# Kinematic Analysis of a Planar Tensegrity Mechanism with Pre-Stressed Springs

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**Abstract** This paper presents the equilibrium analysis of a planar tensegrity mechanism. The device consists of a base and top platform that are connected in parallel by one connector leg (whose length can be controlled via a prismatic joint) and two spring elements whose linear spring constants and free lengths are known. The paper presents three cases: 1) the spring free lengths are both zero, 2) one of the spring free lengths is zero and the other is nonzero, and 3) both free lengths are nonzero. The purpose of the paper is to show the increase in complexity that results from nonzero free lengths. It is shown that six equilibrium configurations exist for case 1, twenty equilibrium configurations exist for case 2, and sixty two configurations exist for case 3.

**Keywords:** planar mechanisms, tensegrity

## 1. Introduction

The word tensegrity is a combination of the words tension and integrity (Edmondson, 1987 and Fuller, 1975). Tensegrity structures are spatial structures formed by a combination of rigid elements in compression (struts) and connecting elements that are in tension (ties). No pair of struts touch and the end of each strut is connected to three non-coplanar ties (Yin et al, 2002). The entire configuration stands by itself and maintains its form solely because of the internal arrangement of the struts and ties (Tobie, 1976).

The development of tensegrity structures is relatively new and the works related have only existed for approximately twenty five years. Kenner, 1976, established the relation between the rotation of the top and bottom ties. Tobie, 1976, presented procedures for the generation of tensile structures by physical and graphical means. Yin, 2002, obtained

Kenner's results using energy considerations and found the equilibrium position for unloaded tensegrity prisms. Stern, 1999, developed generic design equations to find the lengths of the struts and elastic ties needed to create a desired geometry for a symmetric case. Knight, 2000, addressed the problem of stability of tensegrity structures for the design of deployable antennae.

## 2. Problem Statement

The mechanism to be analyzed here is shown in Figure 1. The top platform (indicated by points 4, 5, and 6) is connected to the base platform (indicated by points 1, 2, and 3) by two spring elements whose lengths are $L_1$ and $L_2$ and by a variable length connector whose length is referred to as $L_3$. Although this does not match the exact definition of tensegrity, the device is prestressed in the same manner as a tensegrity mechanism. The exact problem statement is as follows:

Given:

| | |
|---|---|
| $L_{12}$ | distance between points 1 and 2 |
| $p_{3x}$, $p_{3y}$ | coordinates of point 3 in coord. system 1 |
| $L_{45}$ | distance between points 4 and 5 |
| $p_{6x}$, $p_{6y}$ | coordinates of point 6 in coord. system 2 |
| $L_3$ | distance between points 1 and 4 |
| $k_1$, $L_{01}$ | spring constant and free length of spring 1 |
| $k_2$, $L_{02}$ | spring constant and free length of spring 2 |

Find: All static equilibrium configurations.

It is apparent that since the length $L_3$ is given, the device has two degrees of freedom. Thus there are two descriptive parameters that must be selected in order to define the system. For this analysis, the descriptive parameters are chosen as the angles $\gamma_1$, the angle between the $x_1$ axis and the line defined by points 1 and 4, and $\gamma_2$, the angle between

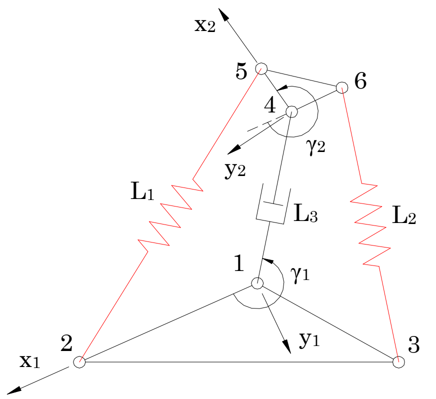


*Figure 1.* Compliant mechanism

the $x_1$ axis and the line defined by points 4 and 5. No other set of tested parameters yielded a less complicated solution than is presented here.

## 3. Solution Approach

Two possible solution approaches were considered, i.e. (1) satisfy force and moment conditions for equilibrium and (2) obtain configurations of minimum potential energy. Each approach was found to realize the same set of constraint equations. As such, obtaining the condition for force and moment balance is presented here.

