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CLOSED-FORM EQUILIBRIUM ANALYSIS OF PLANAR TENSEGRITY STRUCTURES

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ABSTRACT

This paper presents a closed-form analysis of a series of planar tensegrity structures to determine all possible equilibrium configurations for each device when no external forces or moments are applied. The equilibrium position is determined by identifying the configurations at which the potential energy stored in the springs is a minimum. The degree of complexity associated with the solution was far greater than expected. For a two-spring system, a 28th degree polynomial expressed in terms of the length of one of the springs is developed where this polynomial identifies the cases where the change in potential energy with respect to an infinitesimal change in the spring length is zero. Three and four spring systems are also analyzed. These more complex systems were solved using the Continuation Method. Numerical examples are presented.

1. INTRODUCTION

The word tensegrity is a combination of the words tension and integrity (Edmondson, 1987 and Fuller, 1975). Tensegrity structures are spatial structures formed by a combination of rigid elements in compression (struts) and connecting elements that are in tension (ties). In a classic definition, no pair of struts touch and the end of each strut is connected to three non-coplanar ties (Yin et al, 2002). The entire configuration stands by itself and maintains its form solely because of the internal arrangement of the struts and ties (Tobie, 1976).

The development of tensegrity structures is relatively new and the works related have only existed for approximately twenty five years. Kenner, 1976, established the relation between the rotation of the top and bottom ties. Tobie, 1976, presented procedures for the generation of tensile structures by physical and graphical means. Yin, 2002, obtained Kenner's results using energy considerations and found the equilibrium position for unloaded tensegrity prisms. Stern, 1999, developed generic design equations to find the lengths of the struts and elastic ties needed to create a desired geometry for a symmetric case. Knight, 2000, addressed the problem of stability of tensegrity structures for the design of deployable antennae. Duffy, 2000 and 2002, presents static analysis and equilibrium condition for selected tensigrity structures. Roth, 1981, establishes some basis for flexibility and rigidity of tensigrity frameworks. Skelton, 2002, reduces the static model of some tensigrity structures to linear algebra equations by characterizing the problems in vector space. Tibert, 2002, uses the tensegrity structure in deployable space applications. Festl, 2003, explains the adjustable tensigrity structures and their applications.

2. TWO-SPRING SYSTEM

A planar tensegrity system is shown in Figure 1. The device consists of two rigid struts and four ties. For the two-spring system, two of the ties are compliant (the ties between points 4 and 2 and points 1 and 3) and two of the ties are non-compliant. The objective is to determine all equilibrium configurations for the system when given the lengths of the struts and non-compliant ties as well as the free length and spring constant for each of the compliant ties.

The specific problem statement for the two-spring system is written as follows:

given: a_{12}, a_{34} lengths of struts,
a_{23}, a_{41} lengths of non-compliant ties,
k_1, L_{01} spring constant and free length of compliant tie between points 4 and 2,

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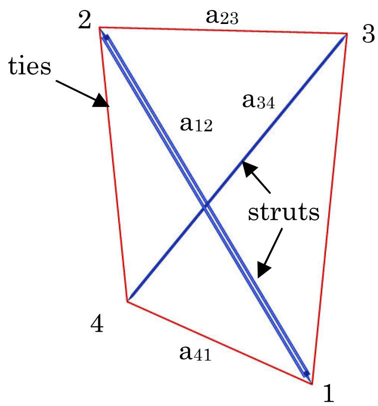

Figure 1: Planar Tensegrity System

k_2, L_{02} spring constant and free length of compliant tie between points 3 and 1.

find: L_1 length of spring 1 at equilibrium position,

L_2 length of spring 2 at equilibrium position corresponding to length of spring 1, i.e. L_1.

It should be noted that the problem statement could be formulated in a variety of ways, i.e. a different variable (such as the relative angle between strut a_{34} and tie a_{41}) could have been selected as the generalized parameter for this problem. Attempts at using this alternate approach did not successfully yield a closed form solution for the equilibrium positions.

2.1 Development of Geometry and Energy Equations

Figure 2 shows the nomenclature that is used. L_1 and L_2 are the extended lengths of the compliant ties between points 4 and 2 and points 3 and 1. A cosine law for the triangle with sides a_{34}, a_{23}, and L_1 can be written as

$$\frac{L_1^{\,2}}{2}+\frac{a_{34}^{\,2}}{2}+L_1\,a_{34}\cos\theta_4{}' = \frac{a_{23}^{\,2}}{2}\ . \tag{1}$$

Solving for $\cos\theta_4{}'$ yields

$$\cos\theta_4{}' = \frac{a_{23}^{\,2}-L_1^{\,2}-a_{34}^{\,2}}{2L_1\,a_{34}}\ . \tag{2}$$

A cosine law for the triangle with sides a_{41}, a_{12}, and L_1 can be written as

$$\frac{L_1^{\,2}}{2}+\frac{a_{41}^{\,2}}{2}+L_1\,a_{41}\cos\theta_4{}'' = \frac{a_{12}^{\,2}}{2}\ . \tag{3}$$

Solving for $\cos\theta_4{}''$ yields

$$\cos\theta_4{}'' = \frac{a_{12}^{\,2}-L_1^{\,2}-a_{41}^{\,2}}{2L_1\,a_{41}}\ . \tag{4}$$

A cosine law for the triangle with sides a_{41}, a_{34}, and L_2 can be written as

$$\frac{a_{34}^{\,2}}{2}+\frac{a_{41}^{\,2}}{2}+a_{34}\,a_{41}\cos\theta_4 = \frac{L_2^{\,2}}{2}\ . \tag{5}$$

Solving for $\cos\theta_4$ yields

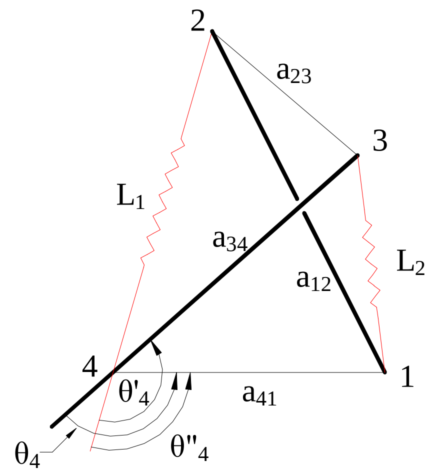

Figure 2: Two-Spring Planar Tensegrity Structure

$$\cos\theta_4 = \frac{L_2^{\,2}-a_{34}^{\,2}-a_{41}^{\,2}}{2a_{34}\,a_{41}}\ . \tag{6}$$

From Figure 2 it is apparent that

$$\theta_4 + \theta_4{}' = \pi + \theta_4{}'' \tag{7}$$

Equating the cosine of the left and right sides of (7) yields

$$\cos(\theta_4 + \theta_4{}') = \cos(\pi + \theta_4{}'') \tag{8}$$

and expanding this equation yields

$$\cos\theta_4 \cos\theta_4{}' - \sin\theta_4 \sin\theta_4{}' = -\cos\theta_4{}''\ . \tag{9}$$

Rearranging (9) yields

$$\cos\theta_4 \cos\theta_4{}' + \cos\theta_4{}'' = \sin\theta_4 \sin\theta_4{}'\ . \tag{10}$$

Squaring both sides of (10) gives

$$(\cos\theta_4)^2 (\cos\theta_4{}')^2 + 2\cos\theta_4 \cos\theta_4{}' \cos\theta_4{}'' + (\cos\theta_4{}'')^2 = (\sin\theta_4)^2 (\sin\theta_4{}')^2\ . \tag{11}$$

Substituting for $(\sin\theta_4)^2$ and $(\sin\theta_4{}')^2$ in terms of $\cos\theta_4$ and $\cos\theta_4{}'$ gives

$$(\cos\theta_4)^2 (\cos\theta_4{}')^2 + 2\cos\theta_4 \cos\theta_4{}' \cos\theta_4{}'' + (\cos\theta_4{}'')^2 = (1-\cos^2\theta_4)(1-\cos^2\theta_4{}')\ . \tag{12}$$

Equations (2), (4), and (6) are substituted into (12) to yield a single equation in the parameters L_1 and L_2 which can be written as

$$A\,L_2^{\,4} + B\,L_2^{\,2} + C = 0 \tag{13}$$

where

$$A = L_1^{\,2}, \quad B = L_1^{\,4} + B_2\,L_1^{\,2} + B_0, \quad C = C_2\,L_1^{\,2} + C_0 \tag{14}$$

and where

$$\begin{aligned} B_2 &= -(a_{23}^{\,2} + a_{34}^{\,2} + a_{41}^{\,2} + a_{12}^{\,2}),\\ B_0 &= (a_{12} - a_{41})(a_{12} + a_{41})(a_{23} - a_{34})(a_{23} + a_{34}),\\ C_2 &= (a_{34} - a_{41})(a_{34} + a_{41})(a_{23} - a_{12})(a_{23} + a_{12}),\\ C_0 &= (a_{41}a_{23} + a_{34}a_{12})(a_{41}a_{23} - a_{34}a_{12})(a_{41}^{\,2} + a_{23}^{\,2} - a_{12}^{\,2} - a_{34}^{\,2}). \end{aligned} \tag{15}$$

