Proceedings of
A Symposium Commemorating the
Legacy, Works, and Life of
Sir Robert Stawell Ball
Upon the 100th Anniversary of
A Treatise on the Theory of Screws
July 9-11, 2000
University of Cambridge, Trinity College

An Initial Investigation into the Geometrical Meaning of the (Pseudo-) Inverses of the Line Matrices for the Edges of Platonic Polyhedra

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Abstract

It is well known that there are five regular (Platonic) polyhedra: the tetrahedron, the hexahedron (cube), the octahedron, the icosahedron and the dodecahedron. Each of these polyhedra has an associated dual polyhedron which is also Platonic. By considering the Platonic polyhedra to be constructed from lines, and then representing the lines in terms of both ray and axis coordinates, a further aspect of this duality is exposed. This is the duality of poles and polars associated with projective configurations of points, lines and planes.

This paper shows that a line matrix may be constructed for any regular polyhedron, in such a way that its columns represent the normalized *ray* coordinates of the edges of the polyhedron. The (pseudo-) inverse of this line matrix may then be constructed whose rows represent the normalized *axis* coordinates of the corresponding dual polyhedron.

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Introduction

It is well known that the dual of any of the Platonic polyhedra, namely, the tetrahedron, octahedron, hexahedron(cube), icosahedron and dodecahedron can be formed by joining the centers of adjacent faces, see for example Coxeter, 1969 & 1963 [1, 2]. The new dual solid constructed in this way is itself a Platonic polyhedron for which each vertex and face correspond respectively to a face and vertex of the original one. The regular tetrahedron is self-dual, and the cube and regular octahedron are dual to each other as are the icosahedron and regular dodecahedron.

In this paper, these dualities are obtained by constructing the inverse (or pseudo-inverse) of the line matrix whose columns are the ray coordinates of the lines along the edges of any given Platonic solid. This is essentially an extension of an examination, by Duffy, 1996 [3], of the geometric meaning of the inverse of the 3x3 line matrix whose columns are the coordinates of the lines along the sides of a plane triangle. By forming the line matrix whose three columns are the normalized ray coordinates and then constructing the inverse, it was shown in that reference that the three rows can be normalized to yield the axis coordinates of the three parallel lines of penetration through the vertices of the triangle and normal to its plane.

Briefly, the six homogeneous ray coordinates (L, M, N ; P, Q, R) for a line joining a pair of points (w_1, x_1, y_1, z_1) and (w_2, x_2, y_2, z_2) are given by the six 2x2 determinants of the array (see for example, Hunt [4]),

$$\begin{bmatrix} w_1 & x_1 & y_1 & z_1 \\ w_2 & x_2 & y_2 & z_2 \end{bmatrix} \tag{1}$$

which yield the three direction ratios of the line

$$L = | w \ \ x | \qquad M = | w \ \ y | \qquad N = | w \ \ z | \tag{2}$$

and the three moments about the three coordinate axes

$$P = | y \ \ z | \qquad Q = | z \ \ x | \qquad R = | x \ \ y | \tag{3}$$

where the abbreviations $|w \quad x| = \begin{vmatrix} w_1 & x_1 \\ w_2 & x_2 \end{vmatrix}$ and $|y \quad z| = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}$ have been introduced. The line is said to be normalized when $L^2 + M^2 + N^2 = 1$.

Analogously the six homogeneous axis coordinates (P, Q, R ; L, M, N) for a line formed by the meet of two planes (D_1, A_1, B_1, C_1) and (D_2, A_2, B_2, C_2) are given by the six 2x2 determinants of the array,

$$\begin{bmatrix} D_1 & A_1 & B_1 & C_1 \\ D_2 & A_2 & B_2 & C_2 \end{bmatrix} \tag{4}$$

which to a scalar multiple yield

$$P = | D \ \ A | \qquad Q = | D \ \ B | \qquad R = | D \ \ C | \tag{5}$$

and

$$L = | B \ \ C | \qquad M = | C \ \ A | \qquad N = | A \ \ B | \tag{6}$$

where the abbreviation $|D \quad A| = \begin{vmatrix} D_1 & A_1 \\ D_2 & A_2 \end{vmatrix}$ has been introduced.