The first step of the analysis is to determine the coordinates of the six points in terms of the base coordinate system as expressed in terms of the descriptive parameters $\gamma_1$ and $\gamma_2$. The coordinates of the three points in the base may be written with respect to the $x_1y_1$ coordinate system as

$$\mathbf{P}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \mathbf{P}_2 = \begin{bmatrix} L_{12} \\ 0 \end{bmatrix}, \mathbf{P}_3 = \begin{bmatrix} p_{3x} \\ p_{3y} \end{bmatrix}. \tag{1}$$

The coordinates of the three points in the top platform may be written as

$$\mathbf{P}_4 = \begin{bmatrix} L_3 c_1 \\ L_3 s_1 \end{bmatrix}, \mathbf{P}_5 = \begin{bmatrix} L_3 c_1 + L_{45} c_2 \\ L_3 s_1 + L_{45} s_2 \end{bmatrix}, \mathbf{P}_6 = \begin{bmatrix} L_3 c_1 + p_{6x} c_2 - p_{6y} s_2 \\ L_3 s_1 + p_{6x} s_2 + p_{6y} c_2 \end{bmatrix} \tag{2}$$

where $s_i$ and $c_i$, i=1,2, represent the sine and cosine of the angle $\gamma_i$.

A free body diagram of the top platform indicates that the sum of the forces along the three connector lines must equal zero at equilibrium. The unitized Plücker coordinates of a connector line can be obtained as

$$\mathbf{\$}_i = \frac{1}{d_i} \begin{bmatrix} x_t - x_b \\ y_t - y_b \\ \left[(x_b \mathbf{i} + y_b \mathbf{j}) \times \left((x_t - x_b)\mathbf{i} + (y_t - y_b)\mathbf{j}\right)\right] \cdot \mathbf{k} \end{bmatrix} \tag{3}$$

where $(x_t, y_t)$ and $(x_b, y_b)$ are respectively the coordinates of the points on the top and bottom platforms that are on the line and $d_i$ is the distance between the points that is calculated as

$$d_i^2 = (x_t - x_b)^2 + (y_t - y_b)^2 . \tag{4}$$

For the two spring connectors, i = 1,2 and

$$d_1^2 = 2L_3 L_{45}(c_1 c_2 + s_1 s_2) - 2L_{12}(L_3 c_1 + L_{45} c_2) + L_{12}^2 + L_3^2 + L_{45}^2 , \tag{5}$$

$$\begin{aligned} d_2^2 &= 2L_3(p_{6x} s_2 + p_{6y} c_2 - p_{3y}) s_1 + 2(p_{6y} p_{3x} - p_{6x} p_{3y} - L_3 p_{6y} c_1) s_2 \\ &+ 2L_3(p_{6x} c_2 - p_{3x}) c_1 + L_3^2 + p_{3x}^2 + p_{3y}^2 + p_{6x}^2 + p_{6y}^2 . \end{aligned} \tag{6}$$

The force in each of the springs can be written as

$$f_1 = k_1 (d_1 - L_{01}) \tag{7}$$

$$f_2 = k_2 (d_2 - L_{02}) \tag{8}$$

The summation of the three forces that are acting on the top platform may be written as

$$f_1 \mathbf{\$}_1 + f_2 \mathbf{\$}_2 + f_3 \mathbf{\$}_3 = \mathbf{0} \tag{9}$$

It is interesting to note that this equation implies that a necessary condition for static equilibrium is that the three line coordinates are linearly dependent.

The three line coordinates which were defined by (3) may now be written as

$$\mathbf{\$}_i = [l_i, m_i, n_i]^T, \ i = 1 \ldots 3 . \tag{10}$$

Equation (9) may be rearranged as

$$f_3 \begin{bmatrix} l_3 \\ m_3 \\ n_3 \end{bmatrix} = - \begin{bmatrix} f_1 l_1 + f_2 l_2 \\ f_1 m_1 + f_2 m_2 \\ f_1 n_1 + f_2 n_2 \end{bmatrix} . \tag{11}$$

In order for a solution to exist, it is necessary that the three scalar equations represented by (11) be satisfied. Since $n_3 = 0$, the equation represented by the third row of (11) may be written as

$$f_1 n_1 + f_2 n_2 = 0 \ . \tag{12}$$

Eliminating the unknown $f_3$, from the two scalar equations obtained from the first two rows of (11) gives

$$l_3 \, (f_1 m_1 + f_2 m_2) - m_3 \, (f_1 l_1 + f_2 l_2) = 0 \ . \tag{13}$$

Equations (12) and (13) represent the conditions that must be satisfied for the mechanism to be in static equilibrium. All the terms in these equations have been defined in terms of the descriptive parameters $\gamma_1$ and $\gamma_2$.