The potential energy of the system can be evaluated as

$$U = \frac{1}{2}k_1(L_1 - L_{01})^2 + \frac{1}{2}k_2(L_2 - L_{02})^2\ . \tag{16}$$

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At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the structure whereby the derivative of the potential energy taken with respect to the length L_1 equals zero, i.e.

$$\frac{dU}{dL_1} = k_1 (L_1 - L_{01}) + k_2 (L_2 - L_{02}) \frac{dL_2}{dL_1} = 0 \quad . \tag{17}$$

The derivative dL_2/dL_1 can be determined via implicit differentiation from equation (13) as

$$\frac{dL_2}{dL_1} = \frac{-L_1 [L_2^2 (L_2^2 + 2L_1^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{23}^2)(a_{41}^2 - a_{34}^2)]}{L_2 [L_1^2 (L_1^2 + 2L_2^2 - a_{23}^2 - a_{41}^2 - a_{34}^2 - a_{12}^2) + (a_{12}^2 - a_{41}^2)(a_{23}^2 - a_{34}^2))]} \quad . \tag{18}$$

Substituting (18) into (17) and regrouping gives

$$D\, L_2^5 + E\, L_2^4 + F\, L_2^3 + G\, L_2^2 + H\, L_2 + J = 0 \tag{19}$$

where

$$\begin{aligned}
D &= D_1 L_1 ,\\
E &= E_1 L_1 ,\\
F &= F_3 L_1^3 + F_2 L_1^2 + F_1 L_1,\\
G &= G_3 L_1^3 + G_1 L_1,\\
H &= H_5 L_1^5 + H_4 L_1^4 + H_3 L_1^3 + H_2 L_1^2 + H_1 L_1 + H_0,\\
J &= J_1 L_1
\end{aligned} \tag{20}$$

and where,

$$\begin{aligned}
&D_1 = k_2,\ \ E_1 = -k_2 L_{02},\\
&F_3 = 2 (k_2 - k_1),\ F_2 = 2 k_1 L_{01},\\
&F_1 = - k_2 (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),\\
&G_3 = -2 k_2 L_{02},\ G_1 = k_2 L_{02} (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),\\
&H_5 = - k_1,\ H_4 = k_1 L_{01},\ H_3 = k_1 (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),\\
&H_2 = - k_1 L_{01} (a_{12}^2 + a_{23}^2 + a_{34}^2 + a_{41}^2),\\
&H_1 = - k_1 (a_{34}^2 - a_{23}^2) (a_{41}^2 - a_{12}^2) + k_2 (a_{34}^2 - a_{41}^2)\\
&\qquad (a_{23}^2 - a_{12}^2),\\
&H_0 = k_1 L_{01} (a_{34}^2 - a_{23}^2) (a_{41}^2 - a_{12}^2),\\
&J_1 = k_2 L_{02} (a_{34}^2 - a_{41}^2) (a_{12}^2 - a_{23}^2) .
\end{aligned} \tag{21}$$

2.2 Solution of Geometry and Energy Equations

Equations (19) and (13) represent two equations in the two unknowns L_1 and L_2. These equations can be solved by using Sylvester's variable elimination procedure by multiplying equation (13) by L_2, L_2^2, L_2^3, and L_2^4 and equation (19) by L_2, L_2^2, L_2^3 to yield a total of nine equations that can be written in matrix form as

$$\begin{bmatrix}
0 & 0 & 0 & D & E & F & G & H & J\\
0 & 0 & 0 & 0 & A & 0 & B & 0 & C\\
0 & 0 & 0 & A & 0 & B & 0 & C & 0\\
0 & 0 & D & E & F & G & H & J & 0\\
0 & 0 & A & 0 & B & 0 & C & 0 & 0\\
0 & D & E & F & G & H & J & 0 & 0\\
0 & A & 0 & B & 0 & C & 0 & 0 & 0\\
D & E & F & G & H & J & 0 & 0 & 0\\
A & 0 & B & 0 & C & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} L_2^8\\ L_2^7\\ L_2^6\\ L_2^5\\ L_2^4\\ L_2^3\\ L_2^2\\ L_2\\ 1 \end{bmatrix}
=
\begin{bmatrix} 0\\0\\0\\0\\0\\0\\0\\0\\0 \end{bmatrix} . \tag{22}$$

A solution to this set of equations can only occur if the determinant of the 9×9 coefficient matrix is equal to zero. Expansion of this determinant yields a 30^{th} degree polynomial in the variable L_1. When the determinant was expanded symbolically, it was seen that the two lowest order coefficients were identically zero. Thus the polynomial can be divided throughout by L_1^2 to yield a 28^{th} degree polynomial. The coefficients of the 28^{th} degree polynomial were obtained symbolically in terms of the given quantities, but are not presented here due to their length and complexity.

Values for L_2 that correspond to each value of L_1 can be determined by first solving (13) for four possible values of L_2. Only one of these four values also satisfies equation (19).

2.3 Numerical Example

The following parameters were selected to show the results of a numerical example:

strut lengths:
a_{12} = 3 in. $\quad$ a_{34} = 3.5 in.

non-compliant tie lengths:
a_{41} = 4 in. $\quad$ a_{23} = 2 in.

spring 1 free length & spring constant:
L_{01} = 0.5 in. $\quad$ k_1 = 4 lbf/in.

spring 2 free length & spring constant:
L_{02} = 1 in. $\quad$ k_2 = 2.5 lbf/in.

Eight real and twenty complex roots were obtained for L_1. The real values for L_1 and the corresponding values of L_2 are shown in Table 1.

Table 1: Eight Real Solutions

Case	L_1, in.	L_2, in.
1	-5.485	2.333
2	-5.322	-2.901
3	-1.741	-1.495
4	-1.576	1.870
5	1.628	1.709
6	1.863	-1.354
7	5.129	-3.288
8	5.476	2.394

The values of L_1 and L_2 listed in Table 1 satisfy the geometric constraints defined by equation (13) and the energy condition defined by equation (19). Each of these eight cases was analyzed to determine whether it represented a minimum or maximum potential energy condition and cases 3, 4, 5, and 6 were found to be minimum states. A free body analysis of struts a_{12} and a_{34} was performed to show that these bodies were indeed in equilibrium. Figure 3 shows the four equilibrium configurations for the numerical case under consideration.

Efforts were undertaken to obtain a numerical example that would yield more than eight real roots. One example of each type of Grashof and non-Grashof four-bar mechanism was analyzed, yet for all these cases a maximum of eight real roots were found. The complex values of L_1 were analyzed and

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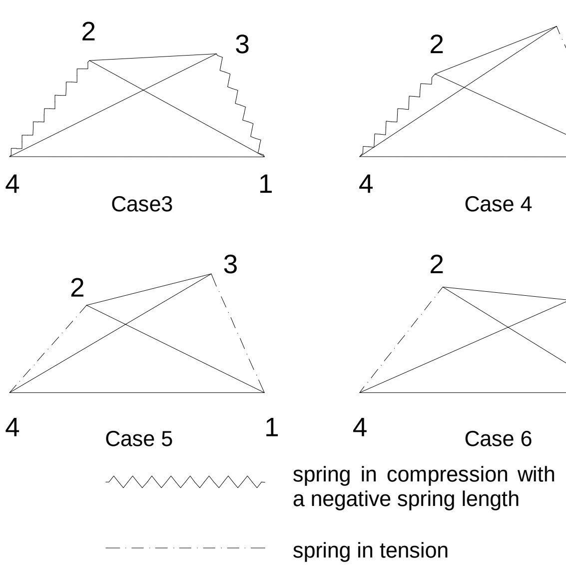

Figure 3: Four Equilibrium Configurations for Two-Spring Tensegrity System

corresponding complex values of L_2 were determined. In every case it was possible to obtain complex pairs of L_1 and L_2 that satisfied the geometric constraint equation, (13), and the derivative of potential energy equation (19). Thus no extraneous roots were introduced during the variable elimination procedure.

3. THREE-SPRING SYSTEM

Figure 4 shows a three-spring tensegrity system. Two parameters must be specified, in addition to the constant mechanism parameters, in order to define the configuration of the device. These two parameters will be referred to as the *descriptive parameters* for the system. One obvious set of descriptive parameters are the angles θ_4 and θ_1. Considering the non-compliant member a_{41} as being fixed to ground, specification of θ_4 will define the location of point 3. Similarly, specification of θ_1 will define the location of point 2.

Two approaches to solve this problem have been analyzed. Both aim to find a set of descriptive parameters that minimize the potential energy in the system. In the first approach, the lengths of the compliant members L_1 and L_2 are chosen as the descriptive parameters. Derivatives of the potential energy equation are obtained with respect to L_1 and L_2 and values for the descriptive parameters are obtained such that these derivatives are zero, corresponding to either a minimum or maximum potential energy state. In the second approach, the cosines of the angles θ_4 and θ_1 were chosen as the descriptive parameters. The cosines of the angles were chosen rather than the angles themselves in the hope that the resulting equations would be of lesser degree in that, for example, a single value of $\cos\theta_4$ accounts for the obvious symmetry in solutions that will occur with respect to the fixed member a_{41}.

