STIFFNESS MAPPING OF PLANAR COMPLIANT PARALLEL MECHANISMS IN A SERIAL ARRANGEMENT

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Abstract This paper presents a stiffness mapping of a mechanism having two planar compliant parallel mechanisms in a serial arrangement. The stiffness matrix of the mechanism is obtained by taking a derivative of the static equilibrium equations. A derivative of spring force connecting two moving bodies is derived and it is applied to obtain the stiffness matrix of the mechanism. A numerical example is presented.

Keywords: Stiffness matrix, compliant coupling, parallel mechanism

1. Introduction

There are many robotic tasks involving contacts of man and machine or the robot and its environment. A small amount of positional error of the robot system, which is almost inevitable, may cause serious damage to the robot or the object with which it is in contact. Compliant couplings which may be inserted between the end effecter and the last link of the robotic manipulator can be a solution to this problem (Whitney, 1982, Peshkin, 1990, and Griffis, 1991).

Dimentberg, 1965, studied properties of an elastically suspended body using Screw theory which was introduced by Ball, 1900. Screw theory is employed throughout this paper to describe the motion of rigid bodies (twist) and the forces applied to rigid bodies (wrench) (Crane et al, 2006). A small twist applied to the compliant coupling generates a small change of the wrench which the compliant coupling exerts on the environment. This relation is well described by the stiffness matrix of the compliant coupling.

Parallel mechanisms have several advantages over serial mechanisms such as high stiffness, compactness, and small positional errors at the cost of a smaller work space and increased complexity of analysis. Griffis, 1991, obtained a global stiffness model for parallel mechanism-based compliant couplings. Huang and Schimmels, 1998, Ciblak and Lipkin, 1999, and Roberts, 1999, studied synthesis of stiffness matrices.

2. Problem Statement

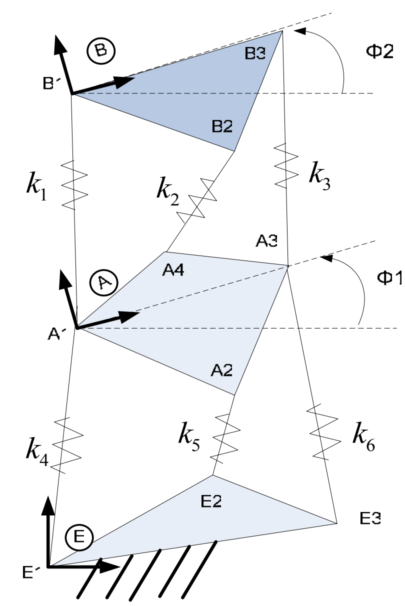

Figure 1. Mechanism having two planar parallel mechanisms in a serial arrangement

Fig. 1 depicts the compliant mechanism whose stiffness matrix will be obtained in this paper. Body A is connected to ground by three compliant couplings and body B is connected to body A in the same way. Each compliant coupling has a revolute joint at each end and a prismatic joint with a spring in the middle. It is assumed that an external wrench $\underline{\mathbf{w}}_{ext}$ is applied to body B and that both body B and body A are in static equilibrium. The poses of body A and body B and the spring constants and free lengths of all compliant couplings are known.

The stiffness matrix $[K]$ which maps a small twist of the moving body B in terms of the ground, ${}^{E}\delta\underline{\mathbf{D}}^{B}$, into the corresponding wrench variation, $\delta\underline{\mathbf{w}}_{ext}$, is desired to be derived. This relationship can be written as

$$\delta \underline{\mathbf{w}}_{ext} = \left[K\right]^{E} \delta \underline{\mathbf{D}}^{B} . \tag{1}$$

The static equilibrium equation of bodies B and A can be written by

$$\begin{aligned} \underline{\mathbf{w}}_{ext} &= \underline{\mathbf{f}}_1 + \underline{\mathbf{f}}_2 + \underline{\mathbf{f}}_3 \\ &= \underline{\mathbf{f}}_4 + \underline{\mathbf{f}}_5 + \underline{\mathbf{f}}_6 \end{aligned} \tag{2}$$

where $\underline{\mathbf{f}}_i$ are the forces from the compliant couplings.