## 4. Case 1 – Both Free Lengths Equal Zero

For this simple case it is assumed that the free lengths of the two springs, i.e. $L_{01}$ and $L_{02}$, are both equal to zero. The forces in the two springs as defined in (7) and (8) now reduce to

$$f_1 = k_1 \, d_1 \tag{14}$$

$$f_2 = k_2 \, d_2 \tag{15}$$

Substituting these expressions as well as the line coordinate terms defined by (3) into (12) and (13) give

$$L_3 \, (k_1 L_{12} + k_2 p_{3x}) \, s_1 + [k_1 L_{12} L_{45} + k_2 (p_{3x} p_{6x} + p_{3y} p_{6y})] \, s_2 - k_2 L_3 p_{3y} \, c_1 + k_2 \, (p_{3x} p_{6y} - p_{3y} p_{6x}) \, c_2 = 0 \tag{16}$$

$$(k_1 L_{45} + k_2 p_{6x}) \, (c_1 s_2 - s_1 c_2) + k_2 p_{6y} \, (c_1 c_2 + s_1 s_2) + (k_2 p_{3x} + k_1 L_{12}) \, s_1 - k_2 p_{3y} \, c_1 = 0 \tag{17}$$

Note that since the free lengths of the springs are zero that the terms $d_1$ and $d_2$ have vanished.

The solution for the values of the angles $\gamma_1$ and $\gamma_2$ that simultaneously satisfy (16) and (17) proceeds by defining their tan-half angles as

$$x_i = \tan \frac{\gamma_i}{2} \tag{18}$$

and then introducing the trigonometric identities

$$s_i = \frac{2x_i}{1+x_i^2}, \quad c_i = \frac{1-x_i^2}{1+x_i^2}. \tag{19}$$

Substituting (19) into (16) and (17) and rearranging yields

$$(A_1x_2^2+A_2x_2+A_3)x_1^2 + (A_4x_2^2+A_5x_2+A_6)x_1 + (A_7x_2^2+A_8x_2+A_9) = 0\ , \tag{20}$$

$$(B_1x_2^2+B_2x_2+B_3)x_1^2 + (B_4x_2^2+B_5x_2+B_6)x_1 + (B_7x_2^2+B_8x_2+B_9) = 0 \tag{21}$$

where the coefficients $A_1$ through $B_9$ can be evaluated in terms of given values.

Crane and Duffy, 1998, show how Bezout's method can be used to yield in general eight solutions for $x_1$ and $x_2$ that satisfy the bi-quadratic equations (20) and (21). In this case the solution was found symbolically to reduce to sixth degree.

Several numerical examples were evaluated. Typically four real solutions and two complex solutions were obtained. All six cases satisfied the equilibrium conditions defined by (12) and (13).

## 5. Case 2 – One Non-Zero Free Length

For this case it is assumed that the free length of spring 1 is nonzero and the free length of spring 2 is zero. The forces in the two springs as defined in (7) and (8) are now written as

$$f_1 = k_1\ (d_1 - L_{01}) \tag{22}$$

$$f_2 = k_2\ d_2\ . \tag{23}$$

Substituting (22) and (23) as well as the line coordinate terms defined by (3) into (12) and (13) and rearranging now gives

$$A_1d_1 + A_2 = 0, \tag{24}$$

$$B_1d_1 + B_2 = 0 \tag{25}$$

where the terms $A_1$ through $B_2$ are expressed in terms of the sines and cosines of $\gamma_1$ and $\gamma_2$. Equations (24) and (25) express the necessary and sufficient condition for an equilibrium configuration. Note that the square of the distance between points 2 and 5, i.e. $d_1^2$, is expressed in terms of the angles $\gamma_1$ and $\gamma_2$ in equation (5).

Equations (24), (25), and (5) are treated simply as a set of three equations in the three unknowns $\gamma_1$, $\gamma_2$, and $d_1$, and no attempt is made to manipulate the equations so that one variable is eliminated by direct substitution. Substituting the tan-half-angle identities, (19), into these three equations and rearranging yields

$$(E_1x_2^2 + E_2x_2 + E_3)\ d_1 + E_4x_2^2 + E_5x_2 + E_6 = 0\ , \tag{26}$$

$$(F_1x_2^2 + F_2x_2 + F_3)\ d_1 + F_4x_2^2 + F_5x_2 + F_6 = 0\ , \tag{27}$$

$$(G_1x_2^2 + G_2x_2 + G_3)\ d_1^2 + G_4x_2^2 + G_5x_2 + G_6 = 0 \tag{28}$$

where the coefficients $E_1$ through $G_6$ are functions of $x_1$.