3.1 Approach 1 – Descriptive Parameters L_1 and L_2

3.1.1 Problem Formulation

The problem statement can be explicitly written as:

given: length of struts (a_{12}, a_{34}) ; length of the non-compliant tie (a_{41}) ; spring constants and free lengths of three springs (k_1, L_{01} ; k_2, L_{02} ; k_3, L_{03})

find: length of springs 1 and 2 (L_1, L_2) and corresponding length of spring 3 (L_3) at equilibrium

The analysis for this case can proceed in a manner similar to that presented for the two-spring system. Specifically, the term a_{23} in (15) can be replaced by L_3 and equation (13) can be factored into the form

$$G_1 L_3^4 + (G_2 L_2^2 + G_3) L_3^2 + (G_4 L_2^4 + G_5 L_2^2 + G_6) = 0 \quad (23)$$

where

$$\begin{aligned} G_1 &= a_{41}^2, \\ G_2 &= G_{2a} L_1^2 + G_{2b}, \\ G_3 &= G_{3a} L_1^2 + G_{3b}, \\ G_4 &= G_{4a} L_1^2, \\ G_5 &= G_{5a} L_1^4 + G_{5b} L_1^2 + G_{5c}, \\ G_6 &= G_{6a} L_1^2 + G_{6b} \end{aligned} \quad (24)$$

and where the remaining coefficients are written in terms of the constant mechanism parameters as

$$\begin{aligned} G_{2a} &= -1, \\ G_{2b} &= a_{12}^2 - a_{41}^2, \\ G_{3a} &= (a_{34}^2 - a_{41}^2), \\ G_{3b} &= a_{41}^2 (a_{41}^2 - a_{12}^2 - a_{34}^2) - a_{12}^2 a_{34}^2, \\ G_{4a} &= 1, \\ G_{5a} &= 1, \\ G_{5b} &= (-a_{12}^2 - a_{34}^2 - a_{41}^2), \\ G_{5c} &= a_{34}^2 (a_{41}^2 - a_{12}^2), \\ G_{6a} &= a_{12}^2 (a_{41}^2 - a_{34}^2), \; G_{6b} = a_{12}^2 a_{34}^2 (a_{12}^2 + a_{34}^2 - a_{41}^2) . \end{aligned} \quad (25)$$

The potential energy of the system can be written as

$$U = \frac{1}{2} k_1 (L_1 - L_{01})^2 + \frac{1}{2} k_2 (L_2 - L_{02})^2 + \frac{1}{2} k_3 (L_3 - L_{03})^2 . \quad (26)$$

At equilibrium, the potential energy will be a minimum. This condition can be determined as the configuration of the

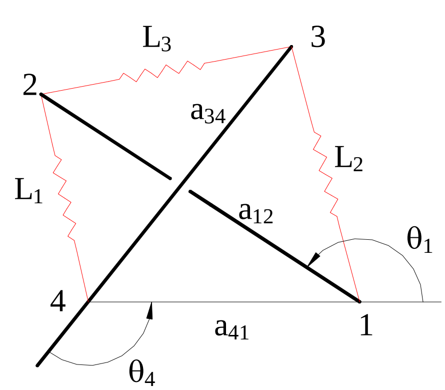

Figure 4: Three-Spring Tensegrity System

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structure whereby the derivative of the potential energy taken with respect to the descriptive parameters L_1 and L_2 both equal zero. The geometric constraint equation, equation (23), contains three unknown terms, L_1, L_2, and L_3. From this equation, L_3 can be considered as a dependent variable of L_1 and L_2. The following two expressions may be written:

$$\frac{\partial U}{\partial L_1} = k_1 (L_1 - L_{01}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_1} = 0 \ , \tag{27}$$

$$\frac{\partial U}{\partial L_2} = k_2 (L_2 - L_{02}) + k_3 (L_3 - L_{03}) \frac{\partial L_3}{\partial L_2} = 0 \ . \tag{28}$$

The derivatives $\delta L_3/\delta L_1$ and $\delta L_3/\delta L_2$ can be determined via implicit differentiation from Equation 23 as

$$\frac{\partial L_3}{\partial L_1} = \frac{-L_1 [2 L_1^2 L_2^2 + G_{2a} L_2^2 L_3^2 + G_{3a} L_3^2 + L_2^4 + G_{5b} L_2^2 + G_{6a}]}{L_3 [2 G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} \tag{29}$$

$$\frac{\partial L_3}{\partial L_2} = \frac{-L_2 [2 L_1^2 L_2^2 + G_{2a} L_1^2 L_3^2 + G_{2b} L_3^2 + L_1^4 + G_{5b} L_1^2 + G_{5c}]}{L_3 [2 G_1 L_3^2 + G_{2a} L_1^2 L_2^2 + G_{2b} L_2^2 + G_{3a} L_1^2 + G_{3b}]} \tag{30}$$

Substituting (29) into (27) and rearranging gives

$$(D_1 L_2^2 + D_2) L_3^3 + (D_3 L_2^2 + D_4) L_3^2 + (D_5 L_2^4 + D_6 L_2^2 + D_7) L_3 + (D_8 L_2^4 + D_9 L_2^2 + D_{10}) = 0 \tag{31}$$

where

$D_1 = D_{1a} L_1$,
$D_2 = D_{2a} L_1 + D_{2b}$,
$D_3 = D_{3a} L_1$,
$D_4 = D_{4a} L_1$,
$D_5 = D_{5a} L_1$,
$D_6 = D_{6a} L_1^3 + D_{6b} L_1^2 + D_{6c} L_1 + D_{6d}$,
$D_7 = D_{7a} L_1^3 + D_{7b} L_1^2 + D_{7c} L_1 + D_{7d}$,
$D_8 = D_{8a} L_1$,
$D_9 = D_{9a} L_1^3 + D_{9b} L_1$
$D_{10} = D_{10a} L_1$ (32)

and

$D_{1a} = - G_{2a} k_3$,
$D_{2a} = 2 G_1 k_1 - G_{3a} k_3$, $D_{2b} = - 2 G_1 k_1 L_{01}$,
$D_{3a} = G_{2a} k_3 L_{03}$,
$D_{4a} = G_{3a} k_3 L_{03}$,
$D_{5a} = - k_3$,
$D_{6a} = G_{2a} k_1 - 2 k_3$,
$D_{6b} = -G_{2a} k_1 L_{01}$, $D_{6c} = G_{2b} k_1 - G_{5b} k_3$, $D_{6d} = -G_{2b} k_1 L_{01}$,
$D_{7a} = G_{3a} k_1$, $D_{7b} = -G_{3a} k_1 L_{01}$, $D_{7c} = G_{3b} k_1 - G_{6a} k_3$,
$D_{7d} = -G_{3b} k_1 L_{01}$,
$D_{8a} = k_3 L_{03}$,
$D_{9a} = 2 k_3 L_{03}$, $D_{9b} = G_{5b} k_3 L_{03}$
$D_{10a} = G_{6a} k_3 L_{03}$ (33)

Substituting (30) into (28) and rearranging gives

$$(E_1 L_2 + E_2) L_3^3 + (E_3 L_2) L_3^2 + (E_4 L_2^3 + E_5 L_2^2 + E_6 L_2 + E_7) L_3 + (E_8 L_2^3 + E_9 L_2) = 0 \tag{34}$$

where

$E_1 = E_{1a} L_1^2 + E_{1b}$,
$E_2 = - 2 G_1 k_2 L_{02}$,
$E_3 = E_{3a} L_1^2 + E_{3b}$,
$E_4 = E_{4a} L_1^2 + E_{4b}$,
$E_5 = E_{5a} L_1^2 + E_{5b}$,
$E_6 = E_{6a} L_1^4 + E_{6b} L_1^2 + E_{6c}$,
$E_7 = E_{7a} L_1^2 + E_{7b}$,
$E_8 = E_{8a} L_1^2$,
$E_9 = E_{9a} L_1^4 + E_{9b} L_1^2 + E_{9c}$ (35)

and

$E_{1a} = -G_{2a} k_3$, $E_{1b} = -G_{2b} k_3 + 2 G_1 k_2$,
$E_{3a} = G_{2a} k_3 L_{03}$, $E_{3b} = G_{2b} k_3 L_{03}$,
$E_{4a} = G_{2a} k_2 - 2 k_3$, $E_{4b} = G_{2b} k_2$,
$E_{5a} = -G_{2a} k_2 L_{02}$, $E_{5b} = -G_{2b} k_2 L_{02}$,
$E_{6a} = - k_3$, $E_{6b} = G_{3a} k_2 - G_{5b} k_3$, $E_{6c} = G_{3b} k_2 - G_{5c} k_3$,
$E_{7a} = -G_{3a} k_2 L_{02}$, $E_{7b} = -G_{3b} k_2 L_{02}$,
$E_{8a} = 2 k_3 L_{03}$,
$E_{9a} = k_3 L_{03}$, $E_{9b} = G_{5b} k_3 L_{03}$, $E_{9c} = G_{5c} k_3 L_{03}$. (36)

3.1.2 Solution of Three Simultaneous Equations in Three Unknowns – Sylvester's Method

Equations (23), (31), and (34) are three equations in the three unknowns L_1, L_2, and L_3. Sylvester's method can be applied in order to obtain sets of values for these parameters that simultaneously satisfy all three equations. In this solution, the parameter L_1 is embedded in the coefficients of the three equations to yield three equations in the apparent unknowns L_2 and L_3. Determining the condition that these new coefficients (which contain L_1) must satisfy such that the three equations can have common roots for L_2 and L_3 will yield a single polynomial in L_1.