Tetrahedron

A line can be considered as the join of two points *A* and *B* or dually as the meet of two planes *a* and *b*. Here the two sets of line coordinates are determined by the six 2x2 determinants of :

$$\begin{bmatrix} W_A & A_x & A_y & A_z \\ W_B & B_x & B_y & B_z \end{bmatrix} \qquad \text{(ray coordinates)} \qquad (7)$$

and:

$$\begin{bmatrix} w_a & a_x & a_y & a_z \\ w_b & b_x & b_y & b_z \end{bmatrix} \qquad \text{(axis coordinates)} \qquad (8)$$

The previous notation has now been changed to illustrate better the relationships of the vertices *A, B, C, D* to the faces *a, b, c, d* of the tetrahedron shown in Figure 1. The four points *A, B, C, D* and the four planes *a, b, c, d* forming the vertices and faces respectively of the tetrahedron (see Figure 1) define the lines which form the six edges of the tetrahedron. These lines may be specified in ray coordinates by using the coordinates

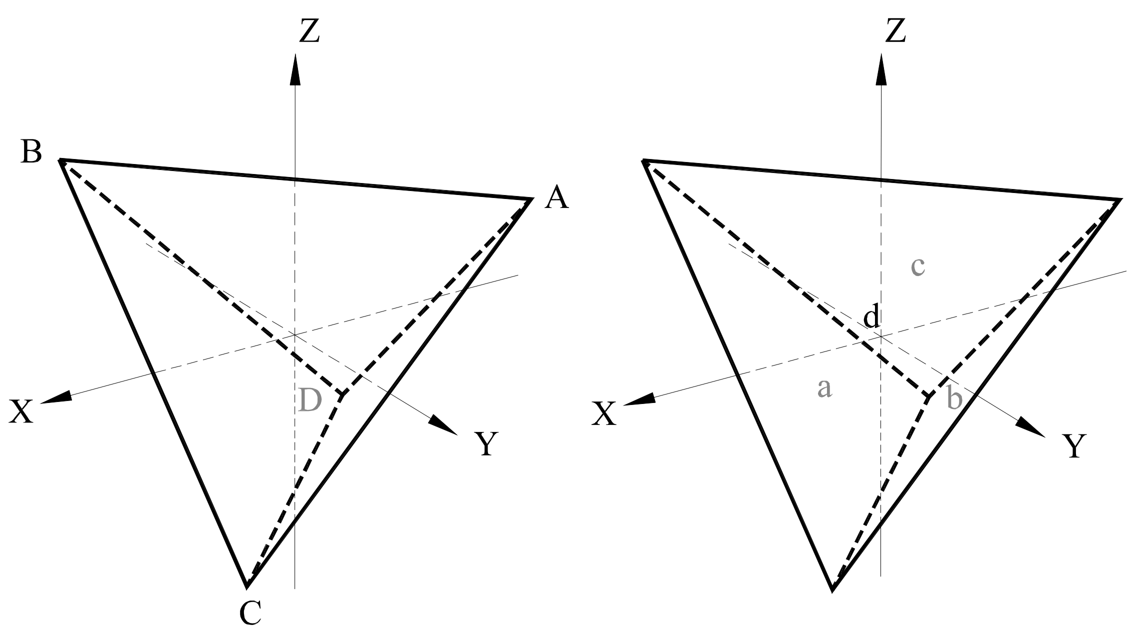

Figure 1 : Original and dual tetrahedrons

of pairs of vertex points, or dually in axis coordinates by using the coordinates of pairs of face planes as illustrated in the Figure. For example the edge joining the vertices *A* and *B* may be labeled *AB*, and, dually, the same edge is the meet of the two planes c and d and so may be labeled *cd*.