Sylvester's elimination procedure is used to solve the set of equations (26) through (28). These three equations are multiplied by $x_2$, $d_1$, and $d_1x_2$. Equations (26) and (27) are multiplied by $d_1^2$ and $d_1^2x_2$. This results in a total of sixteen equations in the set of 'variables' $d_1^3x_2^3$, $d_1^3x_2^2$, $d_1^3x_2$, $d_1^3$, $d_1^2x_2^3$, $d_1^2x_2^2$, $d_1^2x_2$, $d_1^2$, $d_1x_2^3$, $d_1x_2^2$, $d_1x_2$, $d_1$, $x_2^3$, $x_2^2$, $x_2$, and 1. A necessary condition for a solution to exist for these sixteen 'linear' 'homogeneous' equations is that they be linear dependent and thus the determinant of the coefficient matrix must equal zero. Since the coefficients $E_1$ through $G_6$ are 2nd order in the variable $x_1$, expansion of the determinant results in a 32nd degree polynomial in the variable $x_1$. Values for $x_2$ and $d_2$ that correspond to each solution of $x_1$ are then readily obtained.

Several numerical examples for this case were performed. Eight of the 32 solutions for $x_1$ were equal to +i or –i which means that the 32nd degree polynomial in $x_1$ may be divided by the factor $(1+x_1^2)^4$. Four of the solutions correspond to the case where points 2 and 5 are coincident. It can be shown that the value of $x_1$ when the two points are coincident may be determined from

$$(L_3^2+L_{12}^2-L_{45}^2+2L_3L_{12})x_1^2 + (L_3^2+L_{12}^2-L_{45}^2-2L_3L_{12}) = 0 \; . \qquad (29)$$

Thus, the remaining 24th degree polynomial may be divided by this factor to result in a 20th degree polynomial in $x_1$. It is concluded that the correct degree of the solution is twenty.

## 6. Case 3 – Both Free Lengths are Non-Zero

For this case, the free lengths of both springs are nonzero and the forces in the two springs are defined in (7) and (8). Substituting (7) and (8) as well as the line coordinate terms defined by (3) into (12) and (13), substituting the tan-half angle identities for the sines and cosines of $\gamma_1$ and $\gamma_2$, and rearranging now gives

$$(C_1x_2^2+C_2x_2+C_3)\, d_1d_2 + (C_4x_2^2+C_5x_2+C_6)\, d_1 + (C_7x_2^2+C_8x_2+C_9)\, d_2 = 0 \; , \quad (30)$$

$$(D_1x_2^2+D_2x_2+D_3)\, d_1d_2 + (D_4x_2^2+D_5x_2+D_6)\, d_1 + (D_7x_2^2+D_8x_2+D_9)\, d_2 = 0 \quad (31)$$

where the coefficients $C_1$ through $D_9$ are functions of $x_1$. Substituting the tan-half angle identities into (5) and (6) and rearranging gives

$$(M_1x_2^2 + M_2x_2 + M_3)\, d_1^2 + (M_4x_2^2 + M_5x_2 + M_6) = 0 \; , \qquad (32)$$

$$(N_1x_2^2 + N_2x_2 + N_3)\, d_2^2 + (N_4x_2^2 + N_5x_2 + N_6) = 0 \qquad (33)$$

where the coefficients $M_1$ through $N_6$ are functions of $x_1$. The definition of the coefficients $C_1$ through $N_6$ are not listed here for brevity.

Equations (30) and (31) are divided by $d_1d_2$, equation (32) is divided by $d_1^2$, and equation (33) is divided by $d_2^2$ to yield the four equations

$$(C_1x_2^2+C_2x_2+C_3) + (C_4x_2^2+C_5x_2+C_6)\, d_{2i} + (C_7x_2^2+C_8x_2 +C_9)\, d_{1i} = 0 \; , \quad (34)$$

$$(D_1x_2^2+D_2x_2+D_3) + (D_4x_2^2+D_5x_2+D_6)\, d_{2i} + (D_7x_2^2+D_8x_2+D_9)\, d_{1i} = 0 \quad (35)$$

$$(M_1x_2^2 + M_2x_2 + M_3) + (M_4x_2^2 + M_5x_2 + M_6)\, d_{1i}^2 = 0 \; , \qquad (36)$$

$$(N_1x_2^2 + N_2x_2 + N_3) + (N_4x_2^2 + N_5x_2 + N_6)\, d_{2i}^2 = 0 \qquad (37)$$

where $d_{1i} = 1/d_1$ and $d_{2i} = 1/d_2$. Equations (34) through (37) are expressions that must be solved at an equilibrium configuration and are expressed in terms of the four unknowns $x_1$ (which is embedded in the coefficients), $x_2$, $d_{1i}$, and $d_{2i}$.