Equation (23) was multiplied by L_2, L_3, L_2L_3, L_3^2, L_2^2, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, L_3^3, L_2^3, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2^3L_3^3$. Equation (31) was multiplied by L_3, L_2, L_3^2, $L_2L_3^2$, L_3^3, L_2^2, $L_2L_3^3$, $L_2^2L_3$, $L_2^2L_3^2$, L_2^3, L_3^4, $L_2^3L_3$, $L_2^3L_3^2$, and $L_2L_3^4$. Equation (34) was multiplied by L_2, L_3, L_2L_3, L_3^2, L_2^2, $L_2L_3^2$, $L_2^2L_3$, $L_2^2L_3^2$, L_3^3, L_2^3, $L_2L_3^3$, $L_2^2L_3^3$, $L_2^3L_3$, $L_2^3L_3^2$, L_2^4, L_3^4, $L_2^4L_3$, $L_2^4L_3^2$, $L_2L_3^4$, and $L_2^2L_3^4$. This resulted in a set of 52 equations that can be written in matrix for as

$$\mathbf{M} \boldsymbol{\lambda} = \mathbf{0} \ . \tag{37}$$

The vector $\boldsymbol{\lambda}$ is written as

$$\boldsymbol{\lambda} = [L_2^7L_3^3, L_2^5L_3^5, L_2^3L_3^7, L_2^7L_3^2, L_2^6L_3^3, L_2^5L_3^4, L_2^4L_3^5, L_2^3L_3^6, L_2^2L_3^7, L_2^7L_3, L_2^6L_3^2, L_2^5L_3^3, L_2^4L_3^4, L_2^3L_3^5, L_2^2L_3^6, L_2L_3^7, L_2^7, L_2^6L_3, L_2^5L_3^2, L_2^4L_3^3, L_2^3L_3^4, L_2^2L_3^5, L_2L_3^6, L_3^7, L_2^6, L_2^5L_3, L_2^4L_3^2, L_2^3L_3^3, L_2^2L_3^4, L_2L_3^5, L_3^6, L_2^5, L_2^4L_3, L_2^3L_3^2, L_2^2L_3^3, L_2L_3^4, L_3^5, L_2^4, L_2^3L_3, L_2^2L_3^2, L_2L_3^3, L_3^4, L_2^3, L_2^2L_3, L_2L_3^2, L_3^3, L_2^2, L_2L_3, L_3^2, L_2, L_3, 1]^T \ . \tag{38}$$

The coefficient matrix **M** is a 52×52 matrix whose elements are the coefficients G_1 through G_6, D_1 through D_{10}, and E_1 through E_9 which are polynomials in terms of the variable L_1. This matrix is not presented here due to its size. Since the set of 52 simultaneous equations represented by (37) must be linearly dependent, the determinant of the matrix **M** must equal zero. This will yield an equation in terms of the single variable L_1.

It was not possible to symbolically expand the determinant of matrix **M**. A numerical case was analyzed and a polynomial of degree 158 in the variable L_1 was obtained. It was not possible to numerically solve this high degree polynomial for

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the values of L_1, although several commercial and in-house written algorithms were attempted. Because of this, a different method was attempted to solve the set of equations (23), (31), and (34).

3.1.3 Solution of Three Simultaneous Equations in Three Unknowns – Continuation Method

The continuation method (Garcia and Li, 1980, Morgan, 1983, 1986, 1987, Wampler et al., 1990) is a numerical technique to solve a set of equations in multiple variables. This is as opposed to Sylvester's method which would lead to a symbolic solution of the problem.

A concise description of the continuity method is presented by Tsai, 1999. Suppose one wishes to solve the set of equations F(**x**) which are defined by

$$F(x): \begin{cases} f_1(x_1, x_2, \cdots, x_n) = 0 \\ f_2(x_1, x_2, \cdots, x_n) = 0 \\ \vdots \\ f_n(x_1, x_2, \cdots, x_n) = 0 \end{cases} . \quad (39)$$

F(**x**) is called the target system.

The continuation method begins by first estimating the total number of possible solution sets (sets of values for L_1, L_2, and L_3 for our case) that satisfy the given equations. For example, Bezout's theorem states that a polynomial of total degree *n* has at most *n* isolated solutions in the complex Euclidean space. Including solutions at infinity, the Bezout number of a polynomial system is equal to the total degree of the system.

Next, an initial system, G(**x**) =0, is obtained, whose solution will be of the same degree as that of F(**x**), but whose solution set is known in closed form. In other words, G(**x**) maintains the same polynomial structure as F(**x**).

Finally, a homotopy function H(**x**, t) is prepared such as

$$H(\mathbf{x}, t) = \gamma\,(1-t)\,G(\mathbf{x}) + t\,F(\mathbf{x}) \quad (40)$$

where γ is a random complex constant. When t=0, the homotopy function equals the initial system, G(**x**). When t=1, the homotopy function equals the target system, F(**x**). Recall that the solutions to G(**x**) are known. As the parameter *t* is increased in small steps from 0 to 1, the solutions of H(**x**, t) can be tracked (referred to as path tracking) and when t =1, these solutions will be the solutions to the original target system. If the degree of the solution set was overestimated, some of the solutions will track to infinity and these can easily be discarded.

3.1.4 Numerical Example

The following information is given:

strut lengths:

a_{12} = 14 in. a_{34} = 12 in.

non-compliant tie lengths:

a_{41} = 10 in.

spring 1 free length & spring constant:

L_{01} = 8 in. k_1 = 1 lbf/in.

spring 2 free length & spring constant:

L_{02} = 2 in. k_2 = 2.687 lbf/in.

spring 3 free length & spring constant:

L_{03} = 2.5 in. k_3 = 3.465 lbf/in.

Based on these values, the coefficients in equations (23), (31), and (34) were evaluated to yield the three equations

$$100\,L_3^4 + [(-L_1^2 + 96)\,L_2^2 + 44\,L_1^2 - 52224]\,L_3^2 + L_1^2\,L_2^4 + (L_1^4 - 440\,L_1^2 - 13824)\,L_2^2 - 8624\,L_1^2 + 6773760 = 0 \quad (41)$$

$$(2.5\,L_1\,L_2^2 + 90\,L_1 - 1600)\,L_3^3 + (-8.663\,L_1\,L_2^2 + 381.165\,L_1)\,L_3^2 + [-2.5\,L_1\,L_2^4 + (-6\,L_1^3 + 8\,L_1^2 + 1196\,L_1 - 768)\,L_2^2 + 44\,L_1^3 - 352\,L_1^2 - 30664\,L_1 + 417792]\,L_3 + 8.663\,L_1\,L_2^4 + (17.326\,L_1^3 - 3811.651\,L_1)\,L_2^2 - 74708.354\,L_1 = 0 \quad (42)$$

$$[(2.5\,L_1^2 + 160)\,L_2 - 1074.637]\,L_3^3 + (831.633 - 8.663\,L_1^2)\,L_2\,L_3^2 + [(-7\,L_1^2 + 192)\,L_2^3 + (-515.826 + 5.373\,L_1^2)\,L_2^2 + (-2.5\,L_1^4 + 1188\,L_1^2 - 69888)\,L_2 - 236.420\,L_1^2 + 280609.161]\,L_3 + 17.326\,L_1^2\,L_2^3 + (-119755.135 + 8.663\,L_1^4 - 3811.651\,L_1^2)\,L_2 = 0\,. \quad (43)$$

The continuation method was run on this set of three equations in three unknowns to obtain all solution sets for the three spring lengths, L_1, L_2, and L_3, for the particular numerical example. The software PHCpack (Verschelde, 1999) was used to implement the method.

The PHCpack software estimated the number of possible solutions to be 136. Seven real solutions were obtained and these are listed in Table 2.