Assuming this tetrahedron is regular with edge length σ, the coordinates of its four vertices are:

$$A:\begin{pmatrix}1 & \frac{-\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}}\end{pmatrix} \qquad B:\begin{pmatrix}1 & \frac{\sigma}{2\sqrt{2}} & \frac{-\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}}\end{pmatrix}$$

$$C:\begin{pmatrix}1 & \frac{\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}} & \frac{-\sigma}{2\sqrt{2}}\end{pmatrix} \qquad D:\begin{pmatrix}1 & \frac{-\sigma}{2\sqrt{2}} & \frac{-\sigma}{2\sqrt{2}} & \frac{-\sigma}{2\sqrt{2}}\end{pmatrix} \tag{9}$$

and the coordinates of the four faces are:

$$a:\begin{pmatrix}\frac{\sigma}{2\sqrt{2}} & -1 & 1 & 1\end{pmatrix} \qquad b:\begin{pmatrix}\frac{\sigma}{2\sqrt{2}} & 1 & -1 & 1\end{pmatrix}$$

$$c:\begin{pmatrix}\frac{\sigma}{2\sqrt{2}} & 1 & 1 & -1\end{pmatrix} \qquad d:\begin{pmatrix}\frac{\sigma}{2\sqrt{2}} & -1 & -1 & -1\end{pmatrix} \tag{10}$$

The correlation matrix mapping the coordinates of the faces to the coordinates of the vertices is:

$$\begin{bmatrix}\frac{2\sqrt{2}}{\sigma} & 0 & 0 & 0 \\ 0 & \frac{\sigma}{2\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{\sigma}{2\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{\sigma}{2\sqrt{2}}\end{bmatrix} \tag{11}$$

This symmetric diagonal correlation matrix relates the poles (the four vertices) to the corresponding polars (the four faces) of the tetrahedron, which is self-polar (Coxeter, 1974 [5]).

The line matrix for the six lines forming the edges of the tetrahedron, the columns of which are the normalized ray coordinates of the lines, may be written in the form:

$$\begin{array}{cc} & \begin{array}{cccccc} DA & DB & DC & AB & BC & CA \end{array} \\ \boldsymbol{M}= & \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{\sigma}{4} & \frac{\sigma}{4} & \frac{\sigma}{4} & 0 & \frac{\sigma}{4} \\ \frac{\sigma}{4} & 0 & -\frac{\sigma}{4} & \frac{\sigma}{4} & \frac{\sigma}{4} & 0 \\ -\frac{\sigma}{4} & \frac{\sigma}{4} & 0 & 0 & \frac{\sigma}{4} & \frac{\sigma}{4} \end{bmatrix} \end{array} \tag{12}$$

The normalized inverse matrix of the line matrix is,

$$\boldsymbol{M}^{-1} = \begin{bmatrix} 0 & \frac{\sigma}{4} & \frac{\sigma}{4} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{\sigma}{4} & 0 & \frac{\sigma}{4} & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{\sigma}{4} & \frac{\sigma}{4} & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{\sigma}{4} & -\frac{\sigma}{4} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{\sigma}{4} & -\frac{\sigma}{4} & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{\sigma}{4} & 0 & \frac{\sigma}{4} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{matrix} da \\ db \\ dc \\ ab \\ bc \\ ca \end{matrix} \tag{13}$$

The geometrical meaning of $\boldsymbol{M}$ is obvious (the columns represent the normalized ray coordinates of the lines) but the geometrical meaning of $\boldsymbol{M}^{-1}$ is not immediately apparent. However forming $\boldsymbol{M}^{-1}$ and unitizing each of the six rows yields the six lines of the polar tetrahedron (expressed in their axis coordinates). It is clear from Figure 1 that the original tetrahedron and this polar tetrahedron are super-imposable and hence are self-polar. For example, it may be observed in Figure 1 that the line joining vertices A and B in ray coordinates is identical to the line formed by the meet of the planes c and d in axis coordinates. Ray and axis coordinates may be converted one into the other by interchanging the first three components with the last three components.