The stiffness matrix will be derived by taking a derivative of the static equilibrium equation, Eq. 2, to yield

$$\begin{aligned} \delta \underline{\mathbf{w}}_{ext} &= \delta \underline{\mathbf{f}}_1 + \delta \underline{\mathbf{f}}_2 + \delta \underline{\mathbf{f}}_3 \\ &= \delta \underline{\mathbf{f}}_4 + \delta \underline{\mathbf{f}}_5 + \delta \underline{\mathbf{f}}_6 \end{aligned} . \tag{3}$$

Expressions for $\delta \underline{\mathbf{f}}_i$ for the compliant couplings joining body A and ground, i.e., for i=4, 5, 6, were obtained by Griffis, 1991.

The contribution of this new effort is in the analysis of the derivative of the spring force joining bodies A and B which will lead to the derivation of the compliant matrix that will relate the change in the external wrench to the twist of body B with respect to ground.

3. Derivative of Spring Force Connecting Two Moving Bodies

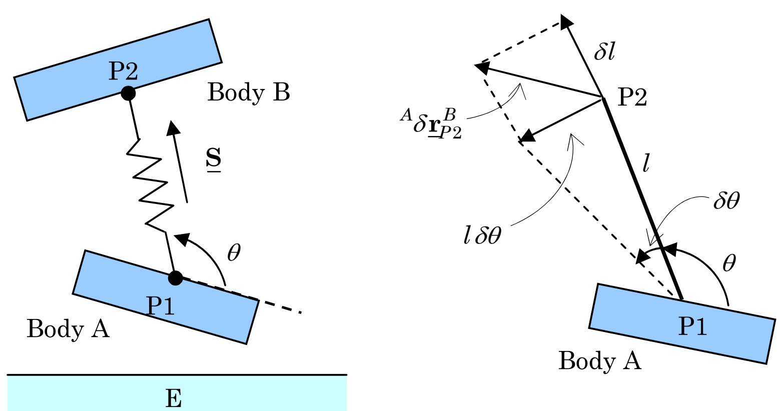

Figure 2. Compliant coupling conneting two moving bodies and variation of point P2 due to twist of body B with respect to body A

Fig. 2 depicts two rigid bodies connected to each other by a compliant coupling with a spring constant k, a free length l_o, and a current length l. The spring force may be written as

$$\underline{\mathbf{f}} = k(l - l_o)\underline{\$} \tag{4}$$

where

$$\underline{\$} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}^{A}_{P1} \times \underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}^{B}_{P2} \times \underline{\mathbf{S}} \end{bmatrix} \tag{5}$$

and where, $\underline{\mathbf{S}}$ is a unit vector along the compliant coupling. ${}^{E}\underline{\mathbf{r}}^{A}_{P1}$ and ${}^{E}\underline{\mathbf{r}}^{B}_{P2}$ are the position vector of the point P1 in body A and that of point P2 in body B, respectively, measured with respect to the reference system embedded in ground (body E).

The derivative of the spring force as in Eq. 4 can be written by

$${}^{E}\delta\underline{\mathbf{f}} = k\,\delta l\,\underline{\$} + k\,(l - l_o)\,{}^{E}\delta\underline{\$}\,. \tag{6}$$

From the twist equation, the variation of position of point P2 in body B with respect to body A can be expressed as

$${}^{A}\delta\underline{\mathbf{r}}^{B}_{P2} = {}^{A}\delta\underline{\mathbf{r}}^{B}_{o} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \times {}^{A}\underline{\mathbf{r}}^{B}_{P2} \tag{7}$$

where ${}^{\mathrm{A}}\underline{\mathbf{r}}^{\mathrm{B}}_{\mathrm{P2}}$ is the position of P2, which is embedded in body B, measured with respect to a coordinate system embedded in body A which at this instant is coincident and aligned with the reference system attached to ground. In addition, ${}^{A}\delta\underline{\boldsymbol{\varphi}}^{B}$ is the differential of angle of body B in terms of body A. It can also be decomposed into two perpendicular vectors along $\underline{\mathbf{S}}$ and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$ which is a known unit vector perpendicular to $\underline{\mathbf{S}}$. These two vectors correspond to the change of the spring length δl and the directional change of the spring $l\delta\theta$ in terms of body A as shown in Fig. 2. Thus Eq. 7 can be rewritten as