Table 2: Seven Real Solutions for Three-Spring Planar Tensegrity System

Case	L_1, in.	L_2, in.	L_3, in.
1	13.000	8.000	7.017
2	-11.376	-10.371	-5.333
3	-7.585	9.097	10.106
4	-11.029	12.557	-3.044
5	13.969	-5.800	9.164
6	14.248	-9.373	-4.774
7	13.181	11.599	-2.488

An equilibrium analysis was conducted for the seven cases and only the first case was indeed in equilibrium, i.e. it

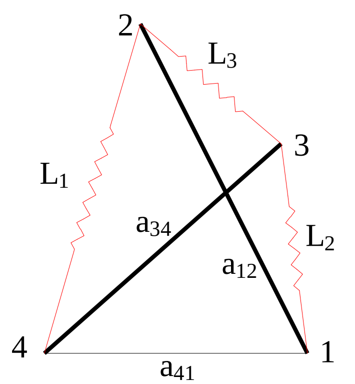

Figure 5: Case 1 ; Equilibrium Solution

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corresponds to a minimum potential energy configuration. Case 1 is shown in Figure 5.

3.2 Approach 2 – Descriptive Parameters $\cos\theta_1$ and $\cos\theta_4$

3.2.1 Problem Formulation

The problem statement is written as

given: a_{41}, a_{12}, a_{34} (strut lengths and non-compliant tie length)
k_1, k_2, k_3, L_{01}, L_{02}, L_{03} (spring constants and spring free lengths)

find: $\cos\theta_4$ and $\cos\theta_1$ when the system is in equilibrium

Figure 4 shows the nomenclature used. Springs L_1, L_2, and L_3 are the extended lengths of the compliant ties between points 4 and 2, 2 and 3, and 1 and 3 respectively. The unloaded lengths of springs are given by L_{01}, L_{02}, and L_{03}. A cosine law for the quadrilateral 1-2-3-4 can be written as:

$$Z_{41} = \frac{L_3^2}{2} \quad (44)$$

where,

$$Z_{41} = -a_{12}(X_4 s_1 + Y_4 c_1) + Z_4 + \frac{a_{12}^2}{2} \quad (45)$$

and where

$$X_4 = a_{34}\, s_4 \quad (46)$$

$$Y_4 = -(a_{41} + a_{34}\, c_4) \quad (47)$$

$$Z_4 = \frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} c_4 \ . \quad (48)$$

Substituting (45) into (44) and rearranging gives

$$a_{12} X_4 s_1 = -a_{12} Y_4 c_1 + Z_4 + \frac{a_{12}^2}{2} - \frac{L_3^2}{2} \ . \quad (49)$$

Substituting (46), (47), and (48) into (49), then squaring it, and substituting for $s_4^2 = 1 - c_4^2$ and $s_1^2 = 1 - c_1^2$ and multiplying the entire equation by 4 yields

$$L_3^4 + A\, L_3^2 + B = 0 \quad (50)$$

where

$$A = A_1 c_4 + A_2 c_1 + A_3 c_1 c_4 + A_4 \ , \quad (51)$$

$$B = B_1 c_1^2 + B_2 c_1^2 c_4 + B_3 c_1 + B_4 c_1 c_4 + B_5 c_1 c_4^2 + B_6 c_4^2 + B_7 c_4 + B_8 \quad (52)$$

and where

$$\begin{aligned}
A_1 &= -4\, a_{34}\, a_{41} \\
A_2 &= -4\, a_{12}\, a_{41} \\
A_3 &= -4\, a_{12}\, a_{34} \\
A_4 &= -2\,(a_{12}^2 + a_{34}^2 + a_{41}^2) \\
B_1 &= 4\, a_{12}^2\,(a_{34}^2 + a_{41}^2) \\
B_2 &= 8\, a_{12}^2\, a_{34}\, a_{41} \\
B_3 &= 4\, a_{12}\, a_{41}\,(a_{41}^2 + a_{12}^2 + a_{34}^2) \\
B_4 &= 4\, a_{12}\, a_{34}\,(a_{34}^2 + a_{12}^2 + 3\, a_{41}^2) \\
B_5 &= 8\, a_{12}\, a_{41}\, a_{34}^2 \\
B_6 &= 4\, a_{34}^2\,(a_{12}^2 + a_{41}^2) \\
B_7 &= 4\, a_{34}\, a_{41}\,(a_{41}^2 + a_{34}^2 + a_{12}^2) \\
B_8 &= (a_{41}^2 + a_{34}^2 - 2\, a_{12}\, a_{34} + a_{12}^2)\,(a_{41}^2 + a_{34}^2 + 2\, a_{12}\, a_{34} + a_{12}^2) \ .
\end{aligned} \quad (53)$$

Equation (50) expresses L_3 as a function of c_4 and c_1. A cosine law for triangle 4-1-2 may be written as

$$\frac{a_{12}^2}{2} + \frac{a_{41}^2}{2} + a_{12} a_{41} c_1 = \frac{L_1^2}{2} \ . \quad (54)$$

A cosine law for the triangle 3-4-1 may be written as

$$\frac{a_{34}^2}{2} + \frac{a_{41}^2}{2} + a_{34} a_{41} c_4 = \frac{L_2^2}{2} \ . \quad (55)$$

Equations (50), (54), and (55) define the three spring lengths in terms of the variables $\cos\theta_4$ and $\cos\theta_1$. From these three equations, each of the three spring lengths L_1, L_2, and L_3 can be determined for a given set of parameters $\cos\theta_4$ and $\cos\theta_1$. Lastly, equation (50) can be regrouped into the format

$$(r_1 c_1 + r_2)\, c_4^2 + (r_3 c_1^2 + r_4 c_1 + r_5)\, c_4 + (r_6 c_1^2 + r_7 c_1 + r_8) = 0 \quad (56)$$

where

$$\begin{aligned}
r_1 &= 8\, a_{12}\, a_{41}\, a_{34}^2 \\
r_2 &= 4\, a_{34}^2\,(a_{12}^2 + a_{41}^2) \\
r_3 &= 8\, a_{34}\, a_{41}\, a_{12}^2 \\
r_4 &= (-4\, a_{12}\, a_{34})\, L_3^2 + 4\, a_{12}\, a_{34}\,(a_{34}^2 + a_{12}^2 + 3\, a_{41}^2) \\
r_5 &= (-4\, a_{34}\, a_{41})\, L_3^2 + 4\, a_{34}\, a_{41}\,(a_{34}^2 + a_{12}^2 + a_{41}^2) \\
r_6 &= 4\, a_{12}^2\,(a_{34}^2 + a_{41}^2) \\
r_7 &= (-4\, a_{12}\, a_{41})\, L_3^2 + 4\, a_{12}\, a_{41}\,(a_{41}^2 + a_{34}^2 + a_{12}^2) \\
r_8 &= L_3^4 - 2\,(a_{34}^2 + a_{12}^2 + a_{41}^2)\, L_3^2 \ .
\end{aligned} \quad (57)$$

The total potential energy stored in all three springs is given by,

$$U = ½\, k_1\, (L_1 - L_{01})^2 + ½\, k_2\, (L_2 - L_{02})^2 + ½\, k_3\, (L_3 - L_{03})^2 \ . \quad (58)$$

Differentiating the potential energy with respect to c_4 and c_1 and then evaluating values for c_4 and c_1 that cause the derivative of the potential energy to equal zero, will identify configurations of either minimum or maximum potential energy. These derivatives may be written as,

$$dU/dc_4 = k_1\, (L_1 - L_{01})\, dL_1/dc_4 + k_2\, (L_2 - L_{02})\, dL_2/dc_4 + k_3\, (L_3 - L_{03})\, dL_3/dc_4 \quad (59)$$

$$dU/dc_1 = k_1\, (L_1 - L_{01})\, dL_1/dc_1 + k_2\, (L_2 - L_{02})\, dL_2/dc_1 + k_3\, (L_3 - L_{03})\, dL_3/dc_1 \quad (60)$$

Since from (54), L_1 is not a function of c_4, $dL_1/dc_4 = 0$. Similarly, from (55), L_2 is not a function of c_1 and thus $dL_2/dc_1 = 0$. Equations (59) and (60) reduce to

$$dU/dc_4 = k_2\, (L_2 - L_{02})\, dL_2/dc_4 + k_3\, (L_3 - L_{03})\, dL_3/dc_4 \ , \quad (61)$$

$$dU/dc_1 = k_1\, (L_1 - L_{01})\, dL_1/dc_1 + k_3\, (L_3 - L_{03})\, dL_3/dc_1 \ . \quad (62)$$

The term dL_2/dc_4 is evaluated from (55) as

$$\frac{dL_2}{dc_4} = \frac{a_{34} a_{41}}{L_2} \quad (63)$$

and the term dL_1/dc_1 is evaluated from (54) as

$$\frac{dL_1}{dc_1} = \frac{a_{12} a_{41}}{L_1} \quad (64)$$

Implicit differentiation of (50) for L_3 with respect to c_4 and c_1 yields

$$\frac{dL_3}{dc_4} = -\frac{(A_1 + A_3 c_1)\, L_3^2 + B_2 c_1^2 + B_4 c_1 + 2 B_5 c_1 c_4 + 2 B_6 c_4 + B_7}{2 L_3 (2 L_3^2 + A_1 c_4 + A_2 c_1 + A_3 c_1 c_4 + A_4)} \quad (65)$$