Octahedron and hexahedron (cube)

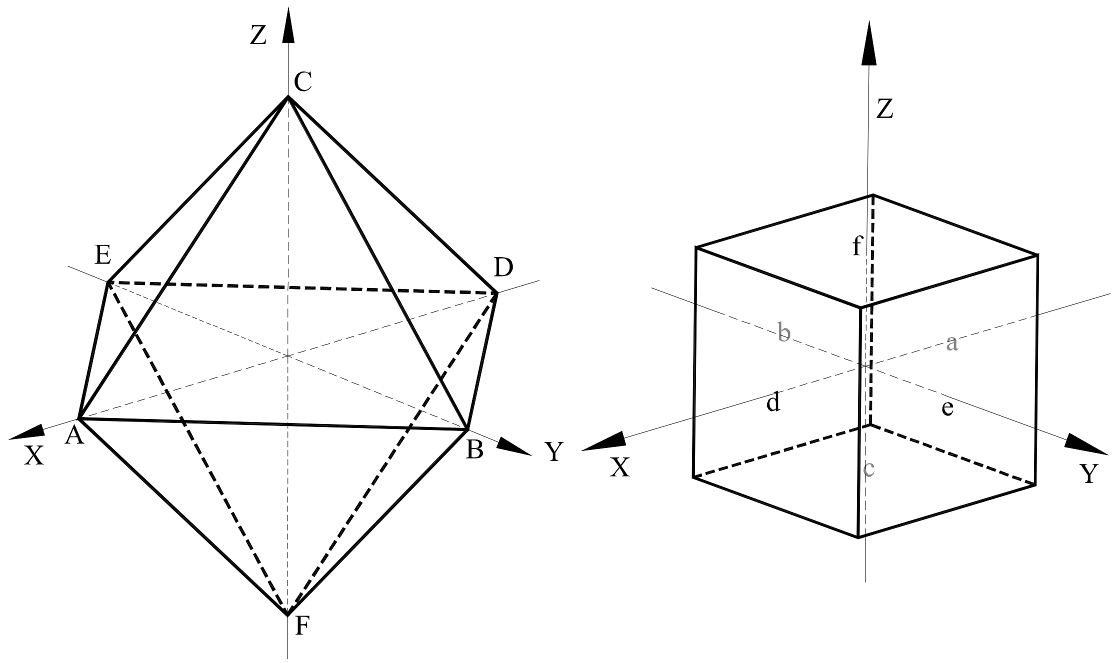

Figure 2 : A regular octahedron and the dual cube

Figure 2 illustrates an octahedron and a cube. Assuming this octahedron is regular with edge length σ, the coordinates of its six vertices are

$$A:\begin{pmatrix}1 & \frac{\sigma}{\sqrt{2}} & 0 & 0\end{pmatrix} \qquad B:\begin{pmatrix}1 & 0 & \frac{\sigma}{\sqrt{2}} & 0\end{pmatrix}$$

$$C:\begin{pmatrix}1 & 0 & 0 & \frac{\sigma}{\sqrt{2}}\end{pmatrix} \qquad D:\begin{pmatrix}1 & -\frac{\sigma}{\sqrt{2}} & 0 & 0\end{pmatrix}$$

$$E:\begin{pmatrix}1 & 0 & -\frac{\sigma}{\sqrt{2}} & 0\end{pmatrix} \qquad F:\begin{pmatrix}1 & 0 & 0 & -\frac{\sigma}{\sqrt{2}}\end{pmatrix} \tag{14}$$

and the coordinates of the six faces of the cube are

$$a:\begin{pmatrix} \frac{\sigma}{2\sqrt{2}} & 1 & 0 & 0 \end{pmatrix} \qquad b:\begin{pmatrix} \frac{\sigma}{2\sqrt{2}} & 0 & 1 & 0 \end{pmatrix}$$

$$c:\begin{pmatrix} \frac{\sigma}{2\sqrt{2}} & 0 & 0 & 1 \end{pmatrix} \qquad d:\begin{pmatrix} \frac{\sigma}{2\sqrt{2}} & -1 & 0 & 0 \end{pmatrix}$$

$$e:\begin{pmatrix} \frac{\sigma}{2\sqrt{2}} & 0 & -1 & 0 \end{pmatrix} \qquad f:\begin{pmatrix} \frac{\sigma}{2\sqrt{2}} & 0 & 0 & -1 \end{pmatrix} \tag{15}$$

The correlation matrix mapping the coordinates of the faces to the coordinates of the vertices is:

$$\begin{bmatrix} \frac{2\sqrt{2}}{\sigma} & 0 & 0 & 0 \\ 0 & \frac{\sigma}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{\sigma}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{\sigma}{\sqrt{2}} \end{bmatrix} \tag{16}$$