$$\begin{aligned} {}^{A}\delta\underline{\mathbf{r}}^{B}_{P2} &= \left({}^{A}\delta\underline{\mathbf{r}}^{B}_{P2} \cdot \underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{A}\delta\underline{\mathbf{r}}^{B}_{P2} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\ &= \delta l\,\underline{\mathbf{S}} + l\delta\theta\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \end{aligned} \tag{8}$$

where

$${}^{A}\frac{\partial\underline{\$}}{\partial\theta} = \begin{bmatrix} \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\ {}^{A}\underline{\mathbf{r}}^{A}_{P1} \times \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \end{bmatrix}. \tag{9}$$

From Eqs. 7 and 8, δl and $l\delta\theta$ can be expressed as

$$\delta l = {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B} \cdot \underline{\mathbf{S}} = {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \cdot \underline{\mathbf{S}} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \times {}^{A}\underline{\mathbf{r}}_{P2}^{B} \cdot \underline{\mathbf{S}}$$
$$= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \cdot \underline{\mathbf{S}} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \cdot {}^{A}\underline{\mathbf{r}}_{P2}^{B} \times \underline{\mathbf{S}} \tag{10}$$
$$= \underline{\$}^{T}\, {}^{A}\delta\underline{\mathbf{D}}^{B}$$

$$l\delta\theta = {}^{A}\delta\underline{\mathbf{r}}_{P2}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} = {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \times {}^{A}\underline{\mathbf{r}}_{P2}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$$
$$= {}^{A}\delta\underline{\mathbf{r}}_{o}^{B} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{A}\delta\underline{\boldsymbol{\varphi}}^{B} \cdot {}^{A}\underline{\mathbf{r}}_{P2}^{B} \times \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \tag{11}$$
$$= \frac{{}^{A}\partial\underline{\$}'^{T}}{\partial\theta}\, {}^{A}\delta\underline{\mathbf{D}}^{B}$$

where

$$\frac{{}^{A}\partial\underline{\$}'}{\partial\theta} = \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\ {}^{A}\underline{\mathbf{r}}_{P2}^{B} \times \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \end{bmatrix}. \tag{12}$$

$\frac{{}^{A}\partial\underline{\$}'}{\partial\theta}$ has the same direction as $\frac{{}^{A}\partial\underline{\$}}{\partial\theta}$ but has a different moment term.

${}^{E}\delta\underline{\$}$ in Eq. 6 is a derivative of the unit screw along the spring in terms of the inertial frame and may be written as

$${}^{E}\delta\underline{\$} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} + {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times {}^{E}\delta\underline{\mathbf{S}} \end{bmatrix}. \tag{13}$$

Using an intermediate frame attached to body A,

$${}^{E}\delta\underline{\mathbf{S}} = {}^{A}\delta\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}}. \tag{14}$$

Then, ${}^{E}\delta\underline{\$}$ may be decomposed into three screws as follows

$${}^{E}\delta\underline{\$} = \begin{bmatrix} {}^{E}\delta\underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} + {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times {}^{E}\delta\underline{\mathbf{S}} \end{bmatrix}$$
$$= \begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} + {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \left({}^{A}\delta\underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}}\right) \end{bmatrix}. \tag{15}$$
$$= \begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times {}^{A}\delta\underline{\mathbf{S}} \end{bmatrix} + \begin{bmatrix} {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}}\right) \end{bmatrix} + \begin{bmatrix} \underline{\mathbf{0}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} \end{bmatrix}$$

Since $\underline{\mathbf{S}}$ is a function of θ alone from the vantage of body A and $l\delta\theta$ is already described in Eq. 11, the first screw in Eq. 15 can be written as

$$\begin{bmatrix} {}^{A}\delta\underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times {}^{A}\delta\underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\delta\theta \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\delta\theta \end{bmatrix} = \frac{{}^{A}\partial\underline{\$}}{\partial\theta}\frac{1}{l}l\delta\theta = \frac{1}{l}\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\frac{{}^{A}\partial\underline{\$}'^{T}}{\partial\theta}\,{}^{A}\delta\underline{\mathbf{D}}^{B} \quad . \tag{16}$$