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$$\frac{dL_3}{dc_1} = -\frac{(A_2 + A_3 c_4) L_3^2 + 2 B_1 c_1 + 2 B_2 c_1 c_4 + B_3 + B_4 c_4 + B_5 c_4^2}{2 L_3 (2 L_3^2 + A_1 c_4 + A_2 c_1 + A_3 c_1 c_4 + A_4)} \quad .(66)$$

Substituting (63) and (65) into (61) and (64) and (66) into (62) and equating to zero yields

$$(M_1 L_3^3 + M_2 L_3^2 + M_3 L_3 + M_4) L_2 + M_5 L_3^3 + M_6 L_3 = 0 \quad (67)$$

$$(N_1 L_3^3 + N_2 L_3^2 + N_3 L_3 + N_4) L_1 + N_5 L_3^3 + N_6 L_3 = 0 \quad (68)$$

where the coefficients M_i and N_i, i=1..6, are functions of c_4 and c_1 as

$M_1 = -k_3 (A_3 c_1 + A_1) + 4 k_2 a_{34} a_{41}$
$M_2 = k_3 L_{03} (A_3 c_1 + A_1)$
$M_3 = -k_3 B_2 c_1^2 + [2 (k_2 a_{34} a_{41} A_3 – k_3 B_5) c_4 + 2 k_2 a_{34} a_{41} A_2 – k_3 B_4) c_1 + 2 (k_2 a_{34} a_{41} A_1 – k_3 B_6) c_4 – k_3 B_7 + 2 k_2 a_{34} a_{41} A_4$
$M_4 = k_3 L_{03} B_2 c_1^2 + k_3 L_{03} B_4 c_1 + 2 k_3 L_{03} B_5 c_1 c_4 + 2 k_3 L_{03} B_6 c_4 + k_3 L_{03} B_7$
$M_5 = -4 k_2 a_{34} a_{41} L_{02}$
$M_6 = -2 k_2 (a_{34} a_{41} L_{02} A_2 c_1 + a_{34} a_{41} L_{02} A_3 c_1 c_4 + a_{34} a_{41} L_{02} A_1 c_4 + a_{34} a_{41} L_{02} A_4)$ (69)

$N_1 = - k_3 (A_3 c_4 + A_2) + 4 k_1 a_{12} a_{41}$
$N_2 = k_3 L_{03} (A_3 c_4 + A_2)$
$N_3 = 2 k_1 a_{12} a_{41} (A_1 c_4 + A_2 c_1 + A_3 c_1 c_4 + A_4) – k_3 (B_4 c_4 + B_5 c_4^2 + 2 B_1 c_1 + 2 B_2 c_1 c_4 + B_3)$
$N_4 = k_3 L_{03} (B_4 c_4 + B_5 c_4^2 + 2 B_1 c_1 + B_2 c_1 c_4 + B_3)$
$N_5 = - 4 k_1 a_{12} a_{41} L_{01}$
$N_6 = -2 k_1 a_{12} a_{41} L_{01} (A_4 + A_1 c_4 + A_2 c_1 + A_3 c_1 c_4)$. (70)

Equations (67) and (68) may be written as

$$(M_1 L_3^3 + M_2 L_3^2 + M_3 L_3 + M_4) L_2 = - M_5 L_3^3 – M_6 L_3 , \quad (71)$$

$$(N_1 L_3^3 + N_2 L_3^2 + N_3 L_3 + N_4) L_1 = - N_5 L_3^3 – N_6 L_3 . \quad (72)$$

Squaring both sides of both equations gives

$$(M_1 L_3^3 + M_2 L_3^2 + M_3 L_3 + M_4)^2 L_2^2 = (M_5 L_3^3 + M_6 L_3)^2 , \quad (73)$$

$$(N_1 L_3^3 + N_2 L_3^2 + N_3 L_3 + N_4)^2 L_1^2 = (N_5 L_3^3 + N_6 L_3)^2 . \quad (74)$$

Using (54) and (55) to substitute for L_1^2 and L_2^2 will yield two equations in the parameters c_1, c_4, and L_3. These two equations can be arranged as

$$(p_1 c_1^2 + p_2 c_1 + p_3) c_4^3 + (p_4 c_1^3 + p_5 c_1^2 + p_6 c_1 + p_7) c_4^2 + (p_8 c_1^4 + p_9 c_1^3 + p_{10} c_1^2 + p_{11} c_1 + p_{12}) c_4 + (p_{13} c_1^4 + p_{14} c_1^3 + p_{15} c_1^2 + p_{16} c_1 + p_{17}) = 0 \quad (75)$$

$$(q_1 c_1 + q_2) c_4^4 + (q_3 c_1^2 + q_4 c_1 + q_5) c_4^3 + (q_6 c_1^3 + q_7 c_1^2 + q_8 c_1 + q_9) c_4^2 + (q_{10} c_1^3 + q_{11} c_1^2 + q_{12} c_1 + q_{13}) c_4 + (q_{14} c_1^3 + q_{15} c_1^2 + q_{16} c_1 + q_{17}) = 0 \quad (76)$$

where

$p_1 = p_{1a} L_3^2 + p_{1b} L_3 + p_{1c}$
$p_2 = p_{2a} L_3^2 + p_{2b} L_3 + p_{2c}$
$p_3 = p_{3a} L_3^2 + p_{3b} L_3 + p_{3c}$
$p_4 = p_{4a} L_3^2 + p_{4b} L_3 + p_{4c}$
$p_5 = p_{5a} L_3^4 + p_{5b} L_3^3 + p_{5c} L_3^2 + p_{5d} L_3 + p_{5e}$
$p_6 = p_{6a} L_3^4 + p_{6b} L_3^3 + p_{6c} L_3^2 + p_{6d} L_3 + p_{6e}$
$p_7 = p_{7a} L_3^4 + p_{7b} L_3^3 + p_{7c} L_3^2 + p_{7d} L_3 + p_{7e}$
$p_8 = p_{8a} L_3^2 + p_{8b} L_3 + p_{8c}$
$p_9 = p_{9a} L_3^4 + p_{9b} L_3^3 + p_{9c} L_3^2 + p_{9d} L_3 + p_{9e}$
$p_{10} = p_{10a} L_3^6 + p_{10b} L_3^5 + p_{10c} L_3^4 + p_{10d} L_3^3 + p_{10e} L_3^2 + p_{10f} L_3 + p_{10g}$
$p_{11} = p_{11a} L_3^6 + p_{11b} L_3^5 + p_{11c} L_3^4 + p_{11d} L_3^3 + p_{11e} L_3^2 + p_{11f} L_3 + p_{11g}$
$p_{12} = p_{12a} L_3^6 + p_{12b} L_3^5 + p_{12c} L_3^4 + p_{12d} L_3^3 + p_{12e} L_3^2 + p_{12f} L_3 + p_{12g}$
$p_{13} = p_{13a} L_3^2 + p_{13b} L_3 + p_{13c}$
$p_{14} = p_{14a} L_3^4 + p_{14b} L_3^3 + p_{14c} L_3^2 + p_{14d} L_3 + p_{14e}$
$p_{15} = p_{15a} L_3^6 + p_{15b} L_3^5 + p_{15c} L_3^4 + p_{15d} L_3^3 + p_{15e} L_3^2 + p_{15f} L_3 + p_{15g}$
$p_{16} = p_{16a} L_3^6 + p_{16b} L_3^5 + p_{16c} L_3^4 + p_{16d} L_3^3 + p_{16e} L_3^2 + p_{16f} L_3 + p_{16g}$
$p_{17} = p_{17a} L_3^6 + p_{17b} L_3^5 + p_{17c} L_3^4 + p_{17d} L_3^3 + p_{17e} L_3^2 + p_{17f} L_3 + p_{17g}$ (77)

$q_1 = q_{1a} L_3^2 + q_{1b} L_3 + q_{1c}$
$q_2 = q_{2a} L_3^2 + q_{2b} L_3 + q_{2c}$
$q_3 = q_{3a} L_3^2 + q_{3b} L_3 + q_{3c}$
$q_4 = q_{4a} L_3^4 + q_{4b} L_3^3 + q_{4c} L_3^2 + q_{4d} L_3 + q_{4e}$
$q_5 = q_{5a} L_3^4 + q_{5b} L_3^3 + q_{5c} L_3^2 + q_{5d} L_3 + q_{5e}$
$q_6 = q_{6a} L_3^2 + q_{6b} L_3 + q_{6c}$
$q_7 = q_{7a} L_3^4 + q_{7b} L_3^3 + q_{7c} L_3^2 + q_{7d} L_3 + q_{7e}$
$q_8 = q_{8a} L_3^6 + q_{8b} L_3^5 + q_{8c} L_3^4 + q_{8d} L_3^3 + q_{8e} L_3^2 + q_{8f} L_3 + q_{8g}$
$q_9 = q_{9a} L_3^6 + q_{9b} L_3^5 + q_{9c} L_3^4 + q_{9d} L_3^3 + q_{9e} L_3^2 + q_{9f} L_3 + q_{9g}$
$q_{10} = q_{10a} L_3^2 + q_{10b} L_3 + q_{10c}$
$q_{11} = q_{11a} L_3^4 + q_{11b} L_3^3 + q_{11c} L_3^2 + q_{11d} L_3 + q_{11e}$
$q_{12} = q_{12a} L_3^6 + q_{12b} L_3^5 + q_{12c} L_3^4 + q_{12d} L_3^3 + q_{12e} L_3^2 + q_{12f} L_3 + q_{12g}$
$q_{13} = q_{13a} L_3^6 + q_{13b} L_3^5 + q_{13c} L_3^4 + q_{13d} L_3^3 + q_{13e} L_3^2 + q_{13f} L_3 + q_{13g}$
$q_{14} = q_{14a} L_3^2 + q_{14b} L_3 + q_{14c}$
$q_{15} = q_{15a} L_3^4 + q_{15b} L_3^3 + q_{15c} L_3^2 + q_{15d} L_3 + q_{15e}$
$q_{16} = q_{16a} L_3^6 + q_{16b} L_3^5 + q_{16c} L_3^4 + q_{16d} L_3^3 + q_{16e} L_3^2 + q_{16f} L_3 + q_{16g}$
$q_{17} = q_{17a} L_3^6 + q_{17b} L_3^5 + q_{17c} L_3^4 + q_{17d} L_3^3 + q_{17e} L_3^2 + q_{17f} L_3 + q_{17g}$. (78)