The 6x12 normalized line matrix of the twelve lines in ray coordinates of the octahedron is,

$$\boldsymbol{M} = \begin{array}{c} \begin{matrix} CA & DC & FD & AF & AB & BD & DE & EA & BC & CE & EF & FB \end{matrix} \\ \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sigma}{2} & \frac{\sigma}{2} & \frac{\sigma}{2} & \frac{\sigma}{2} \\ \frac{\sigma}{2} & \frac{\sigma}{2} & \frac{\sigma}{2} & \frac{\sigma}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\sigma}{2} & \frac{\sigma}{2} & \frac{\sigma}{2} & \frac{\sigma}{2} & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \tag{17}$$

Because this line matrix is not a square matrix, it is not possible to determine its inverse matrix directly. However, a pseudo inverse defined by $\boldsymbol{M}^T\left(\boldsymbol{M}\boldsymbol{M}^T\right)^{-1}$ can apply to determine the inverse of the line matrix.

The normalized pseudo inverse matrix of (17) is,

$$\boldsymbol{M}^{-1}=\begin{bmatrix} \frac{\sigma}{2\sqrt{2}} & 0 & -\frac{\sigma}{2\sqrt{2}} & 0 & 1 & 0 \\ \frac{\sigma}{2\sqrt{2}} & 0 & \frac{\sigma}{2\sqrt{2}} & 0 & 1 & 0 \\ -\frac{\sigma}{2\sqrt{2}} & 0 & \frac{\sigma}{2\sqrt{2}} & 0 & 1 & 0 \\ -\frac{\sigma}{2\sqrt{2}} & 0 & -\frac{\sigma}{2\sqrt{2}} & 0 & 1 & 0 \\ -\frac{\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}} & 0 & 0 & 0 & 1 \\ -\frac{\sigma}{2\sqrt{2}} & -\frac{\sigma}{2\sqrt{2}} & 0 & 0 & 0 & 1 \\ \frac{\sigma}{2\sqrt{2}} & -\frac{\sigma}{2\sqrt{2}} & 0 & 0 & 0 & 1 \\ \frac{\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}} & 0 & 0 & 0 & 1 \\ 0 & -\frac{\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}} & 1 & 0 & 0 \\ 0 & -\frac{\sigma}{2\sqrt{2}} & -\frac{\sigma}{2\sqrt{2}} & 1 & 0 & 0 \\ 0 & \frac{\sigma}{2\sqrt{2}} & -\frac{\sigma}{2\sqrt{2}} & 1 & 0 & 0 \\ 0 & \frac{\sigma}{2\sqrt{2}} & \frac{\sigma}{2\sqrt{2}} & 1 & 0 & 0 \end{bmatrix}\begin{matrix} ca \\ dc \\ fd \\ af \\ fb \\ bd \\ de \\ ea \\ bc \\ ce \\ ef \\ fb \end{matrix} \qquad (18)$$

The rows of the inverse matrix represent axis line coordinates of the twelve edges each determined by two adjacent faces of the cube. It is also important to note that the reverse procedure produces a corresponding result. Hence a cube ends up as the dual octahedron.

