallel mechanisms is not the summation of the compliances of the constituent mechanisms unless the external wrench applied to the mechanism is zero. This result also may be applied for mechanisms having arbitrary number of parallel mechanisms in a serial arrangement. When a preloaded spring rotates or translates other than in the spring axis direction without any deflection, it still causes a wrench variation and it led to the inclusion of the additional term in the compliance matrix. For a system where all bodies are connected serially and constrained to move in a line along the spring axis, there may be only translation along the line and it may be said that the equilibrium compliance of the system is the summation of the compliances of the constituent springs.

Acknowledgements

The authors would like to gratefully acknowledge the support provided by the Department of Energy via the University Research Program in Robotics (URPR), grant number DE-FG04-86NE37967.

References

[1] D.E. Whitney, Quasi-static assembly of compliantly supported rigid parts, ASME Journal of Dynamic Systems, Measurement, and Control 104 (1982) 65–77.
[2] M. Peshkin, Programmed compliance for error corrective assembly, IEEE Transactions on Robotics and Automation 6 (4) (1990) 473–482.
[3] M. Griffis, A novel theory for simultaneously regulating force and displacement, Ph.D. dissertation, University of Florida, Gainesville, 1991.
[4] R.S. Ball, A Treatise on the Theory of Screws, Cambridge University Press, London, 1900.
[5] C.D. Crane, J.M. Rico, J. Duffy, Screw Theory and its Application to Spatial Robot Manipulators, University of Florida, 2007.
[6] F.M. Dimentberg, The screw calculus and its applications in mechanics, Foreign Technology Division, Wright-Patterson Air Force Base, Ohio, Document No. FTD-HT-23-1632-67, 1965.
[7] S. Huang, J.M. Schimmels, The bounds and realization of spatial stiffness achieved with simple springs connected in parallel, IEEE Transactions on Robotics and Automation 14 (3) (1998) 466–475.
[8] N. Ciblak, H. Lipkin, Synthesis of cartesian stiffness for robotic applications, in: Proceedings of the IEEE International Conference on Robotics and Automation, Detroit, MI, May 1999, pp. 2147–2152.
[9] R.G. Roberts, Minimal realization of a spatial stiffness matrix with simple springs connected in parallel, IEEE Transactions on Robotics and Automation 15 (5) (1999) 953–958.
[10] N. Simaan, M. Shoham, Geometric interpretation of the derivatives of parallel robot's Jacobian matrix with application to stiffness control, ASME Journal of Mechanical Design 125 (2003) 33–42.
[11] J. Featherstone, Robot Dynamics Algorithms, Kluwer Academic Publishers, 1985.
[12] N. Ciblak, H. Lipkin, Asymmetric cartesian stiffness for the modeling of compliant robotic systems, in: Proc. ASME 23rd Biennial Mech. Conf., Des. Eng. Div., New York, NY, vol. 72, 1994.