The coefficients p_{1a} through q_{17g} have been expressed symbolically in terms of the given mechanism parameters and as such can be considered as known quantities. The coefficients are not listed here due to the length of the expressions.

3.2.2 Solution of Three Simultaneous Equations in Three Unknowns

Equations (75), (76), and (56) are factored such that the parameter L_3 is embedded in the coefficients p_1 through r_8. Sylvester's method was used in an attempt to solve these equations for all possible sets of values of L_3, c_1, and c_4. Equation (75) was multiplied by c_1, c_1^2, c_4, c_4^2, c_4^3, c_1^4, $c_{12}c_4$, $c_1c_4^2$, and $c_1^2c_4^2$ to obtain 10 equations (including itself).

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Equation (76) was multiplied by c_1, c_1^2, c_1^3, c_4, c_4^2, c_1c_4, $c_1^2c_4$, $c_1^3c_4$, $c_1c_4^2$ and $c_1^2c_4^2$ to obtain 11 equations (including itself). Equation (56) was multiplied by c_1, c_1^2, c_1^3, c_1^4, c_4, c_4^2, c_4^3, c_4^4, c_1c_4, $c_1^2c_4$, $c_1^3c_4$, $c_1^4c_4$, $c_1c_4^2$, $c_1^2c_4^2$, $c_1^3c_4^2$, $c_1^4c_4^2$, $c_1c_4^3$, $c_1^2c_4^3$, $c_1^3c_1^4$, $c_1c_4^4$ and $c_1^2c_4^4$ to obtain 22 equations (including itself). This resulted in a set of 43 "homogeneous" equations in 43 unknowns.

The condition for a solution to this set of equations was that they are linearly dependent, i.e. the determinant of the 43×43 coefficient matrix must equal zero. This would yield a polynomial in the single variable L_3. Due to the complexity of the problem, it was not possible to expand this determinant symbolically. Similarly, it was not possible to root the polynomial that resulted from a particular numerical example. Because of this, the Continuation Method was used again to determine solutions for the same numerical example presented in Section 3.1.4. The continuation method was run on this set of three equations in three unknowns to obtain all solution sets for the three variables, L_3, c_1, and c_4, for the particular numerical example. The software PHC pack (Verschelde [26]) was again used to implement the method.

3.2.3 Numerical Example

The same input parameters used in the numerical example in Section 3.1.4 were used here. The coefficients of equations (75), (76), and (56) were numerically determined and the PHCpack software, which utilizes the continuation method, estimated the number of possible solutions to be 136. A total of 7 real solutions and 129 complex solutions were obtained. The seven real solutions are shown in Table 3. Of these, case number one is the only real solution case that corresponds to a minimum potential energy configuration.

The lengths of the springs L_1 and L_2 were calculated for each of the seven cases and it was determined that the seven real solutions presented in Table 3 correspond with the seven real solutions that were determined from the first problem formulation and listed in Table 2. It is an important verification that that the equilibrium configurations determined from the two solution approaches are identical.

Table 3: Real solutions;
cosine of θ_1, θ_4 in radians and L_3 in inches.

Case	$\cos\theta_1$	$\cos\theta_4$	L_3
1	-0.45357	-0.74999	7.0166
2	-0.59496	-0.56850	-5.3333
3	-0.85169	-0.67187	10.1050
4	-0.62272	-0.35968	-3.0440
5	-0.36020	-0.87651	9.1639
6	-0.33212	-0.65064	-4.7738
7	-.043667	-0.45612	-2.4876

4. FOUR-SPRING SYSTEM

4.1 Problem Formulation

A four-spring planar tensegrity system is shown in Figure 6. The problem statement can be explicitly written as:

Given:

- a_{12}, a_{34} lengths of struts,
- k_1, L_{01} spring constant and its free length between points 4 and 2,
- k_2, L_{02} spring constant and its free length between points 3 and 1,
- k_3, L_{03} spring constant and its free length between points 3 and 2,
- k_4, L_{04} spring constant and its free length between points 4 and 1.

Find at equilibrium position:

- L_1 length of spring 1,
- L_2 length of spring 2,
- L_3 length of spring 3,
- L_4 length of spring 4.

It should be noted that as before, the problem statement could be formulated in a variety of ways. The solution presented here uses L_1, L_2, and L_3 as the three generalized parameters, knowledge of which completely describes the system (i.e. the value for L_4, the last remaining parameter, can be deduced). The geometry equation that relates L_4 in terms of the three generalized parameters is derived first. Following this, a derivative of the potential energy equation is taken with respect to each of the three generalized parameters to yield an additional three equations.

The equation that relates the lengths of the four springs can be obtained directly by substituting the variable L_4 for the length a_{41} in equation (25). The potential energy equation can then be written as

$$U = \tfrac{1}{2}\, k_1 (L_1-L_{01})^2 + \tfrac{1}{2}\, k_2 (L_2-L_{02})^2 + \tfrac{1}{2}\, k_3 (L_3-L_{03})^2 + \tfrac{1}{2}\, k_4 (L_4-L_{04})^2 \quad . \qquad (79)$$

Taking a derivative of the potential energy with respect to L_1, L_2, and L_3 and equating to zero will yield three additional equations that must be satisfied at equilibrium. For brevity, these equations are not presented here, however at this point four equations now exist in the parameters L_1, L_2, L_3, and L_4. It

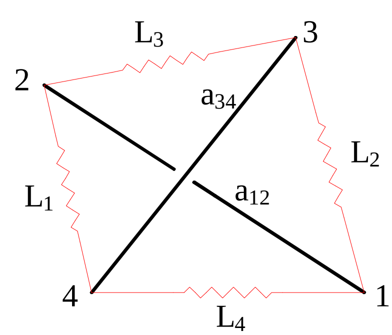

Figure 6: Two Strut, Four-Spring Planar Tensegrity Device

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was not practical to attempt to use Sylvester's method to solve this set of equations. The continuation method, however, was used to solve numerical cases.

4.2 Numerical Example

The following parameters were selected as a numerical example:

Strut lengths: a_{12} = 14 in., a_{34} = 12 in.

Spring free lengths and spring constants:

L_{01} = 8 in.	k_1 = 1 lbf/in.
L_{02} = 2.6866 in.	k_2 = 2.0 lbf/in.
L_{03} = 3.46513 in.	k_3 = 2.5 lbf/in.
L_{04} = 7.3083 in.	k_4 = 1.5 lbf/in.