As to the second screw in Eq. 15, ${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}}$ has the same direction as $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$ with magnitude of ${}_{E}\delta\phi_{A}$ and thus may be written as

$${}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}} = {}_{E}\delta\phi_{A}\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} . \tag{17}$$

Then the second screw in Eq. 15 can be expressed as

$$\begin{bmatrix} {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \left({}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{S}}\right) \end{bmatrix} = \begin{bmatrix} \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\,{}_{E}\delta\phi_{A} \\ {}^{E}\underline{\mathbf{r}}_{P1}^{A} \times \dfrac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\,{}_{E}\delta\phi_{A} \end{bmatrix} = \frac{{}^{A}\partial\underline{\$}}{\partial\theta}\,{}_{E}\delta\phi_{A} = \frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}{}^{E}\delta\underline{\mathbf{D}}^{A} \quad . \tag{18}$$

As to the third screw in Eq. 15, ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$ can be decomposed into two perpendicular vectors along $\underline{\mathbf{S}}$ and $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}$, respectively as

$$\begin{aligned} {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times {}^{E}\underline{\mathbf{r}}_{P1}^{A} \\ &= \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \underline{\mathbf{S}}\right)\underline{\mathbf{S}} + \left({}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \end{aligned} \tag{19}$$

where

$$\begin{aligned} {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \underline{\mathbf{S}} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{r}}_{P1}^{A} \cdot \underline{\mathbf{S}} \\ &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \underline{\mathbf{S}} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \cdot \underline{\mathbf{r}}_{P1}^{A} \times \underline{\mathbf{S}} \\ &= \underline{\$}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{20}$$

$$\begin{aligned} {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \times \underline{\mathbf{r}}_{P1}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \\ &= {}^{E}\delta\underline{\mathbf{r}}_{o}^{A} \cdot \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} + {}^{E}\delta\underline{\boldsymbol{\varphi}}^{A} \cdot \underline{\mathbf{r}}_{P1}^{A} \times \frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta} \quad . \\ &= \frac{{}^{A}\partial\underline{\$}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{21}$$

By combining Eqs. 19, 20, and 21, ${}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}$ can be written as

$$ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A} = \left(\underline{\$}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\underline{\mathbf{S}} + \left(\frac{{}^{A}\partial\underline{\$}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}. \tag{22} $$

Then the third screw in Eq. 15 can be written as

$$ \begin{aligned} \begin{bmatrix} \underline{\mathbf{0}} \\ {}^{E}\delta\underline{\mathbf{r}}_{P1}^{A}\times\underline{\mathbf{S}} \end{bmatrix} &= \begin{bmatrix} \underline{\mathbf{0}} \\ \left\{\left(\underline{\$}^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\underline{\mathbf{S}} + \left(\frac{{}^{A}\partial\underline{\$}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\right\}\times\underline{\mathbf{S}} \end{bmatrix} \\ &= \begin{bmatrix} \underline{\mathbf{0}} \\ \left(\frac{{}^{A}\partial\underline{\$}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A}\right)\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\times\underline{\mathbf{S}} \end{bmatrix} = \begin{bmatrix} \underline{\mathbf{0}} \\ -\frac{{}^{A}\partial\underline{\$}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A} \end{bmatrix} \\ &= -\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\frac{{}^{A}\partial\underline{\$}^{T}}{\partial\theta}\,{}^{E}\delta\underline{\mathbf{D}}^{A} = -\left(\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right)^{T}\,{}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{23} $$

because $\frac{{}^{A}\partial\underline{\mathbf{S}}}{\partial\theta}\times\underline{\mathbf{S}} = -1\left(\underline{\mathbf{k}}\right)$.