Icosahedron and dodecahedron

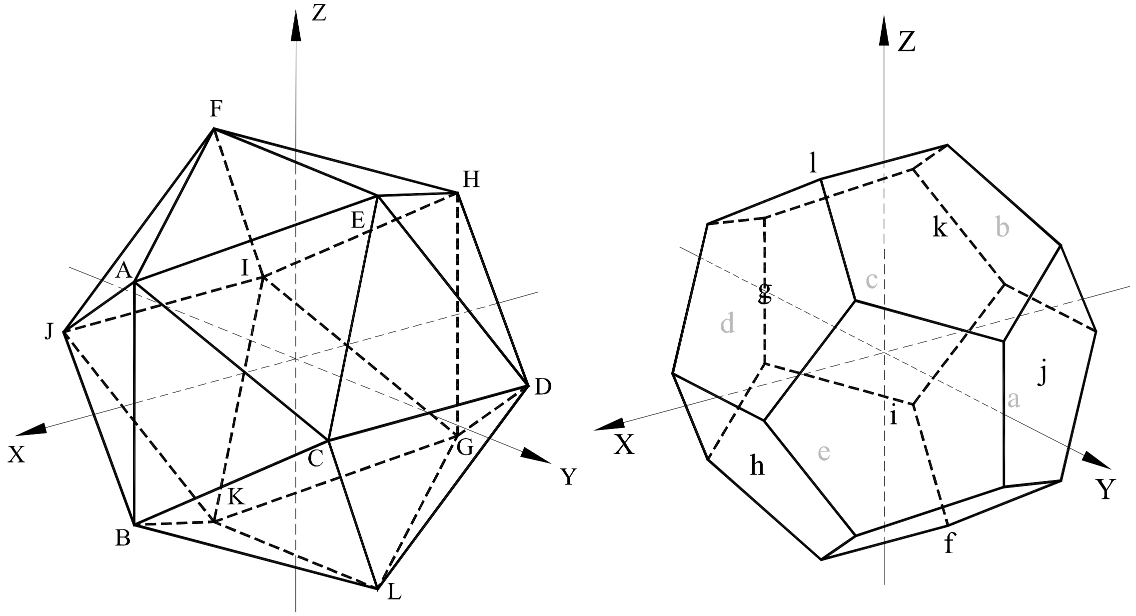

Figure 3 : A regular icosahedron and the dual dodecahedron

Figure 3 illustrates an icosahedron and its dual dodecahedron. Assuming this icosahedron is regular with edge length σ, the coordinates of its twelve vertices are

$$A: \begin{pmatrix} 1 & \left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} & 0 & \frac{\sigma}{2} \end{pmatrix} \qquad B: \begin{pmatrix} 1 & \left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} & 0 & -\frac{\sigma}{2} \end{pmatrix}$$

$$C: \begin{pmatrix} 1 & \frac{\sigma}{2} & \left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} & 0 \end{pmatrix} \qquad D: \begin{pmatrix} 1 & -\frac{\sigma}{2} & \left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} & 0 \end{pmatrix}$$

$$E: \begin{pmatrix} 1 & 0 & \frac{\sigma}{2} & \left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \end{pmatrix} \qquad F: \begin{pmatrix} 1 & 0 & -\frac{\sigma}{2} & \left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \end{pmatrix}$$

$$G: \begin{pmatrix} 1 & -\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} & 0 & -\frac{\sigma}{2} \end{pmatrix} \qquad H: \begin{pmatrix} 1 & -\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} & 0 & \frac{\sigma}{2} \end{pmatrix}$$

$$I:\left(1 \quad -\frac{\sigma}{2} \quad -\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 0\right) \qquad J:\left(1 \quad \frac{\sigma}{2} \quad -\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 0\right)$$

$$K:\left(1 \quad 0 \quad -\frac{\sigma}{2} \quad -\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2}\right) \qquad L:\left(1 \quad 0 \quad \frac{\sigma}{2} \quad -\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2}\right) \tag{19}$$

and the coordinates of the twelve faces of the dodecahedron are

$$a:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 1 \quad 0 \quad \frac{2}{1+\sqrt{5}}\right) \qquad b:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 1 \quad 0 \quad -\frac{2}{1+\sqrt{5}}\right)$$

$$c:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad \frac{2}{1+\sqrt{5}} \quad 1 \quad 0\right) \qquad d:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad -\frac{2}{1+\sqrt{5}} \quad 1 \quad 0\right)$$

$$e:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 0 \quad \frac{2}{1+\sqrt{5}} \quad 1\right) \qquad f:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 0 \quad -\frac{2}{1+\sqrt{5}} \quad 1\right)$$

$$g:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad -1 \quad 0 \quad -\frac{2}{1+\sqrt{5}}\right) \qquad h:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad -1 \quad 0 \quad \frac{2}{1+\sqrt{5}}\right)$$

$$i:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad -\frac{2}{1+\sqrt{5}} \quad -1 \quad 0\right) \qquad j:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad \frac{2}{1+\sqrt{5}} \quad -1 \quad 0\right)$$

$$k:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 0 \quad -\frac{2}{1+\sqrt{5}} \quad -1\right) \qquad l:\left(\left(\frac{1+\sqrt{5}}{2}\right)\frac{\sigma}{2} \quad 0 \quad \frac{2}{1+\sqrt{5}} \quad -1\right) \tag{20}$$