The geometry equation and the three potential derivatives of the potential energy equation can now be expressed in terms of the given quantities as

$$\begin{aligned} & L_2^4 L_1^2 + (L_1^4 - L_3^2 L_1^2 - 440.0\, L_1^2 + 96.0\, L_3^2 \\ & - 13824.0)\, L_2^2 - 8424.0\, L_1^2 - 8624.0\, L_1^2 + 44.0\, L_3^2 L_1^2 \\ & + 6773760.0 - 52224.0\, L_3^2 + L_3^4 = 0 \end{aligned} \quad (80)$$

$$\begin{aligned} & -4.0376\, L_2^4 L_1 + (-1.1520\ 10^4 - 5.9624\, L_3^2 L_1 \\ & + 80.0\, L_1^2 - 18.0751\, L_1^3 + 3216.5300\, L_1 \\ & + 80.0\, L_3^2)\, L_2^2 + 11200.0\, L_3^2 + 1960.0\, L_1^3 - 10\, L_3^2 L_1^3 \\ & - 1577.6530\, L_3^2 L_1 + 10.0\, L_1 L_3^4 - 80.0\, L_3^4 + 0.2257\ 10^7 \\ & - 15680.0\, L_1^2 - 247420.0110\, L_1 + 80.\, L_3^2 L_1^2 = 0 \end{aligned} \quad (81)$$

$$\begin{aligned} & (-28.0751\, L_1^2 + 2880.0 - 20.0\, L_3^2)\, L_2^3 + (-7737.3862 \\ & + 53.7318\, L_3^2 + 53.7318\, L_1^2)\, L_2^2 + (-3187.6065\, L_3^2 \\ & + 5696.5300\, L_1^2 - 15.9624\, L_3^2 L_1^2 + 20.0\, L_3^4 \\ & - 4.0375\, L_1^4 - 508664.6558)\, L_2 + 7522.4588\, L_3^2 \\ & + 0.1517\ 10^7 - 53.7318\, L_3^4 - 10531.4424\, L_1^2 \\ & + 53.7318\, L_3^2 L_1^2 = 0 \end{aligned} \quad (82)$$

$$\begin{aligned} & -494742.0331\, L_3 + 12127.9787\, L_3^2 - 16979.1702\, L_1^2 \\ & + 86.6284\, L_3^2 L_1^2 - 86.6284\, L_3^4 - 4307.5137\, L_3^3 \\ & + 4722.3470\, L_3 L_1^2 + 25.0\, L_3^5 - 25.0\, L_3^3 L_1^2 \\ & + (-20.9624\, L_3 L_1^2 + 86.6284\, L_1^2 + 3212.3934\, L_3 \\ & - 25.0\, L_3^3 - 12474.4924 + 86.6284\, L_3^2)\, L_2^2 \\ & + 0.24450\ 10^7 = 0 \end{aligned} \quad (83)$$

The Polynomial Continuation Method was used to find simultaneous roots for the four equations. Eighteen real solutions and 490 complex solutions were found. All the real and complex solutions were shown to satisfy Equations (80) through (83) with negligible residuals.

Nine of the eighteen real solutions were not physically realizable. For example, L_1 in one of the solutions was greater than the sum of a_{12} and L_4, and hence point 2 cannot be constructed. The remaining real solutions can correspond to either maximum or minimum potential energy configurations. The second derivative of the potential energy with respect to the variables L_1, L_2, and L_3 was evaluated to identity the minimal potential energy cases. Each of these solutions was then analyzed by calculating forces in the struts and elastic ties to

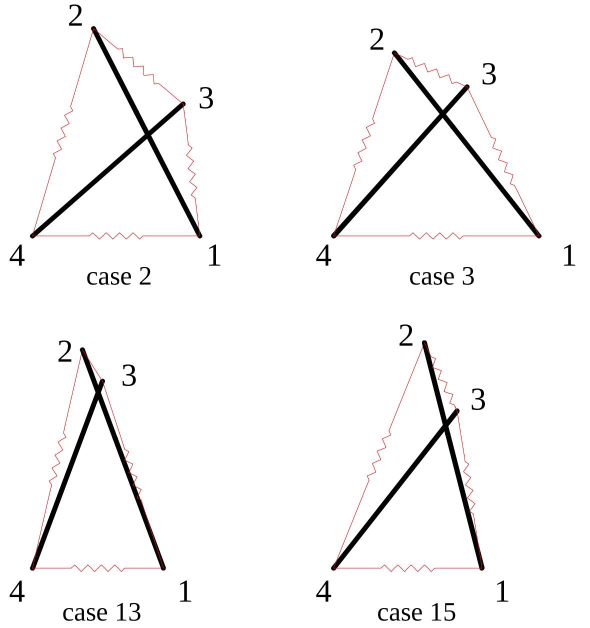

Figure 7: Equilibrium Configurations for Four-Spring Tensegrity Device

determine if it truly corresponds to an equilibrium configuration. This analysis indicates that only four of the real solutions were indeed in equilibrium. The four equilibrium configurations are shown in Figure 7.

5. CONCLUSIONS

The two-spring planar tensegrity system represents the simplest tensegrity device. It was remarkable that such a simple device would have such a complicated solution, i.e. a 28th degree polynomial was obtained in the variable L_1. It is true that conditions corresponding to maximum as well as minimum potential energy states are obtained in this formulation. The coefficients of this polynomial were obtained symbolically.

Two solution approaches were presented for the three-spring system. The complexity of both cases was such that they could not be solved symbolically. Further, an attempt to solve a numerical example using Sylvester's solution method was not successful. The numerical example was solved using the continuation method. However, of the seven real solutions for the three spring lengths, only one solution was found that was in equilibrium.

Lastly, a four-spring planar tensegrity system was analyzed. The continuation method estimate of the solution was degree 508. Four real equilibrium solutions for a specific numerical example were obtained.

The surprising result to the authors was the degree of complexity of the solutions that were obtained for two-, three-, and four-spring planar tensegrity systems. It was not apparent at the outset that such simple planar structures would have such a high degree for the solution polynomial. Other solution

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approaches that would yield only minimum potential energy solutions should be addressed to see if a simpler result can be obtained.

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REFERENCES

[1] Duffy, J., Rooney, J., Knight, B., and Crane, C., (2000), A Review of a Familly of Self-Deploying Tensegrity Structures with Elastic Ties, *The Shock and Vibration Digest*, Vol. 32, No. 2, March 2000, pp. 100-106.

[2] Duffy, J., Crane, C. D., Correa, J. C., (2002), "Static Analysis of Tensegrity Structures PART 1. Equilibrium Equations," Proceedings of DETC'02 ASME 2002 Design Engineering Technical Conferences and Computers and Information in Engineering Conference Montreal, Canada, pp. 1-10.

[3] Duffy, J., Crane, C. D., Correa, J. C., (2002), "Static Analysis of Tensegrity Structures PART 2. Numerical Examples," Proceedings of DETC'02 ASME 2002 Design Engineering Technical Conferences and Computers and Information in Engineering Conference Montreal, Canada, pp. 1-10.

[4] Edmondson, A., (1987) *A Fuller Explanation: The Synergetic Geometry of R. Buckminster Fuller*, Birkhauser, Boston.

[5] Fest1, E., Shea, K., Domer, B., and Smith, I. "Adjustable Tensegrity Structure," journal of structural engineering, pp. 515-526 © ASCE , April 2003.

[6] Fuller, R., (1975), *Synergetics: The Geometry of Thinking*, MacMillan Publishing Co., Inc., New York.

[7] Garcia, C.B. and Li, T.Y., (1980), "On the Number Solutions to Polynomial Systems of Equations," SIAM J. Numer. Anal., Vol. 17, pp. 540-546.

[8] Kenner, H., (1976), *Geodesic Math and How to Use It*, University of California Press, Berkeley and Los Angeles, CA.

[9] Knight, B.F., (2000), Deployable Antenna Kinematics using Tensegrity Structure Design, *Ph.D. thesis*, University of Florida, Gainesville, FL.

[10] Morgan, A.P., (1983), "A Method for Computing All Solutions to Systems of Polynomial Equations," ACM Trans. Math. Software, Vol. 9, No. 1, pp 1-17.

[11] Morgan, A.P. (1986), "A Homotopy for Solving Polynomial Systems," Appl. Math. Comput., Vol. 18, pp. 87-92.

[12] Morgan, A.P., (1987), "Solving Polynomial Systems Using Continuation for Scientific and Engineering Problems", Prentice-Hall, Englewood Cliffs, NJ.

[13] Roth, B., Whiteley, W., "Tensegrity Framework," Transactions American Mathematics Society, Vol.265, pp. 419-446, 1981.

[14] Skelton, R. E., Williamson, D., and Han, J. H., "Equilibrium Conditions of Tensegrity Structure," Proceedings of the Third World Conference on Structural Control (3WCSC) Como, Italy, pp. 1-18, April 7-12, 2002.

[15] Stern, I.P., (1999), Development of Design Equations for Self-Deployable N-Strut Tensegrity Systems, M.S. thesis, University of Florida, Gainesville, FL.

[16] Tibert, G., "Deployable Tensegrity Structures for Space Applications," Ph.D. Thesis. Royal Institute of Technology, Stockholm 2002.

[17] Tobie, R.S., (1976), A Report on an Inquiry into The Existence, Formation and Representation of Tensile Structures, Master of Industrial Design thesis, Pratt Institute, New York.

[18] Tsai, L., (1999), "Robot Analysis; The Mechanics of Serial and Parallel Manipulators," John Wiley.

[19] Verschelde, J. (1999), "PHCpack: a General-Purpose Solver for Polynomial Systems by Homotopy Continuation," Algorithm 795 in ACM Trans. Math. Softw., http://www.math.uic.edu/~jan/PHCpack/phcpack.html .

[20] Wampler, C., Morgan, A., and Sommese, A., (1990), "Numerical Continuation Mehtods for Solving Polynomial Systems Arising in Kinematics," ASME J. Mech. Des., Vol. 112, pp. 59-68.

[21] Yin, J., Duffy, J., and Crane, C., (2002), An Analysis for the Design of Self-Deployable Tensegrity and Reinforced Tensegrity Prisms with Elastic Ties, International Journal of Robotics and Automation, Special Issue on Compliance and Compliant Mechanisms, Volume 17, Issue 1.

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