Sylvester's elimination procedure is used to obtain a single polynomial in $x_1$. Equations (34) through (37) can be treated as four "homogenous" equations in the 15 unknowns $d_{2i}^2x_2^2$, $d_{2i}^2x_2$, $d_{2i}^2$, $d_{1i}^2x_2^2$, $d_{1i}^2x_2$, $d_{1i}^2$, $d_{2i}x_2^2$, $d_{2i}x_2$, $d_{2i}$, $d_{1i}x_2^2$, $d_{1i}x_2$, $d_{1i}$, $x_2^2$, $x_2$, and 1. Equations (34) and (35) are multiplied by $d_{1i}$, $d_{2i}$, $d_{1i}d_{2i}$, $d_{1i}^2$, $d_{2i}^2$, $d_{1i}^2d_{2i}$, and $d_{1i}d_{2i}^2$, equation (36) is multiplied by $d_{1i}$, $d_{2i}$, $d_{1i}d_{2i}$, and $d_{2i}^2$, and equation (37) is multiplied by $d_{1i}$, $d_{2i}$, $d_{1i}d_{2i}$, and $d_{1i}^2$ to yield a set of 26 equations in a total set of 39 unknowns. Multiplying all 26 of these equations by $x_2$ finally results in a total set of 52 "homogeneous" equations in 52 unknowns.

A nontrivial answer will exist for these equations if the determinant of the 52×52 coefficient matrix equals zero. Since the coefficients are functions of the variable $x_1$, a polynomial in $x_1$ will result. An analysis of the degree of the coefficients as functions of $x_1$ indicated that the resulting polynomial would be of degree 104.

Corresponding values for the parameters $x_2$, $d_{1i}$, and $d_{2i}$ can be obtained by evaluating the coefficients $C_1$ through $N_6$ for a particular solution of $x_1$ and then solving any set of 51 of the 52 equations as true non-homogenous equations for the particular unknowns $x_2$, $d_{1i}$, and $d_{2i}$. In this problem this requires that a 51×51 matrix must be inverted for each of the 104 solutions for $x_1$.

A numerical example was run and 104 solution sets of $x_1$, $x_2$, $d_1$, and $d_2$ were obtained. Twenty six of the $x_1$ solutions were equal to ±i. These solutions are often referred to as circular points at infinity and it must be the case that the 104[th] degree polynomial can be divided by $(1+x_1^2)^{13}$. It was not surprising that these solutions occurred as the terms $M_1$, $M_3$, $N_1$, and $N_3$ all equaled $(1+x_1^2)$.

Of the remaining 78 solutions, 40 were real and 38 were complex. Corresponding values for $x_2$, $d_{1i}$, and $d_{2i}$ were obtained for the remaining 78 solutions. All 78 of the solutions were then substituted into (34) through (37) to see if they indeed satisfied the constraint equations. 16 of the real solutions did not satisfy (34) through (37) which means that extraneous roots were indeed introduced in the elimination process. It is concluded from this example that no more than 62 solutions exist. Further analysis must be conducted to obtain an elimination procedure that does not introduce extraneous roots.

## 7. Conclusions

The purpose of this paper was to show the significant increase in complexity that results when springs with nonzero free lengths are incorporated in pre-stressed mechanisms. It has been shown that six equilibrium configurations exist for the case of a simple planar mechanism with two springs where both springs have zero free lengths. Twenty equilibrium configurations were found for the case where one of the springs had a nonzero free length. For the case where both springs had nonzero free lengths, seventy eight solutions sets were obtained once the circular points at infinity were disregarded. Sixteen of these seventy eight, did not satisfy the equation set which means that the presented elimination technique introduced extraneous roots. The remaining sixty two solutions satisfied the equations, but two solutions in the numerical example resulted in cases where the lines along the three legs did not intersect which is puzzling.

Additional work needs to be done before this simple case is fully understood. The approach presented here does however bound the dimension of the solution. The goal of the authors is to extend this work to spatial devices in order to develop a thorough understanding of the nature of these pre-stressed mechanisms.

## 8. Acknowledgements

The authors would like to gratefully acknowledge the support of the Department of Energy, grant number DE-FG04-86NE37967.

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