The correlation matrix mapping the coordinates of the faces to the coordinates of the vertices is:

$$\begin{bmatrix} \frac{2}{\sigma}\left(\frac{2}{1+\sqrt{5}}\right) & 0 & 0 & 0 \\ 0 & \frac{\sigma}{2}\left(\frac{1+\sqrt{5}}{2}\right) & 0 & 0 \\ 0 & 0 & \frac{\sigma}{2}\left(\frac{1+\sqrt{5}}{2}\right) & 0 \\ 0 & 0 & 0 & \frac{\sigma}{2}\left(\frac{1+\sqrt{5}}{2}\right) \end{bmatrix} \tag{21}$$

The 6x30 line matrix is formed by the ray coordinates of the lines of the thirty edges of the icosahedron and the resulting 30x6 pseudo inverse matrix confirms the duality between the icosahedron and the dual dodecahedron (see Appendix). The rows of the pseudo inverse matrix represent the axis line coordinates of the thirty edges of the dual dodecahedron each formed by two adjacent faces.

Acknowledgement

The authors wish to acknowledge the financial support of the Air Force Office of Scientific Research (Grant No. F49620-00-1-0021). The authors also wish to acknowledge the helpful comments of Professor K. H. Hunt.

Appendix

The 6x30 normalized line matrix of the thirty lines forming the edges of the icosahedron is

M =

Columns 1 through 7

BA	*CA*	*EA*	*FA*	*JA*	*CB*	*EC*
0	0.3090	0.8090	0.8090	0.3090	0.3090	0.5000
0	-0.8090	-0.5000	0.5000	0.8090	-0.8090	0.3090
1.0000	0.5000	-0.3090	-0.3090	0.5000	-0.5000	-0.8090
0	0.4045σ	0.2500σ	-0.2500σ	-0.4045σ	-0.4045σ	-0.6545σ
-0.8090σ	-0.2500σ	0.6545σ	0.6545σ	-0.2500σ	0.2500σ	0.4045σ
0	-0.6545σ	-0.4045σ	0.4045σ	0.6545σ	-0.6545σ	-0.2500σ

Columns 8 through 14

FE	*JF*	*BJ*	*LB*	*CL*	*DC*	*ED*
0	-0.5000	-0.3090	0.8090	-0.5000	1.0000	-0.5000
1.0000	0.3090	-0.8090	-0.5000	-0.3090	0	0.3090
0	0.8090	0.5000	0.3090	-0.8090	0	-0.8090
-0.8090σ	-0.6545σ	-0.4045σ	-0.2500σ	-0.6545σ	0	-0.6545σ
0	-0.4045σ	-0.2500σ	-0.6545σ	0.4045σ	0	-0.4045σ
0	-0.2500σ	-0.6545σ	-0.4045σ	0.2500σ	-0.8090σ	0.2500σ

Columns 15 through 21

HE	*FH*	*IF*	*JI*	*KJ*	*BK*	*DL*
0.8090	-0.8090	0.5000	-1.0000	0.5000	-0.8090	0.5000
0.5000	0.5000	0.3090	0	-0.3090	-0.5000	-0.3090
0.3090	-0.3090	0.8090	0	0.8090	-0.3090	-0.8090
-0.2500σ	-0.2500σ	-0.6545σ	0	-0.6545σ	-0.2500σ	-0.6545σ
0.6545σ	-0.6545σ	0.4045σ	0	-0.4045σ	0.6545σ	-0.4045σ
-0.4045σ	-0.4045σ	0.2500σ	-0.8090σ	0.2500σ	-0.4045σ	-0.2500σ