By substituting δl and ${}^{E}\delta\underline{\$}$ in Eq. 6 with Eqs. 10, 16, 18, and 23 and arranging the terms by the twists, the derivative of the spring force can be written as

$$ \begin{aligned} {}^{E}\delta\underline{\mathbf{f}} &= k\delta l\underline{\$} + k(l-l_o)\,{}^{E}\delta\underline{\$} \\ &= \left[K_F\right]{}^{A}\delta\underline{\mathbf{D}}^{B} + \left[K_M\right]{}^{E}\delta\underline{\mathbf{D}}^{A} \end{aligned} \tag{24} $$

where

$$ \left[K_F\right] = k\underline{\$}\underline{\$}^{T} + k\left(1-\frac{l_o}{l}\right)\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\,\frac{{}^{A}\partial\underline{\$}'^{T}}{\partial\theta} \tag{25} $$

$$ \left[K_M\right] = k(l-l_o)\left(\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix} - \left(\frac{{}^{A}\partial\underline{\$}}{\partial\theta}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\right)^{T}\right). \tag{26} $$

As shown in Eq. 24, the derivative of the spring force joining two rigid bodies depends not only on a relative twist between two bodies but also on the twist of the intermediate body, in this case body A, in terms of the inertial frame. $\left[K_F\right]$ is identical to the stiffness matrix of the spring connecting a moving body to the ground which was derived by Griffis, 1991. $\left[K_M\right]$ is newly introduced from this research and results from the motion of the base frame, in this case body A. $\left[K_M\right]$ takes a skew symmetric form in general.

4. Stiffness Matrix of the Mechanism

The stiffness matrix $\left[K\right]$ which maps a small twist of body B in terms of the inertial frame into the corresponding change of the wrench on body B is derived from Eq. 3 (see Fig. 1). The derivatives of spring forces can be written by Eqs. 27 and 28 since Springs 4, 5, and 6 connect body A and ground and Springs 1, 2, and 3 join two moving bodies.

$$\begin{aligned}\delta\underline{\mathbf{f}}_4+\delta\underline{\mathbf{f}}_5+\delta\underline{\mathbf{f}}_6 &= \left[K_F\right]_4 {}^{E}\delta\underline{\mathbf{D}}^{A}+\left[K_F\right]_5 {}^{E}\delta\underline{\mathbf{D}}^{A}+\left[K_F\right]_6 {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= \left[K_F\right]_{R,L} {}^{E}\delta\underline{\mathbf{D}}^{A}\end{aligned} \tag{27}$$

$$\begin{aligned}\delta\underline{\mathbf{f}}_1+\delta\underline{\mathbf{f}}_2+\delta\underline{\mathbf{f}}_3 &= \left[K_F\right]_1 {}^{A}\delta\underline{\mathbf{D}}^{B}+\left[K_F\right]_2 {}^{A}\delta\underline{\mathbf{D}}^{B}+\left[K_F\right]_3 {}^{A}\delta\underline{\mathbf{D}}^{B} \\ &\quad+\left[K_M\right]_1 {}^{E}\delta\underline{\mathbf{D}}^{A}+\left[K_M\right]_2 {}^{E}\delta\underline{\mathbf{D}}^{A}+\left[K_M\right]_3 {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= \left[K_F\right]_{R,U} {}^{A}\delta\underline{\mathbf{D}}^{B}+\left[K_M\right]_{R,U} {}^{E}\delta\underline{\mathbf{D}}^{A}\end{aligned} \tag{28}$$

where

$$\left[K_F\right]_{R,L}=\sum_{i=4}^{6}\left[K_F\right]_i,\ \left[K_F\right]_{R,U}=\sum_{i=1}^{3}\left[K_F\right]_i,\ \left[K_M\right]_{R,U}=\sum_{i=1}^{3}\left[K_M\right]_i$$

and where $\left[K_F\right]_i$ and $\left[K_M\right]_i$ are defined as Eqs. 25 and 26.