Columns 22 through 28

HD	*IH*	*KI*	*LK*	*GL*	*GD*	*GH*
0.3090	-0.3090	-0.5000	0	0.8090	0.3090	0
0.8090	0.8090	-0.3090	-1.0000	0.5000	0.8090	0
-0.5000	0.5000	0.8090	0	-0.3090	0.5000	1.0000
-0.4045σ	-0.4045σ	-0.6545σ	-0.8090σ	0.2500σ	0.4045σ	0
-0.2500σ	0.2500σ	0.4045σ	0	-0.6545σ	0.2500σ	0.8090σ
-0.6545σ	-0.6545σ	-0.2500σ	0	-0.4045σ	-0.6545σ	0

Columns 29 through 30

GI	*GK*
0.3090	0.8090
-0.8090	-0.5000
0.5000	-0.3090
-0.4045σ	-0.2500σ
0.2500σ	-0.6545σ
0.6545σ	0.4045σ

The pseudo inverse, $\boldsymbol{M}^T\left(\boldsymbol{M}\boldsymbol{M}^T\right)^{-1}$, of the line matrix of the lines of the icosahedron is

$\boldsymbol{M}^{-1}$ =

0	0	0.8090σ	0	-1.0000	0	*ba*
0.2500σ	-0.6545σ	0.4045σ	0.5000	-0.3090	-0.8090	*ca*
0.6545σ	-0.4045σ	-0.2500σ	0.3090	0.8090	-0.5000	*ea*
0.6545σ	0.4045σ	-0.2500σ	-0.3090	0.8090	0.5000	*fa*
0.2500σ	0.6545σ	0.4045σ	-0.5000	-0.3090	0.8090	*ja*
0.2500σ	-0.6545σ	-0.4045σ	-0.5000	0.3090	-0.8090	*cb*
0.4045σ	0.2500σ	-0.6545σ	-0.8090	0.5000	-0.3090	*ec*
0	0.8090σ	0	-1.0000	0	0	*fe*
-0.4045σ	0.2500σ	0.6545σ	-0.8090	-0.5000	-0.3090	*jf*
-0.2500σ	-0.6545σ	0.4045σ	-0.5000	-0.3090	-0.8090	*bj*
0.6545σ	-0.4045σ	0.2500σ	-0.3090	-0.8090	-0.5000	*lb*
-0.4045σ	-0.2500σ	-0.6545σ	-0.8090	0.5000	0.3090	*cl*
0.8090σ	0	0	0	0	-1.0000	*dc*
-0.4045σ	0.2500σ	-0.6545σ	-0.8090	-0.5000	0.3090	*ed*
0.6545σ	0.4045σ	0.2500σ	-0.3090	0.8090	-0.5000	*he*
-0.6545σ	0.4045σ	-0.2500σ	-0.3090	-0.8090	-0.5000	*fh*
0.4045σ	0.2500σ	0.6545σ	-0.8090	0.5000	0.3090	*if*
-0.8090σ	0	0	0	0	-1.0000	*ji*
0.4045σ	-0.2500σ	0.6545σ	-0.8090	-0.5000	0.3090	*kj*
-0.6545σ	-0.4045σ	-0.2500σ	-0.3090	0.8090	-0.5000	*bk*
0.4045σ	-0.2500σ	-0.6545σ	-0.8090	-0.5000	-0.3090	*dl*
0.2500σ	0.6545σ	-0.4045σ	-0.5000	-0.3090	-0.8090	*hd*
-0.2500σ	0.6545σ	0.4045σ	-0.5000	0.3090	-0.8090	*ih*
-0.4045σ	-0.2500σ	0.6545σ	-0.8090	0.5000	-0.3090	*ki*
0	-0.8090σ	0	-1.0000	0	0	*lk*
0.6545σ	0.4045σ	-0.2500σ	0.3090	-0.8090	-0.5000	*gl*
0.2500σ	0.6545σ	0.4045σ	0.5000	0.3090	-0.8090	*gd*
0	0	0.8090σ	0	1.0000	0	*gh*
0.2500σ	-0.6545σ	0.4045σ	-0.5000	0.3090	0.8090	*gi*
0.6545σ	-0.4045σ	-0.2500σ	-0.3090	-0.8090	0.5000	*gk*

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