Then from Eqs. 3, 27, and 28 the derivative of the external wrench can be written by

$$\begin{aligned}\delta\,\underline{\mathbf{w}}_{ext} &= \left[K\right] {}^{E}\delta\underline{\mathbf{D}}^{B} \\ &= \left[K_F\right]_{R,L} {}^{E}\delta\underline{\mathbf{D}}^{A} \\ &= \left[K_F\right]_{R,U} {}^{A}\delta\underline{\mathbf{D}}^{B}+\left[K_M\right]_{R,U} {}^{E}\delta\underline{\mathbf{D}}^{A}\end{aligned}\quad . \tag{29}$$

Finally, from Eq. 29 and the twist equation, Eq. 30, the stiffness matrix can be obtained as Eq. 31.

$${}^{E}\delta\underline{\mathbf{D}}^{B} = {}^{E}\delta\underline{\mathbf{D}}^{A} + {}^{A}\delta\underline{\mathbf{D}}^{B} \tag{30}$$

$$\left[K\right]=\left[K_F\right]_{R,L}\left(\left[K_F\right]_{R,L}+\left[K_F\right]_{R,U}-\left[K_M\right]_{R,U}\right)^{-1}\left[K_F\right]_{R,U} \tag{31}$$

5. Numerical Example

The geometry information and spring properties of the mechanism in Fig. 1 and the external wrench $\underline{\mathbf{w}}_{ext}$ are given below.

$$\underline{\mathbf{w}}_{ext}=\begin{bmatrix}0.01 & N \\ -0.02 & N \\ 0.03 & Ncm\end{bmatrix}$$

Table 1. Spring properties (Unit: N/cm for k, cm for l_o)

Spring No.	1	2	3	4	5	6
Stiffness constant k	0.2	0.3	0.4	0.5	0.6	0.7
Free length l_o	5.0040	2.2860	4.9458	5.5145	3.1573	5.2568

Table 2. Positions of pivot points in terms of the inertial frame (Unit: cm)

Pivot points	E1	E2	E3	B1	B2	B3
X	0.0000	1.5000	3.0000	0.0903	1.7063	1.9185
Y	0.0000	1.2000	0.5000	9.8612	8.6833	10.6721

(continue)

A1	A2	A3	A4
0.9036	2.5318	2.7236	1.6063
4.5962	3.4347	5.4255	5.4659

The stiffness matrices $[K]$ is obtained by using Eq. 31.

$$[K] = \begin{bmatrix} 0.0108 \ \ N/cm & -0.0172 \ \ N/cm & -0.0797 \ \ N \\ -0.0172 \ \ N/cm & 0.3447 \ \ N/cm & 0.8351 \ \ N \\ -0.0997 \ \ N & 0.8251 \ \ N & 2.6567 \ \ Ncm \end{bmatrix}$$

To evaluate the result, a small wrench $\delta \underline{\mathbf{w}}_G$ is applied to body B and the static equilibrium pose of the mechanism is obtained by a numerically iterative method. From the equilibrium pose of the mechanism, the twist of body B with respect to ground ${}^E\delta\underline{\mathbf{D}}^B$ is obtained as below.

$$\delta \underline{\mathbf{w}}_G = 10^{-4} \times \begin{bmatrix} 0.5 \ \ N \\ 0.2 \ \ N \\ 0.4 \ \ Ncm \end{bmatrix}$$

$${}^E\delta\underline{\mathbf{D}}^B = \begin{bmatrix} 0.0077 \ \ cm \\ -0.0012 \ \ cm \\ 0.0007 \ \ rad \end{bmatrix}$$

Then the twist ${}^E\delta\underline{\mathbf{D}}^B$ is multiplied by the stiffness matrices to see if the given small wrench $\delta \underline{\mathbf{w}}_G$ results.

$$\delta \underline{\mathbf{w}} = [K]\, {}^E\delta\underline{\mathbf{D}}^B = 10^{-4} \times \begin{bmatrix} 0.4991 \ \ N \\ 0.1988 \ \ N \\ 0.4016 \ \ Ncm \end{bmatrix}$$

The numerical example indicates that $[K]$ produces the given wrench $\delta \underline{\mathbf{w}}_G$ with high accuracy.

6. Conclusion

In this paper, a derivative of spring force connecting two moving bodies was derived by using screw theory and an intermediate frame and applied to obtain a stiffness matrix of a mechanism having two compliant parallel mechanisms serially arranged. A derivative of spring force connecting two moving bodies depends not only on a relative twist between the two bodies but also on the twist of the intermediate body in terms of the inertial frame. This result also can be applied for mechanisms having any arbitrary number of parallel mechanisms in a serial arrangement.

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